MICHA

A Chain of Normalizers, Partitions and a Modular Idealizer CHAin

Norberto Gavioli
Joint work(s) with R. Aragona, R. Civino e C.M. Scoppola

UNIVERSITÀ DEGLI STUDI DELL'AQUILA

Ischia Group Theory – June 2022

The Normalizer Chain

From Riccardo Aragona's talk

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- We consider an elementary abelian regular 2-subgroup T of $\mathrm{Sym}(2^n)$
- T is clearly a normal subgroup of the Sylow 2-subgroup U of the affine group $\mathrm{AGL}(2, n)$
- Given a Sylow 2-subgroup Σ of of $\mathrm{Sym}(2^n)$ containing U we define $N_n^0 = U$ and recursively

$$N_n^i = N_\Sigma(N_n^{i-1})$$

The Normalizer Chain

n	$\log_2\lvert N_n^i : N_n^{i-1}\rvert$ for $1 \le i \le 14$													
3	1	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1	2	1	1	0	0	0	0	0	0	0	0	0	0
5	1	2	4	1	2	2	1	1	1	1	0	0	0	0
6	1	2	4	7	2	4	4	1	1	2	2	2	2	1
7	1	2	4	7	11	4	7	3	4	2	2	4	4	4
8	1	2	4	7	11	16	7	5	6	2	6	6	3	3
9	1	2	4	7	11	16	23	4	9	4	11	4	12	9
10	1	2	4	7	11	16	23	32	4	14	5	20	7	19
11	1	2	4	7	11	16	23	32	43	5	22	7	32	4
12	1	2	4	7	11	16	23	32	43	57	7	32	12	43
13	1	2	4	7	11	16	23	32	43	57	74	12	42	18
14	1	2	4	7	11	16	23	32	43	57	74	95	8	24
15	1	2	4	7	11	16	23	32	43	57	74	95	121	8

TABLE 3. Values of $\log_2\lvert N_n^i : N_n^{i-1}\rvert$ for small i and n. For $i \le n-2$ these numbers do not depend on n and in the table are represented by highlighted digits.

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6	1	2	4	7	2	4	4	1	1	2	2	2	2	1
7	1	2	4	7	11	4	7	3	4	2	2	4	4	4
8	1	2	4	7	11	16	7	5	6	2	6	6	3	3
9	1	2	4	7	11	16	23	4	9	4	11	4	12	9
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The sequence in light blue looks like to be the one of the partial sum of the number of partitions of *i* into distinct parts.

The Normalizer Chain

The highlighted sequence looks like to be the one of the partial sum of the number of partitions of *i* into distinct parts: found via OEIS. Is it a chance?

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Search Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

A317910 Expansion of -1/(1 - x)^2 + (1/(1 - x))*Product_{k>=1} (1 + x^k). 1

0, 0, 0, 1, 2, 4, 7, 11, 16, 23, 32, 43, 57, 74, 95, 121, 152, 189, 234, 287, 350, 425, 513, 616, 737, 878, 1042, 1233, 1454, 1709, 2004, 2343, 2732, 3179, 3690, 4274, 4941, 5700, 6563, 7544, 8656, 9915, 11340, 12949, 14764, 16811, 19114, 21703, 24612, 27875, 31532, 35628, 40209 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,5

COMMENTS Partial sums of A111133.

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Of course it's not

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0 1 3 6 2 7 13 20 12 21 11 22 10 23 OEIS

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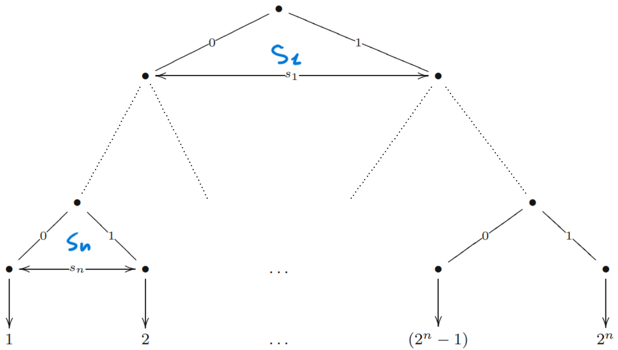
The Sylow 2-Subgroup Σ_n of Sym(2^n)

$\Sigma_n = \langle s_1, \ldots, s_n \rangle$ is the automorphism group of the rooted binary tree with 2^n leaves. It is also the iterated wreath product $\langle s_n \rangle \wr \cdots \wr \langle s_1 \rangle$.



