Hausdorff dimension of some groups of automorphisms of trees

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Measuring the size of subgroups in finite groups

Let G be a finite group, and H a subgroup of G. Then we can use

$$\frac{|H|}{|G|}$$

to measure the relative size of H inside G.

If G is a finite p-group, then

$$\frac{\log |H|}{\log |G|}$$

may be a more suitable choice.

Measuring the size of closed subgroups in profinite groups

Let G be a countably based profinite group, and $\{G_n\}_{n \in \mathbb{N}}$ a base of neighbourhoods of 1 of open normal subgroups. Then

$$G \cong \varprojlim_{n \in \mathbb{N}} G/G_n.$$

If H is a closed subgroup of G, then $H \cong \varprojlim_{n \in \mathbb{N}} HG_n/G_n$.

So H can be recovered from its images in the finite groups G/G_n, and the relative size of these images is

$$\frac{\log |HG_n : G_n|}{\log |G : G_n|}$$

It seems reasonable to take the limit as $n \to \infty$ as the relative size of H in G.

Countably based profinite groups as metric spaces

Let G be a countably based profinite group, and $\mathcal{G} = \{G_n\}_{n\in\mathbb{N}}$ a basis of neighbourhoods of 1 of open normal subgroups.

Then G is metrizable with the following distance $d_{\mathcal{G}}$: for $x, y \in G$, we set

$$d_{\mathcal{G}}(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1/|G : G_n|, & \text{if } x \neq y, \end{cases}$$

where $n \in \mathbb{N}$ is minimum such that $xG_n \neq yG_n$.

Hausdorff dimension in countably based profinite groups

In a metric space (X, d) we can define the Hausdorff dimension of every subset $Y \subseteq X$.

Let $(G, d_{\mathcal{G}})$ be as above. The corresponding Hausdorff dimension is denoted by $\mathrm{hdim}_G^{\mathcal{G}}$.

Theorem (Abercrombie, Barnea-Shalev)

Let H be a closed subgroup of G. Then

$$\mathrm{hdim}_G^{\mathcal{G}}(H) = \liminf_{n\to\infty} \frac{\log |HG_n : G_n|}{\log |G : G_n|}$$

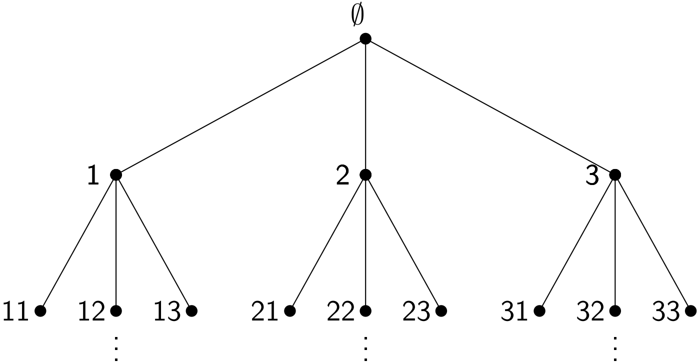
So Hausdorff dimension can be understood as a measure of the relative size of H in G.

Properties of Hausdorff dimension in profinite groups

- It may depend on the choice of $\mathcal{G}$.
- $\mathrm{hdim}^{\mathcal{G}}_{G}(G) = 1$, so Hausdorff dimension takes values in $[0, 1]$.
- Open subgroups always have Hausdorff dimension 1.
- Countable subsets have Hausdorff dimension 0.

Contents

The p-adic tree



- Here $p = 3$. We denote this tree by $\mathcal{T}_p$, or only $\mathcal{T}$ if p is fixed.
- Vertices are words in the alphabet $\{1, \ldots, p\}$, and $\mathcal{T}$ is structured into *levels* of vertices of the same length.

Automorphisms of the p-adic tree $\mathcal{T}$

An automorphism of $\mathcal{T}$ is a bijection of the vertices that preserves incidence. All automorphisms of $\mathcal{T}$ form a group $\operatorname{Aut}\mathcal{T}$.

