Methods of Group Theory in Leibniz Algebras: Some Compelling Results

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Let L be an algebra over a field F with the binary operations + and [,]. Then L is called a ***Leibniz algebra*** (more precisely a ***left Leibniz algebra***) if it satisfies *the Leibniz identity*

$$[[a, b], c] = [a, [b, c]] - [b, [a, c]]$$

for all a, b, c $\in$ A.

Another form of this identity:

$$[a, [b, c]] = [[a, b], c] + [b, [a, c]].$$

Leibniz algebras first appeared in the paper of A.M. Bloh

Bloh A.M. *On a generalization of the concept of Lie algebra*. Doklady AN USSR-**165** (1965), 471-473.

A.M. Bloh in this paper used the term D-algebras. After two decades, a real interest to Leibniz algebras rose. It is happened thanks to the work of J.L. Loday

Loday J.L. *Une version non commutative des algebres de Lie; les algebras de Leibniz.* Enseign. Math. **39** (1993), 269-293.

J.L. Loday "rediscovered" these algebras and used the term ***Leibniz algebras***.

An algebra R over a field F is called ***right Leibniz*** if it satisfies the Leibniz identity

$$[a, [b, c]] = [[a, b], c]]-[[a, c], b]$$

for all a, b, c $\in$ A.

Note at once that the classes of left Leibniz algebras and right Leibniz algebras are different.

We prefer to work with the left Leibniz algebras. Thus, further, the term a Leibniz algebra stands for a left Leibniz algebra.

In **Loday J.L. and Pirashvili T.** *Universal enveloping algebras of Leibniz algebras and (co)homology',* Math. Annalen 296 (1) (1993) 139– 158, J.Lodey and T. Pirashvili began the systematic study of properties of Leibniz algebras.

The Leibniz algebras appeared to be naturally related to several areas such as differential geometry, homological algebra, classical algebraic topology, algebraic K-theory, loop spaces, noncommutative geometry, and so on. They found some applications in physics.

The theory of Leibniz algebras has developed very intensively in many different directions. Some of the results of this theory were presented in the recent book **Ayupov SH.A., Omirov B.A., Rakhimov I.S**. *Leibniz Algebras: Structure and Classification,* CRC Press, Taylor & Francis Group, (2020).

Note that the class of Lie algebras is a subclass of the class of Leibniz algebras. Conversely, if L is a Leibniz algebra, in which the identity [a, a] = 0 is valid for every element a ∈ L, then it is a Lie algebra.

In any algebraic structure, one of the first tasks is the study of substructures generated by a single element. In Lie algebras, the situation is trivial: a cyclic subalgebra of a Lie algebra L generated by an element a coincides with the subspace generated by a. In contrast to Lie algebras, the situation with cyclic subalgebras in Leibniz algebras turned out to be quite difficult. A description of cyclic Leibniz algebras has been obtained in the paper

Chupordya V.A., Kurdachenko L.A., Subbotin I.Ya. *On some minimal Leibniz algebras.* Journal of Algebra and its Applications **-16 (**2017), no. 2. DOI: 10.1142/S0219498817500827

For the case when F = $\mathbb{C}$ is a field of complex number the description of cyclic finite dimensional Leibniz algebras were obtained in the following paper; however, it does not show the structure of cyclic Leibniz algebras

Scofield, D., Mckay Sullivan, S. *Classification of complex cyclic Leibniz algebras.* ArXiv: 1411.0170v2. 2014

Another natural problem that immediately arises is the study of the structure of Leibniz algebras, whose subalgebras are ideals. In group theory, a similar problem was solved a very long time ago in the classical papers of R. Dedekind and R. Baer. The Lie algebras with this property are abelian. But for Leibniz algebras the structure of algebras, whose subalgebras are ideals, is far from being plain. The structure of such Leibniz algebras was described in the paper

Kurdachenko L.A., SEMKO N.N., SUBBOTIN I.Ya. *The Leibniz algebras whose subalgebras are ideals.* Open Mathematics 2017, Volume15, pp. 92–100

Such an algebra L has the following structure: $L = E \oplus Z$ where Z is a subalgebra of the center of L and E is an extraspecial algebra such that $[x, x] \neq 0$ for each element $x \notin \zeta(E)$.

A Leibniz algebra L is called an *extraspecial* algebra, if it satisfies the following condition:

$\zeta(L)$ is non-trivial and has dimension 1,

$L/\zeta(L)$ is abelian.

