Grading switching for modular non-associative algebras

Marina Avitabile

Università di Milano - Bicocca

Ischia Group Theory 2022

Contents

- Grading switching tool:
  - early version based on the Artin-Hasse exponential series (**due to Sandro Mattarei (University of Lincoln, UK))**
  - general version based on certain Laguerre polynomials (**joint work with S. Mattarei**)
- Motivation
- Further directions: generalized finite polylogarithms (**joint work with S. Mattarei**)

The exponential of a derivation

Let A be a (finite-dimensional) non-associative algebra over a field $\mathbb{F}$

- A derivation of A is a linear map $D : A \to A$ such that $D \circ m = m \circ (D \otimes \mathrm{id} + \mathrm{id} \otimes D)$, where $m : A \otimes A \to A$ is the multiplication map.

Lemma

Assume $\mathrm{char}(\mathbb{F}) = 0$. *If D is a nilpotent derivation of A, then* $\exp(D) = \sum_{i=0}^{\infty}(D^i/i!)$ *is an automorphism of A.*

- *Proof:* set $X = D \otimes \mathrm{id}$ and $Y = \mathrm{id} \otimes D$ and use that $\exp(X) \cdot \exp(Y) = \exp(X + Y)$ when X and Y commute.
- For $F(X) \in \mathbb{F}[[X]]$ with $F'(0) = 1$ either the *functional equation* $F(X + Y) = F(X) \cdot F(Y)$ or the *differential equation* $F'(X) = F(X)$ determine $F(X) = \exp(X)$.

The exponential of a derivation

From now on assume $\mathrm{char}(\mathbb{F}) = p > 0$

- if $D^p = 0$

$$\exp(D) = \sum_{i=0}^{p-1} \frac{D^i}{i!}$$

- for every derivation D, consider the *truncated exponential*

$$\mathrm{E}(D) = \sum_{i=0}^{p-1} \frac{D^i}{i!}$$

Direct computation shows that

$$E(D)(a) \cdot E(D)(b) - E(D)(ab) = \sum_{t=p}^{2p-2} \sum_{i=t+1-p}^{p-1} \frac{(D^i a)(D^{t-i} b)}{i!(t-i)!} \qquad (1)$$

for every $a, b \in A$.

In particular, if p is odd and $D^{\frac{p+1}{2}} = 0$, then $\mathrm{E}(D) = \exp(D)$ is an automorphism of A.

The exponential of a derivation

The truncated exponential $E(X) = \sum_{i=0}^{p-1} X^i/i!$ satisfies the congruence

$$\mathrm{E}(X) \cdot \mathrm{E}(Y) \equiv \mathrm{E}(X+Y) \left(1 + \sum_{i=1}^{p-1} (-1)^i X^i Y^{p-i}/i \right) \pmod{X^p, Y^p}$$

in the polynomial ring $\mathbb{F}_p[X, Y]$.
Setting $X = D \otimes \mathrm{id}$ and $Y = \mathrm{id} \otimes D$ we recover Equation (1)

$$E(D)a \cdot E(D)b - E(D)(ab) = \sum_{t=p}^{2p-2} \sum_{i=t+1-p}^{p-1} \frac{(D^i a)(D^{t-i} b)}{i!(t-i)!}.$$

$\mathrm{E}(D)$ has the property of sending a grading of A into another grading of A

Gradings of non-associative algebras

- Grading: direct sum decomposition $A = \oplus_{g \in G} A_g$, G an abelian group, such that $A_g A_h \subseteq A_{g+h}$

Theorem (Grading switching with $D^p = 0$, S. Mattarei)

- *Let $A = \oplus_k A_k$ be a $\mathbb{Z}/m\mathbb{Z}$-grading of A;*
- *let D be a derivation of A, homogeneous of degree d, that is $D(A_k) \subseteq A_{k+d}$, for every k*
- *let $m \mid pd$ and $D^p = 0$*

Then

$$A = \oplus_k \exp(D) A_k$$

is a $\mathbb{Z}/m\mathbb{Z}$-grading of A.

Gradings of non-associative algebras

Theorem (Grading switching with $D^p = 0$, S. Mattarei)

- *Let $A = \oplus_k A_k$ be a $\mathbb{Z}/m\mathbb{Z}$-grading of A;*
- *let D be a derivation of A, homogeneous of degree d, with $m \mid pd$ such that $D^p = 0$.*

Then

$$A = \oplus_k \exp(D) A_k$$

is a $\mathbb{Z}/m\mathbb{Z}$-grading of A.

Proof.