If $f \in \operatorname{Aut}\mathcal{T}$, then

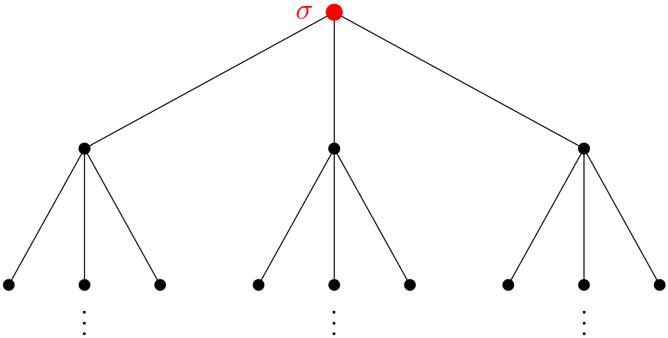
- f preserves the root $\emptyset$, and more generally the levels of $\mathcal{T}$.
- f sends the descendants of a vertex v to the descendants of its image: if $v \overset{f}{\longmapsto} w$ then

$$\begin{array}{ccc} \{vx \mid x = 1, \ldots, p\} & \xrightarrow{f} & \{wx \mid x = 1, \ldots, p\} \\ vx & \longmapsto & w(x\sigma), \end{array}$$

  for some $\sigma \in S_p$. We call σ the label of f at v.
- The labels of f at all vertices fully determine f.

A rooted automorphism

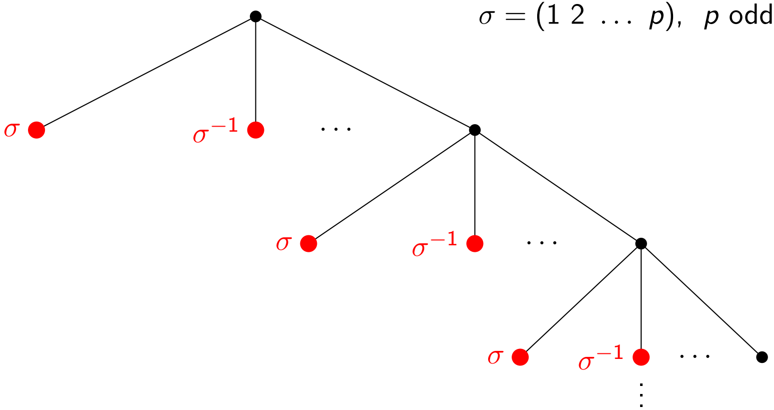
What is for example, the automorphism with label $\sigma = (1\ 2\ \ldots\ p)$ at the root, and 1 elsewhere?



It permutes rigidly the p subtrees under the root as indicated by σ. This is the rooted automorphism corresponding to σ.

Another example

$\sigma = (1\ 2\ \ldots\ p),\ \ p$ odd



All labels are 1 except for vertices of the form $p\overset{i}{\ldots}p1$ and $p\overset{i}{\ldots}p2$ for $i \geq 0$, which are σ and σ^{-1}, respectively.

$\operatorname{Aut}\mathcal{T}$ as a wreath product

Every subtree of $\mathcal{T}$ hanging from the first level is isomorphic to $\mathcal{T}$.

We can define

$$\begin{array}{rccc} \psi & : & \operatorname{Aut}\mathcal{T} & \longrightarrow & \operatorname{Aut}\mathcal{T} \wr S_p \\ & & f & \longmapsto & (f_1,\ldots,f_p)\sigma, \end{array}$$

where f_i comes from the action of f on the ith subtree hanging from the first level and σ is the label of f at the root.

Then ψ is an isomorphism: $\quad \operatorname{Aut}\mathcal{T} \cong \operatorname{Aut}\mathcal{T} \wr S_p$.

Let p be odd, and define $a, b \in \operatorname{Aut} \mathcal{T}$ as follows:

- a is the rooted automorphism corresponding to $(1\ 2\ \ldots\ p)$.
- b is the second automorphism we defined.

Then both a and b are of order p.

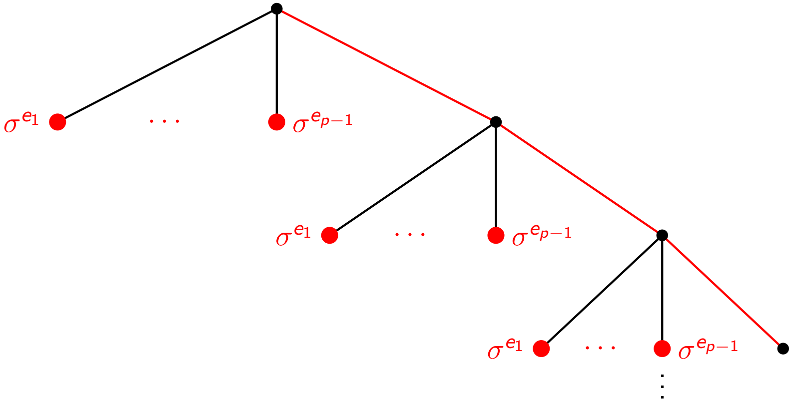
Definition

Then $G = \langle a, b \rangle$ is the Gupta-Sidki group for the prime p.

