Large characteristically simple sections of finite groups [1]

Adolfo Ballester-Bolinches[1]

[1] Departament de Matemàtiques
Universitat de València

Ischia (NA), June, 2022 / Ischia Group Theory 2022

[1] Joint work with R. Esteban-Romero and P. Jiménez-Seral

Introduction

All groups in this talk will be finite.

Main concern

Extension of the notion of crown for isomorphic chief factors.
An upper bound for the number of maximal subgroups of a given index of a group.

Introduction

Gaschütz, 1962:

- If G is a soluble group and A is a G-module, there exists a normal section of G, called the A-*crown* of G, which is a completely reducible and homogeneous G-module and the length of its G-composition series is the number of complemented chief factors of G which are G-isomorphic to A in a given chief series of G.
- The A-crowns are complemented sections of G.

- Every soluble group has a characteristic conjugacy class of subgroups: the *prefrattini* subgroups.

W. Gaschütz
Praefrattinigruppen.
Arch. Math., 13:418–426, 1962.

Introduction

Hawkes, 1973: The notion of crown gives rise to a new closure operator for Schunck classes of finite soluble groups.

T. Hawkes
Closure operations for Schunck classes.
J. Austral. Math. Soc. Ser. A, 16(3):316–318, 1973.

Introduction

Lafuente, 1984:

- Introduction of crowns associated with non-Frattini chief factors of arbitrary groups.
- *G*-connection between non-Frattini chief factors is the key concept in his approach.

J. Lafuente
Nonabelian crowns and Schunck classes of finite groups.
Arch. Math. (Basel), 42(1):32–39, 1984.

Introduction

Definition (Lafuente)

Two chief factors of a group G are *G-connected* (or *G-equivalent*) when they are G-isomorphic or there exists a normal subgroup N of G such that G/N is a primitive group of type 3 whose minimal normal subgroups are G-isomorphic to the given chief factors.

Introduction

Lafuente, 1984, 1985:

- Existence of normal sections associated with non-Frattini chief factors with similar properties to Gaschütz's crowns.
- A new closure operator for Schunck classes of arbitrary groups which allows us to discover new realtions between Schunck classes and saturated formations.

Introduction

J. Lafuente.
Nonabelian crowns and Schunck classes of finite groups.
Arch. Math. (Basel), 42(1):32–39, 1984.

J. Lafuente.
Crowns and centralizers of chief factors of finite groups.
Comm. Algebra, 13(3):657–668, 1985.

J. Lafuente.
Eine Note über nichtalbelsche Hauptfaktoren und maximale Untergruppen einer endlichen Gruppen.
Comm. Algebra, 13(9):2025–2036, 1985.

Introduction

Crowns are have been used to

Förster, 1988: Give an alternative approach of the generalised Jordan-Hölder theorem.

B-B and Ezquerro, 1991: Introduce prefrattini subgroups in every arbitrary group.

P. Förster
Chief factors, crowns, and the generalised Jordan-Hölder theorem.
Comm. Algebra, 16(8):1627–1638, 1988.

A. Ballester-Bolinches and L. M. Ezquerro
On maximal subgroups of finite groups.
Comm. Algebra, 19(8):2373–2394, 1991.

Introduction

Crowns are also very useful in probabilistic group theory and generation of groups.
Hall in 1936 gave a formula for the probability $P_G(t)$ that t random elements generate a group G, t a non-negative integer. If N is a normal subgroup of G and $t \geq d(G/N)$, define $P_{G,N}(t) = P_G(t)/P_{G/N}(t)$; this is the probability that a t-tuple generates G, given that it generates G modulo N. Gaschütz in 1959 gave a formula for $P_{G,N}(t)$, generalising Hall's formula. Detomi and Lucchini obtained in 2003 factorisations of $P_G(t)$. Crowns turned out to crucial in their work.