The center $\zeta(L)$ of a Leibniz algebra L is defined in the following way:

$$\zeta(L) = \{ x \in L \mid [x, y] = 0 = [y, x] \text{ for each element } y \in L \}.$$

Here is the list of some important steps done on the way of developing of Leibniz Algebra systematic theory which is parallel to group theory.

Chupordya V.A.,Kurdachenko L.A., Subbotin I.Ya. *On some minimal Leibniz algebra.* Journal of Algebra and its Applications **-16(**2017), no. 2. DOI: 10.1142/S0219498817500827

A description of the structure of cyclic Leibniz algebras.
A description of minimal Leibniz algebras, i.e. Leibniz algebras, all proper subalgebras of which are Lie algebras.

Kurdachenko L.A., Semko N.N., Subbotin I.Ya. *The Leibniz algebras whose subalgebras are ideals.* Open Mathematics 2017, Volume15, pp. 92–100

A description of Leibniz algebras, all proper subalgebras of which are ideals.

Chupordya V.A., Kurdachenko L.A., Semko N.N. *On the structure of Leibniz algebras whose subalgebras are ideals or core-free.* Algebra and Discrete Mathematics.Volume 29 (2020). Number 2, pp. 180-194

A description of Leibniz algebras, all proper subalgebras of which are either ideals or have a zero kernel.

Kurdachenko L.A., Subbotin I.Ya, Yashchuk V.S. *The Leibniz algebras whose subideals are ideals.* Journal of Algebra and Applications 2018, Volume 17, n. 8, 1850151 (15 p), DOI:10.1142/S0219498818501517

A description of Leibniz algebras in which the relation "to be an ideal" is transitive.

Kurdachenko L.A., Subbotin I.Ya., Yashchuk V.S. On ideals and contraideals in Leibniz algebras. Reports of the National Academy of Sciences of Ukraine, 2020, no. 1, 11-15
Kurdachenko L.A., Subbotin I.Ya., Yashchuk V.S. Some antipodes of ideals in Leibniz algebras. Journal of Algebra and Its Applications Vol. 19, No. 06, 2050113 (2020) https://doi.org/10.1142/S0219498820501133

A description of Leibniz algebras, all proper subalgebras of which are either ideals or contraideals.

Kurdachenko L.A., Subbotin I.Ya., Semko N.N. *On the anticommutativity in Leibniz algebras.* Algebra and Discrete Mathematics 2018, Volume 26, number 1, 97-109

Analysis of the influence of anticommutativity on the structure of Leibniz algebras.

Kurdachenko L.A., Subbotin I.Ya., Semko N.N.*From Groups to Leibniz Algebras: Common Approaches, Parallel Results.* Advances in Group Theory and Applications-5 (2018), pp. 1–31

A proof of the existence of a locally nilpotent radical in Leibniz algebras; generalized nilpotent classes of Leibniz algebras are introduced.

Kurdachenko L.A., Subbotin I.Ya., Yashchuk V.S. Leibniz algebras whose subalgebras are left ideals and contraideals in Leibniz algebras. Serdica Math. J. 2020, v 46, 175–194

A description of Leibniz algebras, all proper subalgebras of which are either left ideals or contraideals.

Kurdachenko L.A., Otal J. , Pypka A.A. Relationships between factors of canonical central series of Leibniz algebras. European Journal of Mathematics -2016, 2, 565-577.

Kurdachenko L.A., Otal J., Subbotin I.Ya. On some properties of the upper central series in Leibniz algebras, Comment.Math.Univ.Carolin. 60,2 (2019) 161–175

An analogue of Schur's theorem and its generalizations

Consider some results related to the concept of nilpotency. This concept arises both in the theory of groups and in the theory of rings and algebras (associative and non-associative). In the theory of Leibniz algebras this concept arises as follows.

Every Leibniz algebra L has the following specific ideal.

Denote by **Leib**(L) the subspace generated by the elements $[a, a]$, $a \in L$. We note that **Leib**(L) is an ideal of L. Indeed, for arbitrary elements $a, x \in L$ we have

$$[a, [a, x]] = [[a, a], x] + [a, [a, x]], \text{ so } [[a, a], x] = 0.$$

Furthermore,

$$[x + [a, a], x + [a, a]] = [x, x] + [x, [a, a]] + [[a, a], x] + [[a, a], [a, a]] = [x, x] + [x, [a, a]].$$

It follows that $[x, [a, a]] = [x + [a, a], x + [a, a]] - [x, x] \in$ **Leib**(L).

