Powerfully nilpotent groups

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The upper powerfully central series. Defined recursively by $\hat{Z}_0(G) = \{1\}$, $\hat{Z}_{n+1}(G) = \{a \in G : [a,x] \in \hat{Z}_{n-1}(G)^p$ for all $x \in G\}$. (Notice that $\hat{Z}_1(G) = Z(G)$).

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Remark. In particular all powerful 2-groups are powerfully nilpotent.

2. Presentations and growth

Let p be an odd prime. Let G be any powerfully nilpotent p-group of rank r, exponent p^e and order p^n.

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Theorem 2.1(T, Williams). We can choose the generators $a_1, \ldots, a_r$ such that $|G| = o(a_1) \cdots o(a_r)$ and such that

$$\begin{array}{ccccccc} \langle a_1, \ldots, a_r\rangle & \geq & \langle a_1^p, a_2, \ldots, a_r\rangle & \geq & \cdots & \geq & \langle a_1^p, \ldots, a_r^p\rangle \\ & & \langle a_1^{p^2}, a_2^p, \ldots, a_r^p\rangle & \geq & \cdots & \geq & \langle a_1^{p^e}, \ldots, a_r^{p^e}\rangle = \{1\} \end{array}$$

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Powerfully nilpotent presentations . With the generators chosen as above we get relations of the form

$$\begin{array}{rcl} [a_i, a_j] & = & a_1^{m_1(i,j)} \cdots a_r^{m_r(i,j)}, \quad 1 \leq j < i \leq r \\ a_i^{n_i} & = & 1, \quad 1 \leq i \leq r \end{array}$$

where $n_i = o(a_i)$ and where $p | m_k(i,j)$. Also $p^2 | m_k(i,j)$ when $k \leq i$.

These relations determine the structure of the group. We call such a presentation a powerfully nilpotent presentation. Conversely any powerfully nilpotent presentation gives us a powerfully nilpotent group G. We say that the presentation is consistent if $|G| = n_1 \cdots n_r$.

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Theorem 2.2 (T, Williams) Let p be an odd prime. The number of powerfully nilpotent groups of exponent p^2 and order p^n is $p^{\alpha n^3 + o(n^3)}$, where $\alpha = \frac{9+4\sqrt{2}}{394}$

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Remark. The number of all powerful p-groups of exponent p^2 and order p^n is on the other hand $p^{\frac{2}{27}n^3 + o(n^3)}$.

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Let p be a fixed prime. For any powerful p-group G we let $r = r(G)$ be the rank of G, $c = c(G)$ be the powerful class and $p^{n(G)}$, $p^{e(G)}$ be the order and exponent of G. As before the coclass is $d(G) = n(G) - c(G)$.

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Theorem 3.1 (T, Williams) For each prime p and non-negative integer d, there are finitely many powerfully nilpotent p-group of powerful coclass d. Furthermore $r \leq d + 1$ and $e \leq d + 1$.

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Remark. As $n \leq re \leq (d + 1)^2$, the two inequalities imply that there are only finitely many G with powerful coclass d.

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Theorem 4.1 (T, Williams). Let r be a positive integer and $p > r$ a prime. There exists a powerfully nilpotent p-group of rank r that is of maximal powerful class.

Theorem 4.2 (T, Williams). Let G be a powerfully nilpotent p-group of rank r where $p > r$ that has maximal powerful class $1 + r(r-1)/2$. There exists generators $b_1, \ldots, b_{r-1}, y$ such that

$$G = \langle y \rangle \cdot \langle b_1 \rangle \cdots \langle b_{r-1} \rangle,$$

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$$\begin{aligned} \langle y, b_1, \ldots, b_{r-1} \rangle \geq \langle y^p, b_1, \ldots, b_{r-1} \rangle \geq \cdots \geq \langle y^p, b_1^p, \ldots, b_{r-1}^p \rangle \\ \langle y^{p^2}, b_1^p, \ldots, b_{r-1}^p \rangle \geq \cdots \geq \langle y^{p^e}, b_1^{p^e} \ldots, b_{r-1}^{p^e} \rangle = 1 \end{aligned}$$

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is powerfully central. Furthermore

(a) $[b_1, y] = b_2^p,\ [b_2, y] = b_3^p, \ldots, [b_{r-2}, y] = b_{r-1}^p, [b_{r-1}, y] = y^{p^2}$.
(b) $H = \Omega_r(G) = \langle y^p \rangle \langle b_1 \rangle \cdots \langle b_{r-1} \rangle$ p.e. G and strongly powerful.
(c) $G^{p^{r-1}} \leq Z(G)$.

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Theorem 2.2 (T, de las Heras) Let p be an odd prime. The number of powerfully solvable groups of exponent p^2 and order p^n is $p^{\alpha n^3 + o(n^3)}$, where $\alpha = \frac{-1+\sqrt{2}}{6}$

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Proposition 2.1.(T, de las Heras) Let G be any powerful p-group of exponent p^2. There exists a powerfully nilpotent group H of exponent p^2 and powerful nilpotency class 2 such that G is powerfully embedded in H.

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Propostion 2.2.(T, de las Heras) Let G be any finite p-group of nilpotency class 2. There exists a powerfully nilpotent group H of powerful nilpotence class 2 that contains G as a subgroup.

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Theorem 5.1(T, de las Heras) (a) Let G be a powerfully nilpotent group in $\mathcal{P}$ and H a powerful subgroup of G. Then H is powerfully nilptotent of powerful nilpotence class at most the powerful nilptotence class of G.

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(b) Let G be a powerfully solvable group in $\mathcal{P}$ and H a powerful subgroup of G. Then H is powerfully solvable of powerful derived length at most the powerful derived length of G.

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Theorem(T, de las Heras) Any two composition series for $G \in \mathcal{P}$ have the same composition factors.