Coverings of finite p-groups by conjugacy classes of cyclic subgroups

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Question (Wu, 2017) Suppose G is a noncyclic finite p group of order p^n, with $p > 2$, and $C_1, \ldots, C_m$ is a set of cyclic subgroups such that for every cyclic subgroup C of G there exists $g \in G$ such that $gCg^{-1} \leq C_i$ for some i then is it true that $m \geq n$?

The question arose from his joint work with von Puttkamer on classifying spaces of familes of subgroups of infinite groups.

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Question (von Puttkamer, 2018) Does the number of conjugacy classes of maximal cyclic subgroups of a noncyclic finite p-group, for $p > 2$, grow with the order of the group?

Von Puttkamer then studied groups where this number is small.

Note this is not true for $p = 2$. Consider the family of dihedral 2-groups

$$D_{2^n} = \langle x, y : x^{2^{n-1}} = 1 = y^2, yxy^{-1} = x^{-1} \rangle.$$

Then, for all $n \geq 2$, there are exactly 3 conjugacy classes of maximal cyclic subgroups, with representatives $\langle x \rangle, \langle y \rangle$ and $\langle xy \rangle$.

Recall a set $\{H_i\}$ of proper subgroups of a group G is a called a *covering* of G if $G = \bigcup H_i$ (H_i called components).

Note a covering has size at least 3.

Mathematicians have studied coverings of groups for a long time.

Scorza 1926 Considered groups with a covering of size 3.

Cohn 1994 Considered groups with a minimal covering of size 3, 4 and 5. For example:

A group has a minimal covering of size 3 iff it posseses at least 2 subgroups of index 2.

In particular people look for coverings of minimal size.

We call a covering a *normal covering* if it is invariant under G-conjugation.

The normal covering number, $\gamma(G)$ is the smallest number of conjugacy classes of proper subgroups in a normal covering of G.

An old result due to Burnside (or Jordan) shows that $\gamma(G)$ is at least 2.

More recently:

Bubboloni & Praeger 2011 Considered normal coverings of finite symmetric and alternating groups.

Crestani & Lucchini 2011 Normal coverings of finite soluble groups (for each $n \geq 2$ there exists a finite soluble group with $\gamma(G) = n$).

They comment that computing the normal covering number seems to require different techniques to studying the covering number.

Our question considers normal coverings of finite p-groups where the components (subgroups) of the partition are required to be cyclic.

We denote the normal covering number of a group G where the components are cyclic by $NCC(G)$ (to stand for *normal cyclic cover*).

Remarks. (i) $NCC(D_{2^n}) = 3$. Similarly $NCC = 3$ for the semidihedral groups and generalised quaternions.

(ii) Let N be a normal subgroup of G then

$$NCC(G/N) \leq NCC(G).$$

(iii) Suppose G is a finite p-group with d generators,

$$NCC(G) \geq NCC(G/G'G^p) = NCC(\underbrace{C_p \times \cdots \times C_p}_{d}) = \frac{p^d - 1}{p-1}.$$

Thus NCC grows with the number of generators of G.

(iv) If $NCC(G/N) = NCC(G)$ we show that $N \leq G'$ and $N \leq G^{\{p\}} = \{g^p : g \in G\}$.

Corollary (MB, RC, ML, EP) *Suppose $|G| = p^n$ with $n \geq 2$ and G noncyclic. Suppose either G has exponent p or G abelian then $NCC(G) \geq n + p - 1$.*

Proof is by induction on n. If $n = 2$, then $G \cong C_p \times C_p$ so $NCC(G) = p + 1$.
For $n > 2$ we can choose $z \in G$ central of order p such that $G/\langle z \rangle$ is noncyclic. Then by previous results $NCC(G/\langle z \rangle) < NCC(G)$ and the result follows.

So the question has an affirmative answer in these cases.

(v) Let $N \trianglelefteq G$. If N is central then $NCC(G) \geq NCC(N)$ and if $|G : N| = k$ then $NCC(G) \geq NCC(N)/k$.

Where to look next? Given the examples in (i) we decided to look at metacyclic p-groups more generally.

Theorem (MB, RC, ML 2022) *Let G be a metacyclic p-group of order p^n that is not a dihedral, generalised quaternion or semidihedral group. Then $NCC(G) \geq n - 2$.*

For metacyclic p-groups of positive type we show that $NCC(G) = NCC(G/G')$.

There is also a nice correspondence with nilpotency class.

Theorem (MB, RC, ML 2022) *Let G be a noncyclic p-group of nilpotency class c then* $NCC(G) \geq (p-1)(n/c-2)+p+1$.

- *Conjugacy classes of maximal cyclic subgroups, MB, RC, EP, ML, arxiv.*

- *Conjugacy classes of maximal cyclic subgroups of metacyclic p-groups, MB, RC, ML, arxiv*

- *Conjugacy classes of maximal cyclic subgroups and nilpotence class of p-groups, MB, RC, ML, Bull. Aust. Math. Soc. online.*

However we were failing to prove the result in general, but yet didn't have any counterexamples for p odd. Maybe considering pro-p groups would be useful.