Put $K =$ **Leib**(L). Then in the factor-algebra L/K we have

$$[a + K, a + K] = [a, a] + K = K$$

for each element $a \in L$. By above, we obtain that L/K is a Lie algebra.

Conversely, suppose that H is an ideal of L such that L/H is a Lie algebra. Then

$$H = [a + H, a + H] = [a, a] + H,$$

which implies that $[a, a] \in H$ for every element $a \in L$. Then **Leib**(L) $\leq H$.

The ideal **Leib**(L) is called the *Leibniz kernel* of algebra L.

We note the following important property of the Leibniz kernel:

$$[[a, a], x] = [a, [a, x]]-[a, [a, x]] = 0.$$

This property shows that **Leib**(L) is an abelian subalgebra of L.

Let L be a Leibniz algebra. Define the lower central series of L

$$L = \gamma_1(L) \geq \gamma_2(L) \geq . . . \geq \gamma_\alpha(L) \geq \gamma_{\alpha + 1}(L) \geq . . . \gamma_\delta(L)$$

by the following rule: $\gamma_1(L) = L$, $\gamma_2(L) = [L, L]$, and, recursively, $\gamma_{\alpha + 1}(L) = [L, \gamma_\alpha(L)]$ for all ordinals α and $\gamma_\lambda(L) = \cap_{\mu < \lambda} \gamma_\mu(L)$ for the limit ordinals λ. The last term $\gamma_\delta(L)$ is called the *lower hypocenter* of L. We have $\gamma_\delta(L) = [L, \gamma_\delta(L)]$.

Since $\zeta(L)$ is an ideal of L, we can consider the factor-algebra $L/\zeta(L)$.

A Leibniz algebra L is called *nilpotent* if there exists a positive integer k such that $\gamma_k(L) = <0>$. More precisely, L is said to be *nilpotent of nilpotency class c* if $\gamma_{\mathbf{c} + 1}(L) = <0>$, but $\gamma_{\mathbf{c}}(L) \neq <0>$. We denote by **ncl**(L) the nilpotency class of L.

In some algebraic structures another definition of nilpotency based on the concept of the (upper) central series is used. In fact, suppose that L is a nilpotent Leibniz algebra and $\gamma_{k + 1}(L) = <0>$. For each factor $\gamma_j(L)/\gamma_{j + 1}(L)$ we have $[L, \gamma_j(L)] = \gamma_{j + 1}(L)$ and $[\gamma_j(L), L] \leq \gamma_{j + 1}(L)$, and this leads us to the following concepts.

Let A, B be ideals of L such that $A \leq B$. The factor B/A is called *central* (in L) if $[L, B], [B, L] \leq A$.

Starting from the center we can define the upper central series

$$<0> = \zeta_0(L) \leq \zeta_1(L) \leq \zeta_2(L) \leq . . . \leq \zeta_\alpha(L) \leq \zeta_{\alpha + 1}(L) \leq . . . \zeta_\gamma(L) = \zeta_\infty(L)$$

of the Leibniz algebra L by the following rule: $\zeta_1(L) = \zeta(L)$ is the center of L, and recursively $\zeta_{\alpha + 1}(L)/\zeta_\alpha(L) = \zeta(L/\zeta_\alpha(L))$ for all ordinals α, and $\zeta_\lambda(L) = \cup_{\mu < \lambda} \zeta_\mu(L)$ for limit ordinals λ. By definition, each term of this series is an ideal of L. The last term $\zeta_\infty(L)$ of this series is called the ***upper hypercenter*** of L. A Leibniz algebra L is said to be ***hypercentral*** if it coincides with the upper hypercenter. Denote by **zl**(L) the length of upper central series of L.

It is a well–known that in nilpotent Lie algebras and nilpotent groups the lower and the upper central series have the same length.

Let

$$<0> = C_0 \leq C_1 \leq . . . \leq C_\alpha \leq C_{\alpha + 1} \leq . . . C_\gamma = L$$

be an ascending series of ideals of a Leibniz algebra L. This series is called ***central*** if $C_{\alpha + 1}/C_\alpha \leq \zeta(L/C_\alpha)$ for each ordinal $\alpha < \gamma$. In other words, $[C_{\alpha + 1}, L], [L, C_{\alpha + 1}] \leq C_\alpha$ for each ordinal $\alpha < \gamma$.

