# A BRIEF SURVEY OF F-SUBNORMAL SUBGROUPS

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- For our conference organizers
- Represents some joint work with Maria Ferrara and Marco Trombetti

# Well-known theorem of B. H. Neumann

**Theorem**

*Let G be a group in which* $|H^G : H| < \infty$ *for all subgroups H of G. Then G is finite-by-abelian.*

# Definitions

- $H \leq G$ is *$f$-ascendant* if we have

$$H = G_0 \leq G_1 \leq \ldots G_\alpha \leq \ldots G_\lambda = G$$

with $G_\alpha \lhd G_{\alpha+1}$ or $|G_{\alpha+1} : G_\alpha| < \infty$ (Phillips 1972).

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- $H$ is almost subnormal in $G$ if

$$H \leq H_n \leq H_{n-1} \leq \cdots \leq H_1 \leq H_0 = G$$

  where $H_1 = H^G, H_i = H^{G,i} = H^{H_{i-1}}$ and $|H_n : H| < \infty$. (Lennox 1977)

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where $H_1 = H^G, H_i = H^{G,i} = H^{H_{i-1}}$ and $|H_n : H| < \infty$. (Lennox 1977)
- $H$ is subnormal-by-finite in $G$ if $H$ contains a subnormal subgroup $S$ of $G$ such that $|H : S| < \infty$. May assume $S \triangleleft H$.

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- In $S_3$, $(1\,2)$ has near defect $(1, 3)$ but is not subnormal
- (Casolo-Mainardis) $G \in \mathbf{L}\mathfrak{N}$ implies every f-sn subgp is sn.

# More Background

- Suppose $K = \gamma_{r+1}(G)$ is finite. If $H \leq G$, then $HK$ is sn of defect at most $r$ in $G$, so $H$ is of near defect at most $(r, |K|)$. ie all subgroups of $G$ are of bounded near defect.

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**Theorem**

*(Lennox, 1977)*
*Let $r, s$ be fixed natural numbers. If $|H_r : H| \leq s$ for all subgroups $H$ of $G$, then $|\gamma_{f(r+s)}(G)| \leq s!$, for some function $f$.*

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- Compare this result with the well-known theorem of Roseblade,1965.

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Lennox deduced:

**Theorem**

*Let G be a finitely generated group. TFAE:*

1. *Every f.g. subgroup of G is almost sn of bounded near defect*
2. *G is finite-by-nilpotent*
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Such groups have the maximum condition; for $D_\infty$ every subgroup is subnormal-by-finite. Thus even for f.g. groups, if all subgroups are sn-by-fte this does not imply all subgroups almost sn.

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Casolo-Mainardis, 2001 construct a group which is not hypercentral in which all subgroups $H$ satisfy $|H_2 : H| < \infty$.

# Work of Casolo-Mainardis

**Theorem**

*(Casolo-Mainardis, 2001) Let G be a group. TFAE*

1. *Every subgp of G is f-sn.*
2. *Every subgp of G is almost sn*
3. *Every subgp H of G is contained in a subgp K such that H sn K and* $|G : K| < \infty$

*Every subgroup is subnormal-by-finite.*

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**Theorem**

*Let G be a gp in which every subgp is f-sn. Then*

1. *Every subgp of $G/D(G)$ is sn*
2. *$D(G)$ is fte-by-nilpt*
3. *G is fte-by-soluble*
4. *$D(G) \cap G^{\mathfrak{F}} \leq \zeta_\omega(G)$*
5. *every element of $G^{\mathfrak{F}}$ is right Engel in G*

*In particular, if G is torsion-free, then G is hypercentral*

Recall the theorem of Möhres that a group in which every subgp is sn is soluble

# Work of Detomi

**Theorem**

*(Detomi 2004) Let $G$ be a periodic group such that $|H_n : H| < \infty$ for all subgroups $H$ of $G$. There is a function $g$ of $n$ such that $\gamma_{g(n)}$ is finite.*

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**Theorem**

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**Theorem**

*(Casolo-Mainardis, Detomi) Let G be a torsion-free group. If $|H_n : H| < \infty$ for all subgroups H, then there is a function h of n such that G is nilpotent of class at most $h(n)$.*