It is a counterexample to the General Burnside Problem: it is finitely generated, periodic, but infinite.

Generalising the automorphism b

Let $\boldsymbol{e} = (e_i)$ be a vector in $\mathbb{F}_p^{p-1}$. We define $b_{\boldsymbol{e}}$ via labels:



All labels are 1 except for vertices $p \overset{n}{\ldots} pj$, which are σ^{e_j}.

Note that b corresponds to $\boldsymbol{e} = (1, -1, 0, \ldots, 0)$.

Generalising the Gupta-Sidki group: Multi-GGS-groups

Let $\boldsymbol{E}$ be a subspace of $\mathbb{F}_p^{p-1}$. We define the multi-GGS-group corresponding to $\boldsymbol{E}$ as

$$G_{\boldsymbol{E}} = \langle a, b_{\boldsymbol{e}} \mid \boldsymbol{e} \in \boldsymbol{E} \rangle.$$

If $\dim \boldsymbol{E} = r$ then $G_{\boldsymbol{E}}$ can be generated by $r+1$ elements.

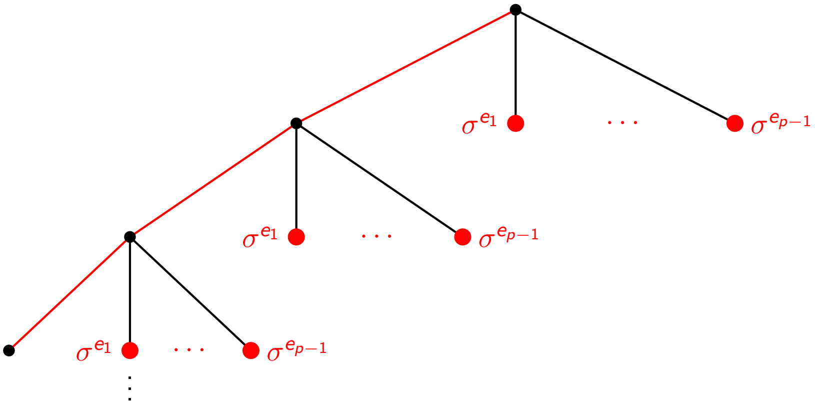
When $\dim \boldsymbol{E} = 1$ then we say that $G_{\boldsymbol{E}}$ is a GGS-group. If $\boldsymbol{e} \in \boldsymbol{E}$ is a non-zero vector then

$$G_{\boldsymbol{E}} = \langle a, b_{\boldsymbol{e}} \rangle.$$

The ordinary Gupta-Sidki group is the GGS-group corresponding to $\boldsymbol{E} = \langle (1, -1, 0, \ldots, 0) \rangle$.

A variation of the automorphism $b_{\boldsymbol{e}}$

Let $\boldsymbol{e} = (e_i)$ be a vector in $\mathbb{F}_p^{p-1}$. We define $c_{\boldsymbol{e}}$ via labels:



All labels are 1 except for vertices $1 \overset{n}{\ldots} 1j$, $j \neq 1$, which are $\sigma^{e_1}, \ldots, \sigma^{e_{p-1}}$.

Variations of the automorphism $b_{\boldsymbol{e}}$

In all generality, given $\boldsymbol{e} \in \mathbb{F}_p^{p-1}$ and $\ell \in \{1, \ldots, p\}$, we can similarly define

$$b_{\boldsymbol{e}}^{(\ell)} \in \Gamma$$

by using the path with vertices $\ell \overset{n}{\ldots} \ell$.

So all labels are 1, except at vertices $\ell \overset{n}{\ldots} \ell j$ for $j \neq \ell$, where we place in order the labels $\sigma^{e_1}, \ldots, \sigma^{e_{p-1}}$.

For every $\ell \in \{1, \ldots, p\}$, let $\boldsymbol{E}(\ell)$ be a subspace of $\mathbb{F}_p^{p-1}$, and set

$$\boldsymbol{E}(*) = (\boldsymbol{E}(1), \ldots, \boldsymbol{E}(p)).$$

We define the multi-EGS group corresponding to $\boldsymbol{E}(*)$ as

$$G_{\boldsymbol{E}(*)} = \langle a, b^{(\ell)}_{\boldsymbol{e}(\ell)} \mid \boldsymbol{e}(\ell) \in \boldsymbol{E}(\ell) \rangle.$$

Let $\boldsymbol{C}$ be the subspace of $\mathbb{F}_p^{p-1}$ consisting of constant vectors.

