On groups with restricted centralizers of commutators

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Part 1: Introduction

A group G is said to have restricted centralizers if for each $g \in G$ the centralizer $C_G(g)$ either is finite or has finite index in G.

If G is profinite, then this is equivalent to saying that for each $g \in G$ the centralizer $C_G(g)$ either is finite or open.

This notion of restricted centralizers was introduced by Shalev in 1994. He showed that a profinite group with restricted centralizers is abelian-by-finite.

In a joint work with Detomi and Morigi we obtained a "verbal" version of this result.

Let $w = w(x_1, \ldots, x_k)$ be a group-word, that is, a nontrivial element of the free group F on free generators $x_1, x_2, \ldots$.

In any group G we define the verbal subgroup $w(G)$ of G, that is, the subgroup generated by the set of all values $w(g_1, \ldots, g_k)$, where $g_1, \ldots, g_k$ are elements of G.

If G is profinite, $w(G)$ is the closed subgroup generated by the set of all values $w(g_1, \ldots, g_k)$, where $g_1, \ldots, g_k$ are elements of G.

Say that a word is multilinear commutator if it can be written in the form of multilinear Lie monomial, ex.

$$[[x_1, x_2, x_3], [x_4, x_5]].$$

Such words are also known under the name of outer commutator words.

Many important words are multilinear commutators.

Examples include the lower central words

$$\gamma_k = [x_1, \dots, x_k]$$

and the derived words δ_k, which are defined recursively by

$$\delta_0 = x_1, \qquad \delta_k = [\delta_{k-1}(x_1, \dots, x_{2^{k-1}}), \delta_{k-1}(x_{2^{k-1}+1}, \dots, x_{2^k})].$$

Of course, $\gamma_k(G)$ is the familiar kth term of the lower central series of the group G while $\delta_k(G) = G^{(k)}$ is the k-th commutator subgroup of G.

We proved the following theorem.

Let w be a multilinear commutator word and G a profinite group in which all centralizers of w-values are either finite or open. Then $w(G)$ is abelian-by-finite.

I will now describe some steps in the proof.

Part 2: FC-stuff

Recall that a group G is an FC-group if the centralizer $C_G(g)$ has finite index in G for each $g \in G$.

Equivalently, G is an FC-group if each conjugacy class g^G is finite.

A group G is a BFC-group if all conjugacy classes in G are finite and have bounded size.

A famous theorem of B. H. Neumann says that the commutator subgroup of a BFC-group is finite.

In general, an FC-group need not be a BFC-group. However Shalev proved that a profinite FC-group has finite commutator subgroup (and so is a BFC-group).

In a recent joint work with G. Dierings we deduced a BFC-type theorem for commutators:

Theorem (Dierings, Shumyatsky) *Let m be a positive integer and G a group. If $|x^G| \leq m$ for any commutator x, then $|G''|$ is finite and $|m|$-bounded.*

Later, with Detomi and Morigi, we extended this to arbitrary multilinear commutator words.

Theorem (E. Detomi, M. Morigi, Shumyatsky) *Let m be a positive integer, w a multilinear commutator word and G a group such that $|x^G| \leq m$ for any w-value x, then $|w(G)'|$ is finite and $|m|$-bounded.*

We have also established a nonquantitative version for profinite groups:

If w is a multilinear commutator word and G is a profinite group in which all w-values have open centralizers, then $w(G)$ has finite commutator subgroup.

This extends Shalev's result on profinite FC-groups but... it turns out this is not sufficient for our purposes.

Actually we require something much stronger.

PROPOSITION: *Let w be a multilinear commutator word, G a profinite group and T an open normal subgroup of G such that every w-value of G contained in T is an FC-element. Let K be the subgroup generated by all w-values contained in T. Then K is open in $w(G)$ and K' is finite.*

Now let G be a profinite group in which the centralizers of w-values are either finite or open. We want to prove that $w(G)$ is abelian-by-finite.

Suppose that g is a w-value having infinite order. Then $C_G(g)$ is open and each w-value of G contained in $C_G(g)$ has infinite centralizer. Thus, each w-value of G contained in $C_G(g)$ has open centralizer.

Let T be an open normal subgroup of G contained in $C_G(g)$, and let K be the subgroup generated by all w-values contained in T.

Using the above show that K is open in $w(G)$ and K' is finite. Then we easily deduce that $w(G)$ is abelian-by-finite, as required.

Hence, from now on we assume that all w-values in G are of finite order.

Part 3: On local finiteness of $w(G)$

Suppose G is a profinite group in which every w-value is of finite order. Does it follow that $w(G)$ is locally finite?

We have no counter-examples to that question. On the other hand, positive results related to that question are also scarce.

The answer is positive if $w(x) = x$ (famous result of Zelmanov).

We obtained the following result.