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- The i-th base subgroup

$$S_i = \langle s_i \rangle^{\Sigma_i}$$

is the normal closure of $\langle s_i \rangle$ in $\Sigma_i = \langle s_i \rangle \wr \cdots \wr \langle s_1 \rangle$ and it is an elementary abelian 2-group generated by commuting conjugates of s_i. So that

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$$\Sigma_n = S_1 \ltimes \cdots \ltimes S_n.$$

- The subgroup S_i has a special set of independent generators, i.e. the left normed commutators

$$[s_i, s_{i_2} \ldots, s_{i_k}] \,,$$

where $n \geq i_1 > \cdots > i_1 \geq 1$, that are called rigid commutators.

Rigid Commutators

- For the sake of simplicity we denote a rigid commutator only by the indices, i.e.

$$[6,5,2] := [s_6, s_5, s_2] \ ,$$

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$$\vee[6;4,3,1] := [s_6, s_5, s_2] \ ,$$

also written as $\vee[6; X]$, where $X = \{1,2,4\}$ is the set of missing digits.

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- RIGID COMMUTATOR MACHINERY. Suppose that $a \geq b$ then

$$[\vee[a;X], \vee[b;Y]] = \begin{cases} 1 = [\,] & \text{if } b \notin X \\ \vee[a; Y \cup (X \setminus \{b\})] & \text{if } b \in X \end{cases}$$

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Saturated Subgroups

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- A subgroup H of Σ_n is saturated if it is generated by rigid commutators.
- The regular elementary abelian subgroup T is saturated being generated by $[1], [2, 1], \ldots, [n, \ldots, 2, 1]$.
- The Sylow 2-subgroup U of $\mathrm{AGL}(2, n)$ contained in Σ_n is saturated generated by T and the rigid commutators of the form $\vee[a, \{b\}]$, where $1 \leq b < a \leq n$.

Theorem

If H is a saturated subgroup of Σ_n containing exactly m nontrivial rigid commutators then $|H| = 2^m$.

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In particular, as expected, $|T| = 2^n$ and $|U| = 2^{\binom{n+1}{2}}$.

Back to the Normalizer Chain: an example $n = 5$

- The group T is generated by the rigid commutators $\vee[5;\emptyset], \vee[4;\emptyset], \vee[4;\emptyset], \vee[3;\emptyset], \vee[1;\emptyset]$

Back to the Normalizer Chain: an example $n = 5$

- The group T is generated by the rigid commutators
  $\vee[5;\emptyset], \vee[4;\emptyset], \vee[4;\emptyset], \vee[3;\emptyset], \vee[1;\emptyset]$
- The group U is generated by adding to the previous the rigid commutators
  $\vee[5;\{1\}], \vee[5;\{2\}], \vee[5;\{3\}], \vee[5;\{4\}]$,
  $\vee[4;\{1\}], \vee[4;\{2\}], \vee[4;\{3\}]$,
  $\vee[3;\{1\}], \vee[3;\{2\}]$,
  $\vee[2;\{1\}]$.

Back to the Normalizer Chain: an example $n = 5$

- The group $N_5^1 = N_{\Sigma_5}(U)$ by adding to the previous the rigid commutator $\vee[5; \{1, 2\}]$ Partition(s) of 3 into distinct parts

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- The group $N_5^2 = N_{\Sigma_5}(N_5^1)$ by adding to the previous the rigid commutator $\vee[5; \{1, 3\}]$ Partition(s) of 4 into distinct parts
  $\vee[4; \{1, 2\}]$ Partition(s) of 3 into distinct parts

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- The group $N_5^3 = N_{\Sigma_5}(N_5^2)$ by adding to the previous the rigid commutator
  $\vee[5;\{1,4\}]$, $\vee[5;\{2,3\}]$ Partitions of 5 into distinct parts
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  $\vee[3;\{1,2\}]$ Partition(s) of 3 into distinct parts

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  $\vee[4; \{1, 2\}]$ Partition(s) of 3 into distinct parts
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  $\vee[4; \{1, 3\}]$ Partition(s) of 4 into distinct parts
  $\vee[3; \{1, 2\}]$ Partition(s) of 3 into distinct parts

By way of the rigid commutators machinery it is possible to show that in general $|N_n^i : N_n^{i-1}| = 2^{b_{i+2}}$, for $i = 1, \ldots n-2$, where b_i is the i-th term of the partial sum sequence of the sequence $\{a_i\}$ of partitions into distinct parts.