Theorem

Let $G_{\boldsymbol{E}(*)}$ be a multi-EGS-group and assume that not all subspaces $\boldsymbol{E}(\ell)$ are equal to $\boldsymbol{C}$. Then:

$$\gamma_3(G) \times \cdots \times \gamma_3(G) \subseteq \psi(\gamma_3(G)). \tag{1}$$

If furthermore one of the $\boldsymbol{E}(\ell)$ contains a non-symmetric vector, or else $\dim\langle \boldsymbol{E}(1), \ldots, \boldsymbol{E}(p)\rangle \geq 2$*, then*

$$G' \times \cdots \times G' \subseteq \psi(G'). \tag{2}$$

We say $G_{\boldsymbol{E}(*)}$ is regular branch over $\gamma_3(G)$ or over G' according as (1) or (2) holds.

Consider the automorphisms $s, t \in \operatorname{Aut} \mathcal{T}$ defined recursively via

$$\psi(s) = (1, \overset{p-1}{\ldots}, 1, t), \quad \text{and} \psi(t) = (1, \overset{p-1}{\ldots}, 1, s)\sigma.$$

Definition

The group $B = \langle s, t \rangle$ is called the Basilica p-group.

If $p = 2$ then we obtain the well-known Basilica group.

Theorem

If $\mathcal{T}(n)$ is the tree $\mathcal{T}$ truncated at the nth level then

$$\operatorname{Aut}\mathcal{T} \cong \varprojlim_{n\to\infty} \operatorname{Aut}\mathcal{T}(n).$$

So $\operatorname{Aut}\mathcal{T}$ *is a profinite group.*

Definition

The *n*th level stabiliser, $\operatorname{st}(n)$, is the subgroup of $\operatorname{Aut}\mathcal{T}$ of all automorphisms that fix every vertex at the nth level.

Theorem

We have $\operatorname{Aut}\mathcal{T}(n) \cong \operatorname{Aut}\mathcal{T}/\operatorname{st}(n)$ *and* $\{\operatorname{st}(n)\}_{n\in\mathbb{N}}$ *is a base of neighbourhoods of* 1 *in* $\operatorname{Aut}\mathcal{T}$.

Hausdorff dimension of subgroups of Aut $\mathcal{T}$

We can study Hausdorff dimension of subgroups of Aut $\mathcal{T}$ with respect to the canonical base of neighbourhoods $\mathcal{S} = \{\mathrm{st}(n)\}_{n\in\mathbb{N}}$. We then delete $\mathcal{S}$ from $\mathrm{hdim}^{\mathcal{S}}_{\mathrm{Aut}\,\mathcal{T}}$.

If H is a closed subgroup of Aut $\mathcal{T}$ and $\mathrm{st}_H(n) = \mathrm{st}(n) \cap H$:

$$\mathrm{hdim}_{\mathrm{Aut}\,\mathcal{T}}(H) = \liminf_{n\to\infty} \frac{\log|H : \mathrm{st}_H(n)|}{\log|\,\mathrm{Aut}\,\mathcal{T} : \mathrm{st}(n)|}.$$

If H is an arbitrary subgroup of Aut $\mathcal{T}$ then

$$\mathrm{hdim}_{\mathrm{Aut}\,\mathcal{T}}(\overline{H}) = \liminf_{n\to\infty} \frac{\log|H : \mathrm{st}_H(n)|}{\log|\,\mathrm{Aut}\,\mathcal{T} : \mathrm{st}(n)|}.$$

Thus the Hausdorff dimension of the closure $\overline{H}$ can be determined just from H, without knowing explicitly the closure.

Let Γ be the set of $f \in \operatorname{Aut}\mathcal{T}$ with the property that all labels of f are powers of $\sigma = (1\ 2\ \ldots\ p)$. All multi-EGS-groups and the Basilica p-groups are subgroups of Γ.

Theorem

Γ *is a Sylow pro-p subgroup of* $\operatorname{Aut}\mathcal{T}$.