Check that $\exp(D)A_s \cdot \exp(D)A_t \subseteq \exp(D)A_{s+t}$. Let $a \in A_s$ and $b \in A_t$, then $D^i(a) \cdot D^{p-i}(b) \in A_{s+t+pd} = A_{s+t}$. Then

$$\exp(D)a \cdot \exp(D)b = \underbrace{\exp(D)(ab)}_{\exp(D)(A_{s+t})} + \underbrace{\exp(D)\left(\sum_{i=0}^{p-1} \frac{(-1)^i}{i} D^i a \cdot D^{p-i} b\right)}_{\exp(D)(A_{s+t})}$$

Grading switching

Grading switching tool: a technique for modular, non-associative algebras, whose aim is to produce a new grading of an algebra from a given one

- $\exp(D) = \mathrm{E}(D)$ with $D^p = 0$
- Tensor product device: $X = D \otimes \mathrm{id}$ and $Y = \mathrm{id} \otimes D$
- The congruence

$$\mathrm{E}(X) \cdot \mathrm{E}(Y) \equiv \mathrm{E}(X+Y)\left(1 + \sum_{i=1}^{p-1} (-1)^i X^i Y^{p-i}/i\right) \pmod{X^p, Y^p}$$

in the polynomial ring $\mathbb{F}_p[X, Y]$.

Artin-Hasse exponential of a derivation

The Artin-Hasse exponential series is defined as

$$\mathrm{E}_p(X) = \exp\left(\sum_{i=0}^{\infty} X^{p^i}/p^i\right) = \prod_{i=0}^{\infty} \exp(X^{p^i}/p^i) \in \mathbb{Z}_{(p)}[[X]]$$

Theorem (S. Mattarei)

There exist $a_{ij} \in \mathbb{F}_p$ with $a_{ij} = 0$ unless $p \mid i + j$, such that

$$\mathrm{E}_p(X) \cdot E_p(Y) = \mathrm{E}_p(X + Y)\Big(1 + \sum_{i,j=1}^{\infty} a_{ij} X^i Y^j\Big)$$

in the power series ring $\mathbb{F}_p[[X, Y]]$.

Grading-switching with D nilpotent

Let D be a nilpotent derivation of A, thus $\mathrm{E}_p(D)$ is a finite sum

Theorem (Grading-switching with D nilpotent, S. Mattarei)

- *Let $A = \oplus_k A_k$ be a $\mathbb{Z}/m\mathbb{Z}$-grading of A;*
- *let D be a nilpotent derivation of A, homogeneous of degree d with $m \mid pd$.*

Then

$$A = \oplus_k \mathrm{E}_p(D) A_k$$

is a $\mathbb{Z}/m\mathbb{Z}$-grading of A.

Laguerre polynomials

The classical (generalized) Laguerre polynomial of degree $n \geq 0$ is defined as

$$L_n^{(\alpha)}(X) = \sum_{k=0}^{n} \binom{\alpha + n}{n - k} \frac{(-X)^k}{k!} \in \mathbb{Q}[\alpha, X].$$

- for $\alpha > -1$ and $n \neq m$

$$\int_0^{\infty} X^{\alpha} \exp(-X) L_n^{(\alpha)}(X) L_m^{(\alpha)}(X) dX = 0$$

- $Y = L_n^{(\alpha)}(X) \in \mathbb{R}[X]$ satisfies the differential equation

$$XY'' + (\alpha + 1 - X)Y' + nY = 0$$

Laguerre polynomials modulo p

Fixed a prime p, set $n = p - 1$

$$L_{p-1}^{(\alpha)}(X) = (1 - \alpha^{p-1}) \sum_{k=0}^{p-1} \frac{X^k}{(1+\alpha)(2+\alpha)\cdots(k+\alpha)} \in \mathbb{F}_p[\alpha, X].$$

Special cases:

- $\alpha = 0$

$$L_{p-1}^{(0)}(X) = \sum_{k=0}^{p-1} \frac{X^k}{k!} = \mathrm{E}(X)$$

- $\alpha = -\sum_{i=1}^{\infty} X^{p^i}$

$$L_{p-1}^{(-\sum_{i=1}^{\infty} X^{p^i})}(X) = \mathrm{E}_p(X) G(X^p)$$

for some $G(X) \in 1 + X\mathbb{F}_p[[X]]$

A modular differential equation

The polynomial $L_{p-1}^{(\alpha)}(X)$ satisfies the modular differential equation

$$\boxed{X\frac{d}{dX}(L_{p-1}^{(\alpha)}(X)) = (X-\alpha)L_{p-1}^{(\alpha)}(X) + X^p - (\alpha^p - \alpha)}$$

- Special cases:
  - $\alpha = 0$

$$X\frac{d}{dX}\,\mathrm{E}(X) \equiv X\,\mathrm{E}(X) \pmod{X^p}$$

  - $\alpha = -\sum_{i=1}^{\infty} X^{p^i}$

$$X\frac{d}{dX}L_{p-1}^{(-\sum_{i=1}^{\infty} X^{p^i})}(X) = \left(\sum_{i=0}^{\infty} X^{p^i}\right) L_{p-1}^{(-\sum_{i=1}^{\infty} X^{p^i})}(X)$$

- Taking a further derivative we recover for $Y = L_{p-1}^{(\alpha)}(X)$ the classical differential equation

$$XY'' + (\alpha + 1 - X)Y' - Y = 0$$

A modular functional equation

An analogue of the functional equation $\exp(X) \cdot \exp(Y) = \exp(X + Y)$.