We note the following properties of central series (**Kurdachenko L.A., Otal J., Pypka A.A.** *Relationships between factors of canonical central series of Leibniz algebras*. European Journal of Mathematics -2016, 2, 565-577.)

1. *Let L be an Leibniz algebra over a field F, and*

$$<0> = C_0 \leq C_1 \leq \ . \ . \ . \leq C_n = L$$

be a finite central series of L. Then

(i) $\gamma_j(L) \leq C_{n-j+1}$, *so that* $\gamma_{n+1}(L) = <0>$.

(ii) $C_j \leq \zeta_j(L)$, *so that* $\zeta_n(L) = L$.

(iii) If j, k are positive integer such that $k \geq j$, *then* $[\gamma_j(L), \zeta_k(L)], [\zeta_k(L), \gamma_j(L)] \leq \zeta_{k-j}(L)$.

As a corollary we obtain

2. *Let L be an Leibniz algebra over a field F and suppose that L has a finite central series*

$$<0> = C_0 \leq C_1 \leq \ . \ . \ . \leq C_n = L.$$

Then L is nilpotent and **ncl**$(L) \leq n$. *Furthermore, the upper central series of L is finite,* $\zeta_\infty(L) = L$, **zl**$(L) \leq n$. *Moreover,* **ncl**(L) = **zl**(L).

The last result shows that a Leibniz algebra L is nilpotent if and only if there is a positive integer k such that $L = \zeta_k(L)$. *The least positive integer having this property coincides with nilpotency class of L.* So, as in the cases of Lie algebras and groups, the definition of nilpotency can be given here using the notion of the upper central series.

It will be appropriate to note that the Leibniz algebra L can be associative.

3. *Let L be a Leibniz algebra over a field F. Then L is associative if and only if $[L, L] \le \zeta(L)$.*

The concepts of upper and lower central series immediately leads us to the mentioned classes of Leibniz algebras.

A Leibniz algebra L is said to be ***hypercentral*** if it coincides with the upper hypercenter.

A Leibniz algebra L is said to be ***hypocentral*** if its lower hyporcenter is trivial.

In the case of finite dimensional algebras, these two concepts coincide, but, in general, these two classes are very different.

Thus, for finitely generated hypercentral Leibniz algebras we have (see

Kurdachenko L.A., Subbotin I.Ya., Semko N.N.*From Groups to Leibniz Algebras: Common Approaches, Parallel Results.* Advances in Group Theory and Applications-5 (2018), pp. 1–31)

4. *Let L be a finitely generated Leibniz algebra over a field F. If L is hypercentral, then L is nilpotent. Moreover, L has finite dimension. In particular, a finitely generated nilpotent Leibniz algebra has finite dimension.*

This result is an analog of a similar group theoretical result proved by A. I. Mal'cev

Maltsev A.I. *Nilpotent torsion-free groups.* Izvestiya AN USSR, series math.-**13**(1949), no. 3, 201-212.

At the same time, a finitely generated hypocentral Leibniz algebra can have infinite dimension. A simple example, which shows it is a cyclic Leibniz algebra, generated by an element of infinite depth.

A Leibniz algebra L is said to be ***locally nilpotent*** if every finite subset of L generates a nilpotent subalgebra.

That is why, hypercentral Leibniz algebras give us examples of locally nilpotent algebras.

We obtained the following characterization of hypercentral Leibniz algebras.

5. *Let L be a Leibniz algebra over a field F. Then L is hypercentral if and only if for each element $a \in L$ and every countable subset $\{ x_n \mid n \in \mathbf{N} \}$ of elements of L there exists a positive integer k such that all commutators $[x_1, \ldots, x_j, a, x_{j+1}, \ldots, x_k]$ are zeros for all j, $0 \leq j \leq k$.*

As a corollary we obtain

6. *Let L be a Leibniz algebra over a field F. Then L is hypercentral if and only if every subalgebra of L having finite or countable dimension is hypercentral.*

These results are analogs to some group-theoretical results of S.N. Chernikov.

Let L be a Leibniz algebra. If A, B are nilpotent ideals of L, then their sum A + B is a nilpotent ideal of L. This result has been proved in the paper

Barnes D. *Schunck classes of soluble Leibniz algebras*. Communications in Algebra, 41(2013), 4046-4065.