For an arbitrary $G \leq \Gamma$ we have

$$\operatorname{hdim}_{\operatorname{Aut}\mathcal{T}}(\overline{G}) = \frac{1}{\log_p(p!)} \operatorname{hdim}_{\Gamma}(\overline{G}),$$

where $\operatorname{hdim}_{\Gamma}$ is calculated wrt $\{\operatorname{st}_{\Gamma}(n)\}_{n\in\mathbb{N}}$, and

$$\operatorname{hdim}_{\Gamma}(\overline{G}) = (p-1)\liminf_{n\to\infty} \frac{\log_p |G/\operatorname{st}_G(n)|}{p^n}.$$

Let G be a GGS-group defined by a non-zero vector $\boldsymbol{e}$.

Theorem (F-A & Zugadi-Reizabal, 2014)

If C is the circulant matrix whose first row is $(\boldsymbol{e}, 0)$ then

$$\text{hdim}\,\overline{G_{\boldsymbol{e}}} = \frac{(p-1)t}{p^2} - \frac{\delta}{p^2} - \frac{\varepsilon}{(p-1)p^2},$$

where t is the rank of C, and

$$\delta = \begin{cases} 1, & \textit{if } \mathrm{e} \textit{ is symmetric,} \\ 0, & \textit{otherwise,} \end{cases} \qquad \textit{and} \qquad \varepsilon = \begin{cases} 1, & \textit{if } \mathrm{e} \textit{ is constant,} \\ 0, & \textit{otherwise.} \end{cases}$$

We actually calculate the order of the quotients of G by its level stabilisers, an interesting result on its own.

Definition

A subspace of $\mathbb{F}_p^p$ is circulant if it is invariant under the shift $(a_1, a_2, \ldots, a_p) \mapsto (a_p, a_1, \ldots, a_{p-1})$.

- There is exactly one circulant subspace C_i of every dimension $0 \leq i \leq p$, and they form a chain.
- If V is an arbitrary subspace of $\mathbb{F}_p^p$, of dimension r, then the chain
$$\{0\} = V \cap C_0 \subseteq V \cap C_1 \subseteq \cdots \subseteq V \cap C_p$$
contains exactly r strict inclusions.
- If $t_1 < \cdots < t_r$ are the indices for which $V \cap C_{t_i} \subsetneq V \cap C_{t_i+1}$ then we call $(t_1, \ldots, t_r)$ the circulant weight of V.

hdim_Γ of multi-EGS-groups: regular branch over G'

Let $G_{\boldsymbol{E}(*)}$ be a multi-EGS-group that is regular branch over G'. If $V = \langle \boldsymbol{E}(1), \ldots, \boldsymbol{E}(p) \rangle$ then we say that $G_{\boldsymbol{V}}$ is the multi-GGS group associated to $G_{\boldsymbol{E}(*)}$.

Theorem (F-A & Gül & Thillaisundaram, 2022)

$\text{hdim}_\Gamma(\overline{G_{\boldsymbol{E}(*)}}) = \text{hdim}_\Gamma(\overline{G_{\boldsymbol{V}}})$. *So we reduce to multi-GGS-groups in this case.*

Theorem (F-A & Gül & Thillaisundaram, 2022)

Let $G_{\boldsymbol{E}}$ be a multi-GGS group that is regular branch over G', with $\dim \boldsymbol{E} = r$. *Then*

$$\text{hdim}_\Gamma(\overline{G_{\boldsymbol{E}}}) = (p-1)\Big(\frac{t_1}{p^2} + \frac{t_2}{p^3} + \cdots + \frac{t_r}{p^{r+1}}\Big),$$

where $(t_1, \ldots, t_r)$ is the circulant weight of $\boldsymbol{E}$.

hdim$_\Gamma$ of multi-EGS-groups: regular branch over $\gamma_3(G)$

Let $G_{\boldsymbol{E}(*)}$ be a multi-EGS group regular branch only over $\gamma_3(G)$.

Then every subspace $E(\ell)$ has dimension ≤ 1 and consists of symmetric vectors.

We may assume there are at least two non-trivial subspaces (otherwise we have a GGS-group).

Theorem (F-A & Gül & Thillaisundaram, 2022)

In these circumstances, we have

$$\operatorname{hdim}_\Gamma(\overline{G_{\boldsymbol{E}(*)}}) = \frac{(p-1)t}{p^2} + \frac{1}{p^3}.$$

Note that in all cases we actually obtain the order of $|G/\operatorname{st}_G(n)|$.

Theorem (Di Domenico & F-A & Noce & Thillaisundaram)

If B is the Basilica p-group then

$$\mathrm{hdim}_\Gamma(\overline{B}) = \frac{p}{p+1}.$$

This case is substantially different from multi-EGS-groups, since Basilica p-groups are not regular branch over any subgroup of finite index.