Theorem (Exponential-like property of $L_{p-1}^{(\alpha)}(X)$)

Let α, β, X, Y be indeterminates over $\mathbb{F}_p$. There exist rational expressions $c_i(\alpha, \beta) \in \mathbb{F}_p(\alpha, \beta)$, such that

$$L_{p-1}^{(\alpha)}(X) \cdot L_{p-1}^{(\beta)}(Y) \equiv L_{p-1}^{(\alpha+\beta)}(X + Y) \cdot \left(c_0(\alpha, \beta) + \sum_{i=1}^{p-1} c_i(\alpha, \beta) X^i Y^{p-i} \right)$$

in $\mathbb{F}_p(\alpha, \beta)[X, Y]$, modulo the ideal generated by $X^p - (\alpha^p - \alpha)$ and $Y^p - (\beta^p - \beta)$.

- $\alpha = 0 = \beta$, $L_{p-1}^{(0)}(X) = E(X)$, we recover

$$\mathrm{E}(X) \cdot \mathrm{E}(Y) \equiv \mathrm{E}(X + Y) \left(1 + \sum_{i=1}^{p-1} (-1)^i X^i Y^{p-i} / i \right) \pmod{X^p, Y^p}$$

Characterization of the Laguerre polynomials of degree $p-1$

Theorem

Let α, β, X, Y be indeterminates over $\mathbb{F}_p$ and let $P^{(\alpha)}(X)$ be a polynomial in $\mathbb{F}_p[\alpha][X]$ of degree less than p with $P^0(0) \neq 0$. The following conditions are equivalent:

i) *There exist rational expressions $\overline{c}_i(\alpha, \beta) \in \mathbb{F}_p(\alpha, \beta)$, defined even specializing α or β to zero, such that*

$$P^{(\alpha)}(X) \cdot P^{(\beta)}(Y) \equiv P^{(\alpha+\beta)}(X+Y) \cdot \left(\overline{c}_0(\alpha, \beta) + \sum_{i=1}^{p-1} \overline{c}_i(\alpha, \beta) X^i Y^{p-i}\right)$$

in $\mathbb{F}_p(\alpha, \beta)[X, Y]$ modulo the ideal generated by $X^p - (\alpha^p - \alpha)$ and $Y^p - (\beta^p - \beta)$.

ii) *$P^{(\alpha)}(X) = d(\alpha) L_{p-1}^{(c\alpha)}(cX)$ for some $d(\alpha) \in \mathbb{F}_p[\alpha]$ with $d(0) \neq 0$, and for some $c \in \mathbb{F}_p$.*

Grading switching: a model special case

Theorem

- *Let* $A = \bigoplus_k A_k$ *be a* $\mathbb{Z}/m\mathbb{Z}$*-grading of* A*;*
- *let* $D \in \mathrm{Der}(A)$*, homogeneous of degree* d*, with* $m \mid pd$*, such that* $D^{p^2} = D^p$*;*
- *let* $A = \bigoplus_{a \in \mathbb{F}_p} A^{(a)}$ *be the decomposition of* A *into generalized eigenspaces for* D*;*
- *assuming* $\mathbb{F}_{p^p} \subseteq \mathbb{F}$*, fix* $\gamma \in \mathbb{F}$ *with* $\gamma^p - \gamma = 1$*;*
- *let* $\mathcal{L}_{\mathcal{D}} : A \to A$ *be the linear map on* A *whose restriction to* $A^{(a)}$ *coincides with* $L_{p-1}^{(a\gamma)}(D)$*.*

Then $A = \bigoplus_k \mathcal{L}_{\mathcal{D}}(A_k)$ *is a* $\mathbb{Z}/m\mathbb{Z}$*-grading of* A*.*

General case: the only assumption on D is D^{p^r} semisimple with finitely many eigenvalues, for some r

Toral switching

- Replacing a torus T of a restricted Lie algebra L with another torus, which is more suitable for further study of L
- It relies an a delicate adaptation of the exponential of a derivation
- D. J. Winter, R. E. Block, R. L. Wilson, A. A. Premet
- Grading of L: root space decomposition of L attached to any torus
- The toral switching process can be viewed as a special instance of the grading switching