In this connection, the following question arises: *If a similar assertion is valid for locally nilpotent ideals?* For Lie algebras this assertion takes place. It was shown by B. Hartley in the paper

Hartley B. *Locally nilpotent ideals of a Lie algebras*. Proc. Cambridge Phil. Society, 63(1967), 257-272.
Our next result gives an affirmative answer to this question.

7. *Let L be a Leibniz algebra over a field F, A, B be locally nilpotent ideals of L. Then A + B is locally nilpotent.*

As a corollaries we obtain

8. *Let L be a Leibniz algebra over a field F and **S** be a family of locally nilpotent ideals of L. Then a subalgebra generated by **S** is locally nilpotent.*

9. *Let L be a Leibniz algebra over a field F. Then L has the greatest locally nilpotent ideal.*

Let L be a Leibniz algebra over field F. The greatest locally nilpotent ideal of L is called the ***locally nilpotent radical*** of L and will be denoted by **Ln**(L).

These results are analogues of the results in groups proven by K.A. Hirsch

Hirsch K.A. *Über local-nilpotente Gruppen,* Math. Z.- **63** (1955), 290-291.
and B.I. Plotkin

Plotkin B.I. *Radical groups.* Math. sbornik, **37** (1955), 507-526.
Plotkin B.I. *Generalized soluble and generalized nilpotent groups.* Uspekhi mat. nauk, **13** (1958), no. 4, 89-172.

The subalgebra $\mathbf{Nil}(L)$ generated by all nilpotent ideals of L is called the ***nil radical*** of L. If $L = \mathbf{Nil}(L)$, then L is called a Leibniz ***nil-algebra***. Every nilpotent Leibniz algebra is a nil-algebra, but converse is not true even for a Lie algebra. Every Leibniz nil-algebra is locally nilpotent, but converse is not true even for a Lie algebra. Moreover, there exists a Lie nil-algebra, which is not hypercentral. There is a corresponding example in Chapter 6 of the book

Amayo R.K., Stewart I. *Infinite Dimensional Lie Algebras.* Noordhoff Intern. Publ.: Leyden, 1974.

Note the following important properties of locally nilpotent Leibniz algebras.

10. *Let L be a locally nilpotent Leibniz algebra over a field F.*

(i) If $A, B, A \leq B$ are the ideals of L such that factor B/A is L-chief, then B/A is central in L (that is $B/A \leq \zeta(L/A)$). In particular, $\mathbf{dim}_F(B/A) = 1$.

(ii) If A is a maximal subalgebra of L, then A is an ideals of L.

Let L be a Leibniz algebra over the field F and H a subalgebra of L. The ***idealizer*** of H is defined by the following rule:

$$\mathbf{I}_L(H) = \{ x \in L \mid [h, x], [x, h] \in H \text{ for all } h \in H \}.$$

It is possible to prove that the idealizer of H is a subalgebra of L. If L is a hypercentral (in particular, nilpotent) Leibniz algebra, then $H \neq \mathbf{I}_L(H)$. This leads us to the following class of Leibniz algebras.

Let L be a Leibniz algebra over field F. We say that L ***satisfies the idealizer condition*** if $I_L(A) \neq A$ for every proper subalgebra A of L.

A subalgebra A is called ***ascendant*** in L, if there is an ascending chain of subalgebras

$$A = A_0 \leq A_1 \leq \ldots A_\alpha \leq A_{\alpha+1} \leq \ldots A_\gamma = L$$

such that A_α is an ideal of $A_{\alpha+1}$ for all $\alpha < \gamma$.

It is possible to prove that L satisfies the idealizer condition if and only if every subalgebra of L is ascendant.

The last our result is the following

11. *Let L be a Leibniz algebra over a field F. If L satisfies the idealizer condition then L is locally nilpotent.*

This result is an analog to the following result proved by B.I. Plotkin for groups.

Plotkin B.I. *To the theory of locally nilpotent groups.* Doklary AN USSR **76** (1951), 655-657.

Again, it should be noted that the Leibniz algebras with the idealizer condition will form a proper subclass of the class of locally nilpotent Leibniz algebras. It happens since this is already the case for Lie algebras. A corresponding example could be found in Chapter 6 of the book

Amayo R.K., Stewart I. *Infinite dimensional Lie algebras*. Noordhoff Intern. Publ.: Leyden, 1974.