Recall a pro-p group G is an inverse limit of finite p-groups. That is given an inverse system of finite p-groups, i.e. a family of finite p-groups P_i such that there exists homomorphisms $\pi_{i,j} : P_i \to P_j$ whenever $i > j$, such that $\pi_{i,i} = id$ and $\pi_{i,j}\pi_{j,k} = \pi_{i,k}$, then you can construct

$$G = \lim_{\leftarrow} P_i = \{(g_i) \in \Pi P_i : \pi_{i,j}(g_i) = g_j\}.$$

The P_i are given the discrete topology and G the induced product topology.

Standard example is the p-adic integers $\mathbb{Z}_p = \lim_{\leftarrow} \mathbb{Z}/p^n\mathbb{Z}$.

Also have $\lim_{\leftarrow} D_{2^n}$, the pro-2 completion of the infinite dihedral group.

Thus if we study a pro-p group then we are studying a whole family of p-groups at the same time.

For G a pro-p group we consider coverings by procyclic pro-p groups, an infinite procyclic pro-p group is isomorphic to $\mathbb{Z}_p$.

The following are equivalent (call the property $(*)$):

(i) there are infinitely many noncyclic finite p-groups P with $NCC(P) \leq k$.

(ii) there exists an infinite nonprocyclic pro-p group G with $NCC(G) \leq k$.

Lemma *Suppose that for some k there are infinitely many noncyclic finite p-groups P with $NCC(P) \leq k$. Then there exists an infinite non-procyclic pro-p group G with $NCC(G) \leq k$.*

Sketch. Let $\Gamma_k(p)$ be the oriented graph with vertices noncyclic finite p-groups with $NCC(P) \leq k$. There is an oriented edge from P to Q iff $Q \cong P/Z$ with $|Z| = p$. We claim that $\Gamma_k(p)$ has finitely many connected components. Since there are only finitely many abelian finite p-groups with NCC bounded by k, but each component contains an abelian group.

So we choose an infinite connected component and an infinite path in this graph, $P_1 \leftarrow P_2 \leftarrow P_3 \leftarrow \cdots$, then the inverse limit of these P_i is a pro-p group G. Furthermore $NCC(G) \leq k$ (since the covers form an inverse system). □

Recall a pro-p group is just infinite if all its proper continuous quotients are finite.

Lemma *An infinite pro-p group with finite NCC is just infinite.*

Sketch: First note such a G must be finitely generated. Suppose we can find a normal subgroup H such that G/H is infinite, then we can find an element of infinite order in this quotient. Let g be the preimage of this element and let $h \in H$. We consider the infinite sequence of elements $\{g^{p^k} h\}$ and claim they cannot lie in conjugate procyclic subgroups. $\square$

A pro-p group is p-adic analytic if it has an analytic structure over $\mathbb{Q}_p$, but there are also algebraic characterisations.

For example a pro-p group is p-adic analytic iff $D_n(G) = D_{n+1}(G)$ for some n where $D_n(G)$ are the dimension subgroups.

That is, $D_n(G) = \{g \in G : g \equiv 1 \mod I^n\}$ where I is the augmentation ideal of the group algebra $\mathbb{F}_p[G]$.

In particular:

$[D_n, D_m] \subseteq D_{n+m}$ and

$D_n^p \subseteq D_{np}$.

Theorem (YB, RC, ME, ML 2022) *Let G be a pro-p group with finite NCC. Then G is p-adic analytic.*

Sketch: We show that $D_n(G) = D_{n-1}(G)$. Suppose G has finite NCC then there exists $\{x_1, \dots, x_k\}$ such that every element of G is conjugate to x_i^λ for some i and some $\lambda \in \mathbb{Z}_p$.
For $1 \neq x$ there exists n such that $x \in D_n \setminus D_{n+1}$, call this $\deg(x)$.
Let $\deg(x_i^{p^j}) = d_{i,j}$, it follows that the degree of any nonidentity element of G is $d_{i,j}$ for some i and j.
However $d_{i,j} \geq p^j d_i$. So for $N \in \mathbb{N}$ there are at most $k(\lfloor \log_p(N) \rfloor + 1)$ possible degrees of elements of G of degree $\leq N$.
For sufficiently large N, $k(\lfloor \log_p(N) \rfloor + 1) < N$ so there exists an n which is not a degree of an element of G and thus $D_n(G) = D_{n-1}(G)$ and G is p-adic analytic. $\square$

Theorem (YB, RC, ME, ML 2022) *Let p be a prime and G a pro-p group. Then G has finite NCC iff one of the following holds:*

(i) G is finite.

(ii) G is infinite procyclic or $p = 2$ and G is infinite prodihedral (that is the pro-2 completion of the infinite dihedral group).

(iii) G is isomorphic to an open torsion-free subgroup of $PGL_1(D)$ where D is the unique degree 2 central division algebra over $\mathbb{Q}_p$.

Thus the answer to Wu and von Puttkamer's question is No.

Furthermore, let NCC_{min} denote the smallest k such that $(*)$ holds.

Theorem (YB, RC, ME, ML 2022)

$$NCC_{min}(p) = \begin{cases} 3 & \text{if } p = 2 \\ 9 & \text{if } p = 3 \\ p+2 & \text{if } p > 3. \end{cases}$$

- *On groups that can be covered by conjugates of finitely many cyclic or procyclic subgroups, YB, RC, ME, ML.*