Introduction

P. Hall
The Eulerian functions of a group.
Quart. J. Math., 7(1):134–151, 1936.

W. Gaschütz
Die Eulersche Funktion endlicher auflösbarer Gruppen.
Illinois J. Math., 3(4):469–476, 1959.

E. Detomi and A. Lucchini.
Crowns and factorization of the probabilistic zeta function of a finite group.
J. Algebra, 265(2):651–668, 2003.

Introduction

Lucchini, Marion and Tracy, 2020: Crowns' machinery to determine the minimal number of generators requiered to generate a maximal subgroup of an almost simple group with simple socle an excepcional group of Lie type. It improves a result of T. C. Burness, M. W. Liebeck and A. Shalev [Adv. Math. 248, 59-95 (2013)].

A. Lucchini, C. Marion and G. Tracey
Generating maximal subgroups of finite almost simple groups.
Forum Math. Sigma, 8:67 pp., 2020.

Large characteristically simple sections of a group

Definition

A primitive group is a group with a core-free maximal subgroup.

If M is a maximal subgroup of G, then M/M_G is a core-free maximal subgroup of G/M_G and so G/M_G is primitive.

Large characteristically simple sections of a group

Theorem (Baer, 1957)

Let G be a primitive group and let U be a core-free maximal subgroup of G. Exactly one of the following statements holds:

1. $\mathrm{Soc}(G) = S$ *is a self-centralising abelian minimal normal subgroup of* G, $G = US$ *and* $U \cap S = 1$ *(type 1).*
2. $\mathrm{Soc}(G) = S$ *is a non-abelian minimal normal subgroup of* G, $G = US$. *In this case,* $C_G(S) = 1$ *(type 2).*
3. $\mathrm{Soc}(G) = A \times B$, *where* A *and* B *are the two unique minimal normal subgroups of* G, $G = AU = BU$ *and* $A \cap U = B \cap U = A \cap B = 1$. *In this case,* $A = C_G(B)$, $B = C_G(A)$, *and* $A \cong B \cong AB \cap U$ *are non-abelian (type 3).*

Large characteristically simple sections of a group

R. Baer
Classes of finite groups and their properties.
Illinois J. Math., 1:115-187, 1957.

Large characteristically simple sections of a group

We say that a maximal subgroup M of a group G is of *type i*, if the primitive group G/M_G is of type i, $1 \leq i \leq 3$; if M is of type 1 or 2, we say that M is a *monolithic maximal* subgroup of G and G/M_G is a monolithic primitive group.

Large characteristically simple sections of a group

Definition

The primitive group $[H/K] * G$ associated with a chief factor H/K of G is:

1. the semidirect product $[H/K]\big(G/C_G(H/K)\big)$ if H/K is abelian, or
2. the quotient group $G/C_G(H/K)$ if H/K is non-abelian.

Large characteristically simple sections of a group

Definition

Let H/K be a non-Frattini chief factor of a group G. Let $\mathcal{E}$ denote the set of all cores M_G of all monolithic maximal subgroups M of G such that M supplements chief factors G-connected with H/K, let

$$R = \bigcap \{N \mid N \in \mathcal{E}\},$$

and let $C = H\,\mathrm{C}_G(H/K)$. We say that the factor C/R is the *crown* of G associated with H/K.

Large characteristically simple sections of a group

Theorem

Let C/R be the crown of G associated with the non-Frattini chief factor H/K. Then $C/R = \mathrm{Soc}(G/R)$. Furthermore,

1. *every minimal normal subgroup of G/R is a n on-Frattini chief factor of G which is G-connected with H/K, and*
2. *no non-Frattini chief factor of G over C or below R is G-connected with H/K.*

Large characteristically simple sections of a group

Dalla Volta and Lucchini, 1998: Given a monolithic primitive group L with a unique minimal normal subgroup A, for each positive integer k we consider the direct product L^k of k copies of L. The subgroup

$$L_k = \{(l_1, \dots, l_k) \in L^k \mid l_1 \equiv \cdots \equiv l_k \pmod{A}\},$$

is called the *kth crown-based power* of L.