Grading switching *versus* toral switching

- applies to non-associative algebras
- is not restricted to gradings over groups of exponent p

Motivation: thin Lie algebras

Thin Lie algebra: an infinite-dimensional Lie algebra $L = \bigoplus_{i=1}^{\infty} L_i$ such that

1) $\dim(L_1) = 2$
2) for every graded ideal I of L, $L^{i+1} \subseteq I \subseteq L^i$, for some i (*covering property*)

A. Caranti, S. Mattarei, M. F. Newman, C. M. Scoppola
Thin groups of prime-power order and thin Lie algebras
Quart. J. Math. Oxford Ser. (2) **47** (1996), 279–296

- Example: the graded Lie algebra associated to the Nottingham group $\mathcal{N}(\mathbb{F}_p)$ w.r.t. its lower central series
- Periodic ones: *loop algebras* of certain simple, finite-dimensional Lie algebras of Cartan type

Further directions: finite polylogarithms

The *truncated logarithm* is defined in $\mathbb{F}_p[X]$ as

$$£_1(X) = \sum_{k=1}^{p-1} \frac{X^k}{k}$$

- Truncated version of the series $\mathrm{Li}_1(X) = -\log(1-X) = \sum_{k=1}^{\infty} X^k/k$
- For p odd

$$-£_1(\mathrm{E}(X)) \equiv X \pmod{X^p}$$

The *finite polylogarithm* of order $d \in \mathbb{Z}$ is defined in $\mathbb{F}_p[X]$ as

$$£_d(X) = \sum_{k=1}^{p-1} \frac{X^k}{k^d}$$

- Truncated version of the ordinary polylogarithm of order d

$$\mathrm{Li}_d(X) = \sum_{k=1}^{\infty} \frac{X^k}{k^d}.$$

Basic properties

- $$\pounds_d(X) = \pounds_{p-1+d}(X) \leadsto 0 \leq d < p-1;$$
- $$X \frac{d}{dX} \pounds_d(X) = \pounds_{d-1}(X)$$
- Inversion relation
$$\pounds_d(X) = -X^p \cdot \pounds_d(1/X)$$
- (A special instance of the) Distribution relation:
$$\pounds_d(X^h) \equiv h^d \pounds_d(X) \mod X^p - 1, \qquad 0 < d, h < p-1$$
- Powers
$$\pounds_1(X)^d \equiv (-1)^{d-1} d! \, \pounds_d(X) \mod X^p - 1, \qquad 0 < d < p-1$$

M. Mirimanoff, 1900, Fermat's Last Theorem

Generalized truncated logarithm

Inverting the Laguerre polynomials

$$\mathrm{E}(X) = L_{p-1}^{(0)}(X) \quad \rightsquigarrow \quad L_{p-1}^{(\alpha)}(X)$$

$$\pounds_1(X) \quad \rightsquigarrow \quad \pounds_1^{(\alpha)}(X)$$

Theorem

There is a unique polynomial $\pounds_1^{(\alpha)}(X)$ of degree less than p in $\mathbb{F}_p(\alpha)[X]$ such that

$$-\pounds_1^{(\alpha)}(L_{p-1}^{(\alpha)}(X)) \equiv X \pmod{X^p - (\alpha^p - \alpha)}.$$

- $\pounds_1^{(\alpha)}(X)$ gives also rise to a functional right-inverse of $L_{p-1}^{(\alpha)}(X)$, w.r.t. an appropriate modulus
- Special case: when $\alpha = 0$, $\pounds_1^{(0)}(X) = \pounds_1(X)$

Generalized finite polylogarithms

- For every $d \in \mathbb{Z}$, define the *generalized finite polylogarithm* of order d $\pounds_d^{(\alpha)}(X) \in \mathbb{F}_p(\alpha)[X]$ according to

$$X \frac{d}{dX} \pounds_d^{(\alpha)}(X) = \pounds_{d-1}^{(\alpha)}(X)$$

- Special case: when $\alpha = 0$

$$\pounds_d^{(0)}(X) = \pounds_d(X)$$

- (Congruential) functional equations for $\pounds_d^{(\alpha)}(X)$: inversion relation, distribution relation, powers

References

S. Mattarei
Artin-Hasse exponentials of derivations
J. Algebra **294** (2005), 1–18

M. Avitabile and S. Mattarei
Laguerre polynomials of derivations
Israel J. Math. **205** (2015), 109–126

M. Avitabile and S. Mattarei
Grading switching for modular non-associative algebras
Contemp. Math. **652** (2015), 1–14

M. Avitabile and S. Mattarei
A generalized truncated logarithm
Aequationes Math. **93** (2019), 711–734

M. Avitabile and S. Mattarei
Generalized finite polylogarithms
Glasg. Math. J. **63** (2021), 66–80