# Recent work concerning f-subnormal subgroups etc.

**Theorem**

*(M. Ferrara, M. Trombetti, MD)*
*Let G be a group. Then*

(i) *G satisfies min-fsn* $\iff$ *G satisfies min-sn;*
(ii) *G satisfies max-fsn* $\iff$ *G satisfies max-sn;*
(iii) *G satisfies min-*$\infty$*-fsn* $\iff$ *G satisfies min-*$\infty$*-sn;*
(iv) *G satisfies max-*$\infty$*-fsn* $\iff$ *G satisfies max-*$\infty$*-sn;*
(v) *G satisfies double chain condition on subnormal subgroups* $\iff$ *G satisfies double chain condition on f-subnormal subgroups;*
(vi) *G satisfies weak double chain condition on subnormal subgroups* $\iff$ *G satisfies weak double chain condition on f-subnormal subgroups.*

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$\bar{\omega}(G) \leq \omega(G)$. Equality does not hold in general ($S_3$). For locally nilpotent groups $G$ we have $\bar{\omega}(G) = \omega(G)$.

**Theorem**

*Let G be a group satisfying the minimal condition on subnormal subgroups. Then $|G : \bar{\omega}(G)|$ is finite.*

# Wielandt Subgroup

Generalized Wielandt subgroup $\omega_i(G)$ of a group $G$:

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Generalized f-Wielandt subgroup

$$\overline{\omega}_i(G) = \bigcap \{N_G(H) | H \text{ is an infinite } f\text{-subnormal subgroup of } G\},$$

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$\overline{\omega}(G) \leq \overline{\omega}_i(G)$ in general.

# The case when $\overline{\omega}_i(G) \neq \overline{\omega}(G)$

**Theorem**

*Let G be a group satisfying $\overline{\omega}_i(G) \neq \overline{\omega}(G)$. Then the Baer radical of G is Prüfer-by-finite and nilpotent.*

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**Proposition**

*Let G be a group with $\overline{\omega}_i(G) \neq \overline{\omega}(G)$. Then every abelian normal subgroup of G is Prüfer-by-finite.*

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**Proposition**

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**Corollary**

*Let $G$ be a group such that $\overline{\omega}_i(G) \neq \overline{\omega}(G)$ and suppose that $H$ is a Chernikov f-subnormal subgroup of $G$. Then $H$ is Prüfer-by-finite.*

# Structure of $\overline{\omega}_i(G)/\overline{\omega}(G)$

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**Lemma**

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**Lemma**

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(i) *$\overline{\omega}_i(G)/\overline{\omega}(G)$ is finite;*

(ii) *There exists a finite normal subgroup $N$ of $G$ such that every $f$-subnormal subgroup of $\overline{\omega}_i(G)/N$ is a normal subgroup.*

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## Theorem

*For all groups $G$ the quotient group $\overline{\omega}_i(G)/\overline{\omega}(G)$ is residually finite. Furthermore, $\overline{\omega}_i(G)$ is either finite or $\overline{\omega}_i(G)/\overline{\omega}(G)$ is Dedekind.*

# Structure of $\overline{\omega}_i(G)/\overline{\omega}(G)$ in generalized soluble groups

**Theorem**

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(ii) *$G/\overline{\omega}(G)$ has finite exponent;*

(iii) *$\overline{\omega}_i(G)/\overline{\omega}(G)$ is a finite abelian $\{p, p-1\}$-group.*

By contrast, there is a periodic soluble group in which $\omega_i(G)/\omega(G)$ is nonabelian an example due to de Giovanni and Franciosi, 1985

**Corollary**

*Let G be an infinite subsoluble group.*

# Structure of $\overline{\omega}_i(G)/\overline{\omega}(G)$ in generalized soluble groups

## Corollary

*Let G be an infinite subsoluble group.*

(i) *If G is finitely generated, then* $\overline{\omega}_i(G) = \overline{\omega}(G)$.

# Structure of $\overline{\omega}_i(G)/\overline{\omega}(G)$ in generalized soluble groups

**Corollary**

*Let G be an infinite subsoluble group.*

(i) *If G is finitely generated, then* $\overline{\omega}_i(G) = \overline{\omega}(G)$.