F. Dalla Volta and A. Lucchini
Finite groups that need more generators than any proper quotient.
J. Austral. Math. Soc. Ser. A, 64(1):82-91, 1998.

Large characteristically simple sections of a group

Theorem

Let H/K be a non-Frattini chief factor of a group G and let C/R be its crown. Then G/R is isomorphic to a crown-based power L_k, where $L = [H/K] * G$ and k is the number of chief factors of G that are G-related to H/K in a given chief series of G.

E. Detomi and A. Lucchini.
Crowns and factorization of the probabilistic zeta function of a finite group.
J. Algebra, 265(2):651–668, 2003.

Large characteristically simple sections of a group

- To present an extension of the notion of crown for isomorphic chief factors, not necessarily related by connectedness.
- To establish a relation between the number of non-Frattini chief factors isomorphic to a characteristically simple group A in a given chief series and the A-rank $\mathrm{rk}_A(G)$, defined as the largest number k such that G has a normal section that is the direct product of k non-Frattini chief factors of G that are isomorphic to A.

Large characteristically simple sections of a group

Theorem B

Let A be a non-Frattini chief factor of a group G and suppose that in a given chief series of G there are k non-Frattini chief factors isomorphic to A. Then there exist two normal subgroups C and R of G such that $R \leq C$ and C/R is isomorphic to a direct product of k minimal normal subgroups of G/R isomorphic to A.

In particular, $\mathrm{rk}_A(G)$ is the number of non-Frattini chief factors of G isomorphic to A in a given chief series of G.

Large characteristically simple sections of a group

Theorem

Let G be a monolithic primitive group with a unique minimal normal subgroup B. Then G/B has no chief factors isomorphic to B.

Large characteristically simple sections of a group

Let G be a primitive group of type 2.

- $B = \mathrm{Soc}(G) = S_1 \times \cdots \times S_n$, $S_i \cong S$ simple groups.
- $N = \mathrm{N}_G(S_1)$, $C = \mathrm{C}_G(S_1)$.
- $X = N/C$ is almost simple, $\mathrm{Soc}(X) = S_1C/C$.
- There exists a transitive subgroup $P_n \leq \mathrm{Sym}(n)$ with G isomorphic to a subgroup of $X \wr P_n$.

Large characteristically simple sections of a group

Theorem

If G is a primitive group with a unique minimal normal subgroup of order $q = p^d$, where p is a prime, then the number of composition factors of G of order p is at most $d + \frac{\varepsilon_p d - 1}{p-1}$, where

$$\varepsilon_p = \begin{cases} \frac{p}{p-1} & \text{if } p \text{ is a Fermat prime,} \\ 1 & \text{otherwise.} \end{cases}$$

M. Giudici, S. P. Glasby, C. H. Li, and G. Verret
The number of composition factors of order p in completely reducible groups of characteristic p.
J. Algebra, 490:241–255, 2017.

The upper bound

Since

$$\langle x_1, \ldots, x_r \rangle \neq G \iff \exists M \lessdot G \text{ such that } \langle x_1, \ldots, x_r \rangle \leq M.$$

In fact,

$$\begin{aligned} \mathrm{Prob}(\langle x_1, \ldots, x_r \rangle \leq M) &= \prod_{i=1}^{r} \mathrm{Prob}(x_i \in M) \\ &= \left(\frac{|M|}{|G|} \right)^r = \frac{1}{|G : M|^r} \end{aligned}$$

The number $\mathrm{m}_n(G)$ of maximal subgroups of G of a given index n is relevant here.

The upper bound

Theorem (Lubotzky, 2002)

If G is a group with r chief factors in a given chief series, then

$$\mathrm{m}_n(G) \leq r^2 n^{\mathrm{d}(G)+2}.$$

A. Lubotzky.
The expected number of random elements to generate a finite group.
J. Algebra, 257:452–459, 2002.

The upper bound

Theorem

Let G be a non-cyclic group with r chief factors in a given chief series. For every natural $n \geq 2$, $\mathrm{m}_n(G) \leq rn^{\mathrm{d}(G)+2}$.