Milnor-Wolf 's Theorem for group endomorphisms

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It all begins with the famous question of Milnor:
"Are there finitely generated groups of intermediate growth?"

Let G be a finitely generated group and let $S = S^{-1}$ be a finite set of generators for G. For each $n \in \mathbb{N}$, consider $|S^n|$, where

$$S^n := \{s_1 \cdots s_n \mid s_1, \ldots, s_n \in S\}.$$

It is easy to see that the mapping $n \mapsto |S^n|$ grows at most exponentially in n. There are groups having exponential grow: for instance free groups. On the other side, the abelian group $\mathbb{Z}^n$ has polynomial grow. Incidentally, the grow type does not depend on the generating set and hence it is a property of the group only.

This can be interpreted geometrically. The Cayley graph of G with connection set S is the graph having vertex set G and where two distinct vertices g and h are declared to be adjacent if and only if $gh^{-1} \in S$.

In particular, S is the neighbourhood of the identity of G and S^n is the set of vertices that are visited in the graph following a path starting at the identity and having length n.

Milnor is therefore asking how the ball S^n grows.

The famous examples of Grigorchuk give finitely generated groups having "intermediate growth", that is, neither polynomial nor exponential.

However, before these striking examples were discovered, Milnor and Wolf have proved that a finitely generated soluble group has either exponential or polynomial growth.

Here we are interested in a dynamic generalization of this milestone.

Let G be a group and let $\phi : G \to G$ be group homomorphism. For each finite subset F of G, consider the mapping

$$n \to |F\phi(F)\phi^2(F)\cdots\phi^{n-1}(F)|,$$

where

$$F\phi(F)\phi^2(F)\cdots\phi^{n-1}(F) = \{f_0\phi(f_1)\cdots\phi^{n-1}(f_{n-1}) \mid f_0, f_1, \ldots, f_{n-1} \in F\}.$$

Broadly speaking, we are computing the cardinality of the group elements visited via a dynamical system starting from a finite set F. Observe that, when ϕ is the identity automorphism, we obtain the cardinality of the ball F^n. (Here F is not necessarily inverse-closed and hence the Cayley graph is actually directed and hence not a metric space.)

The growth of $n \to |F\phi(F)\phi^2(F)\cdots\phi^{n-1}(F)|$ is used to define an entropy of the dynamical system. The entropy of ϕ is defined as

$$h(\phi) := \sup_{\substack{F \subseteq G \\ F \text{ finite}}} \left(\lim_{n\to\infty} \frac{\log(|F\phi(F)\phi^2(F)\cdots\phi^{n-1}(F)|)}{n} \right).$$

We say that ϕ is exponential, or intermediate, or polynomial if so is the growth $n \to |F\phi(F)\phi^2(F)\cdots\phi^{n-1}(F)|$. (Observe that one has to be more careful than that because this growth might depend on F.)

Theorem

Let G be a locally virtually soluble group and let $\phi : G \to G$ be a group endomorphism, then ϕ has either exponential or polynomial growth.

Locally virtually soluble means that, for every finite subset F of G, the subgroup $\langle F \rangle$ contains a finite index (which might depend on F) which is soluble.

We actually proved this theorem for elementary emenable groups.

Here is in my opinion the most interesting case of the previous theorem (where we actually prove something stronger).

Theorem

Let G be a finitely generated virtually soluble group, let $\phi : G \to G$ be an automorphism and let $\langle G, \phi \rangle$ the subgroup of the holomorph $G \rtimes \mathrm{Aut}(G)$ of G generated by G and ϕ. Then either ϕ has exponential growth or $\langle G, \phi \rangle$ is virtually nilpotent. In the latter case, ϕ is polynomial.

A pivotal ingredient in this proof is the characterisation of Gromov of finitely generated groups having polynomial growth. However, there is another important ingredient due to Grigorchuk on cancellative semigroups.

Without going in too many details, Grigorchuk has defined the growth in cancellative semigroups and has shown that a cancellative semigroup S of polynomial growth forces a polynomial growth in the group of left quotients $S^{-1}S$.

From this we are also able to show the following:

Theorem

Let G be a finitely generated virtually soluble group and let $\phi : G \to G$ be an automorphism of polynomial growth. Then there exists $d \in \mathbb{N}$ (which depends on G and ϕ only) such that for every finite generating set F, the function $n \to |F\phi(F)\phi^2(F)\cdots\phi^{n-1}(F)|$ is asymptotic to n^d.

From the work of Grigorchuk, we obtain something very important and useful in our work. Let G be a finitely generated group and let F be a finite generating set for G (not necessarily inverse-closed). Grigorchuk has proved that if

$$n \mapsto F^n$$

grows polynomially, then

$$n \mapsto (F \cup F^{-1})^n$$

also grows polynomially. Therefore, we are now in the context of the classic growth in groups and we may apply the work of Gromov.

Is there an analogous result for intermediate growth?

If

$$n \mapsto |F^n|$$

has intermediate growth, is it true that also

$$n \mapsto |(F \cup F^{-1})^n|$$

has intermediate growth?