Let p be a prime, w a multilinear commutator word and G a profinite group in which all w-values have finite p-power order. Let K be the abstract subgroup of G generated by all w-values. Then K is a locally finite p-group.

We do not know whether the closed subgroup of G generated by all w-values is a locally finite p-group.

The above proposition enables us to deduce our result in the case where G is pronilpotent.

Let w be a multilinear commutator word and let G be a pronilpotent group with restricted centralizers of w-values in which every w-value has finite order. Then $w(G)$ is abelian-by-finite.

The proof of this is not difficult using that if a locally nilpotent group H is residually finite and has an element with finite centralizer, then H is necessarily finite.

Go back to our question.

Suppose G is a profinite group in which every w-value is of finite order. Does it follow that $w(G)$ is locally finite?

If G is soluble and w is a multilinear commutator, the answer is positive.

Theorem (Detomi, Morigi, Shumyatsky, 2015): *Let w be a multilinear commutator word and G a soluble-by-finite profinite group in which all w-values have finite order. Then $w(G)$ is locally finite and has finite exponent.*

Another criterion for local finiteness of $w(G)$ was established by Khukhro and the speaker in 2014.

Given a word w and a subgroup P of G, let P_w be the set of w-values on elements of P. Set

$$W(P) = \langle {P_w}^G \cap P \rangle.$$

For example, if $w = [x, y]$, then $W(P)$ is generated by all commutators $[a, b]^g \in P$, where $a, b \in P$ and $g \in G$.

Here a^g and b^g need not belong to P.

Theorem (Khukhro, Shumyatsky, 2014) *Let w be a multilinear commutator word, and let G be a profinite group in which w-values have finite order and the subgroup $W(P)$ is torsion for any Sylow subgroup $P \leq G$. Then $w(G)$ is locally finite.*

The proof uses the Hall-Higman theory as well as its profinite version developed by J. S. Wilson.

The cited results on local finiteness of $w(G)$ enable us to show that if G is a profinite group with restricted centralizers of w-values and if each w-value in G has finite order, then $w(G)$ is locally finite.

At this point a whole range of tools (in particular, those using the classification of finite simple groups) become available.

Part 4: The easier part

We appeal to Wilson's theorem on the structure of compact torsion groups which implies that in our situation $w(G)$ has a finite series of closed characteristic subgroups in which each factor either is a pro-p group for some prime p or is isomorphic to a Cartesian product of finite simple groups.

Cartesian product of finite simple groups are easy to deal with.

In view of the famous Ore's conjecture, proved by Liebeck, O'Brien, Shalev and Tiep, for each multilinear commutator word w every element of a nonabelian finite simple group is a w-value.

If a group K is isomorphic to a Cartesian product of nonabelian finite simple groups and has restricted centralizers of w-values, then actually all centralizers of elements in K are either finite or open and so, by Shalev's theorem, K is finite.

Thus, we conclude that under our assumptions the verbal subgroup $w(G)$ is (locally soluble)-by-finite.

So $w(G)$ possesses a characteristic open subgroup which has finite series of closed characteristic subgroups in which each factor is a pro-p group for some prime p.

Eventually we prove that $w(G)$ is abelian-by-finite by induction on the length of that series.

Part 5: Follow-up

A straightforward corollary of the theorem is that if G is a profinite group in which all centralizers of nontrivial w-values are finite, then either G is finite or $w(G) = 1$.

Indeed, we know that $w(G)$ is abelian-by-finite. So $w(G)$ has an open characteristic abelian subgroup N. If N contains a nontrivial w-value, then N is finite. If N contains no nontrivial w-values, then N is contained in $w^*(G)$. Since the marginal subgroup centralizes $w(G)$, we again deduce that N is finite. This proves that $w(G)$ is finite. Hence, $C_G(w(G))$ has finite index in G. We see that $C_G(w(G))$ is both finite and of finite index, which proves that G is finite.

A variation of the above result for finite groups should of course be of quantitative nature. So we considered the following general question.

Let m be a positive integer, w a group-word, and G a finite group such that $w(G) \neq 1$ and $C_G(g)$ has order at most m for each nontrivial w-value $g \in G$. Does it follow that the order of G is bounded in terms of m and w only?

For some words the answer to the question is negative. For example, for $w = x^n$.

Indeed, let G be a Frobenius group with an abelian kernel K of exponent n and a cyclic complement of order dividing $n - 1$. Every nontrivial nth power in G belongs to a conjugate of the complement and therefore has centralizer of order at most $n - 1$. On the other hand, the kernel K can be chosen of arbitrarily large order. Hence, we cannot bound the order of G in terms of orders of centralizers of nth poswers.

Yet, for many words the answer is positive.

In particular, the answer is positive for multilinear commutators.

Further, the answer is positive for the Engel words $[x, y, \ldots, y]$ and words like $[x^p, y^p]$, $[x^p, y^p, z^p]$, etc.

GRAZIE!

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