(ii) *If G contains no Prüfer subgroups, then* $\overline{\omega}_i(G) = \overline{\omega}(G)$.

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**Corollary**

*Let G be a group such that* $\overline{\omega}_i(G)/\overline{\omega}(G)$ *is an infinite nonabelian group. Then* $\overline{\omega}_i(G)/\overline{\omega}(G)$ *has finite exponent.*

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There is an infinite group $G = \omega_i(G)$ such that $\overline{\omega}_i(G) = \overline{\omega}(G)$ is finite, but $\omega(G) \neq \omega_i(G)$. Furthermore, $\omega_i(G)/\omega(G)$ is not Dedekind.

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Let $A$ be a nontrivial finite abelian group. Then there is a finitely generated infinite group $G$ such that $\overline{\omega}_i(G)/\overline{\omega}(G) \cong A \times A$.

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If $G$ is a finite Dedekind group, then there is a group $R$ such that $\overline{\omega}_i(R)/\overline{\omega}(R) \cong G \cong \omega_i(R)/\omega(R)$.

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If $\omega_i(G)$ is finite, then so is $\overline{\omega}_i(G)$, but if $\overline{\omega}_i(G)$ is finite,then $\omega_i(G)$ can be infinite.

If $G$ is a finite Dedekind group, then there is a group $R$ such that $\overline{\omega}_i(R)/\overline{\omega}(R) \cong G \cong \omega_i(R)/\omega(R)$.

In fact, can $\overline{\omega}_i(G)/\overline{\omega}(G)$ ever be infinite?

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*If $G$ is an infinite group, then in any case $\overline{\omega}_i(G)/\overline{\omega}(G)$ is Dedekind.*

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**Theorem**

*If $G$ is an infinite group, then in any case $\overline{\omega}_i(G)/\overline{\omega}(G)$ is Dedekind.*

**Lemma**

*Let $G$ be a group. Then $\overline{\omega}_i(G)$ is finite if and only if $\overline{\omega}(G)$ is finite.*

# Bounded Near Defect

**Theorem**

*(De Falco, de Giovanni, Musella 2014) Let $G$ be a radical group in which every subgroup of infinite rank is nearly normal. Then either $G$ has finite rank or $G'$ is finite.*

Note that nearly normal means $|H^G : H| < \infty$ (what I would call almost normal).

# Bounded Near Defect

Let $\mathfrak{Y}_0$ denote the class of periodic locally graded groups.

Let $L$, $R$, $\acute{P}$, $\grave{P}$ denote the usual closure operations.

For each ordinal $\alpha$ let

$$\mathfrak{Y}_{\alpha+1} = L\mathfrak{Y}_\alpha \bigcup R\mathfrak{Y}_\alpha \bigcup \acute{P}\mathfrak{y}_\alpha \bigcup \grave{P}\mathfrak{Y}_\alpha,$$

and as usual let $\mathfrak{Y}_\gamma = \bigcup_{\beta<\gamma} \mathfrak{Y}_\beta$, for limit ordinals $\gamma$.

Set $\mathfrak{X} = \bigcup_\gamma \mathfrak{Y}_\gamma$.

# Bounded Near Defect

## Theorem

1. *Let $G \in \mathfrak{X}$ have infinite rank. Suppose all subgroups of G of infinite rank are almost subnormal of bounded near defect at most $(r, s)$. Then G is finite-by-nilpotent.*

# Bounded Near Defect

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2. *If $G$ is a periodic group and all subgroups of infinite rank are almost subnormal of bounded near defect at most $(r, s)$, then $|\gamma_{h(r+s^2)}| < (s^2)!$.*

# Bounded Near Defect

## Theorem

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2. *If $G$ is a periodic group and all subgroups of infinite rank are almost subnormal of bounded near defect at most $(r, s)$, then $|\gamma_{h(r+s^2)}| < (s^2)!$.*
3. *If $G$ is a locally nilpotent group and all subgroups of infinite rank are almost subnormal of bounded near defect at most $(r, s)$, then there is a function $k$ such that $G$ is nilpotent of class at most $k(r+s)$.*

Thank you very much.

Enjoy the rest of the conference and have safe journeys home!