We set

$$A_m = \begin{cases} (\mathbb{Z}/m\mathbb{Z}) & \text{if } m \neq 0 \\ \mathbb{Z} & \text{if } m = 0 \end{cases}$$

Let $\Lambda = \{\lambda_i\}_{i\geq 1}$ be a sequence of non-negative integers such that $\lambda_i = 0$ for $i \geq k$ and let L_n be the free A_m-module spanned by the non-trivial symbols

$$x^\Lambda \partial_k = \left(\prod_{i=1}^{k-1} x_i^{\lambda_i} \right) \partial_k$$

where $1 \leq k \leq n$ and $x_i^m = 0$ if $m > 0$.
The weight of Λ is defined as $\mathrm{wt}(\lambda) = \sum_{i\geq 1} i\lambda_i$.

- The set L_n can be made into a Lie ring by A_m-bilinearly extending the Lie-product

$$[x^{\Lambda}\partial_k, x^{\Theta}\partial_h] = \left(\frac{\partial}{\partial_h}(x^{\Lambda})x^{\Theta}\right)\partial_k - \left(x^{\Lambda}\frac{\partial}{\partial_k}(x^{\Theta})\right)\partial_h$$

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- If $m = p$ is a prime then L_p is actually the Lie algebra associated to the lower central series of the iterated wreath product $\Sigma_n = \wr^n C_p$, i.e. the Sylow p-subgroup of $\mathrm{Sym}(p^n)$.

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- The analog of the regular elementary abelian subgroup is the subalgebra

$$T = \langle \partial_1, \ldots, \partial_n \rangle$$

- As above we let

Modular Idealizer CHAin

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Theorem (M.I.CHA. $m > 2$)

Let $m > 2$ and $1 \leq i \leq n-1$, then $|I^i : I^{i-1}| = m^{b_{p,i+1}}$, where $\{b_{m,i}\}_{i \geq 2}$ is the partial sums sequence of the sequence $\{a_{m,i}\}_{i \geq 2}$ of the number of partitions of i in at least 2 parts, every part occurring with multiplicity at most $m-1$.

- As above we let
  - $I_n^0 = U$ to be the idealizer of T in L_n,
  - I_n^1 to be the idealizer of U in L_n,
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Theorem (M.I.CHA. $m = 2$)

Let $m = 2$ and $1 \leq i \leq n-2$, then $|I^i : I^{i-1}| = 2^{b_{i+2}}$.

Modular Idealizer CHAin

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  - $I_n^0 = U$ to be the idealizer of T in L_n,
  - I_n^1 to be the idealizer of U in L_n,
  - I_n^i to be the idealizer of I_n^{i-1} in L_n for $i \geq 2$.

Theorem (M.I.CHA. $m = 2$ and $i = n - 1$)

Let $m = 2$ then $|I^{n-1} : I^{n-2}| = 2^u$, where u is the number of base elements $x^\Lambda \partial_k$ such that Λ is an unrefinabile partition of $k + 1$ into distinct parts not larger than $k - 1$, and such that $k \geq n - e$, where e is the minimum excludant of Λ.

Finding, for $m = 2$, the same result as in the case of the normalizer chain in Σ_n.

M.I.CHA. an example: $n = 5, m = 0$

```
index 1, grades [ [ 4, 1 ] ]
[ "x1^2*D5" ]
1
index 2, grades [ [ 4, 2 ], [ 3, 1 ] ]
[ "x1^3*D5", "x1^1*x2^1*D5", "x1^2*D4" ]
3
index 3, grades [ [ 4, 4 ], [ 3, 2 ], [ 2, 1 ] ]
[ "x1^4*D5", "x1^2*x2^1*D5", "x1^1*x3^1*D5", "x2^2*D5", "x1^3*D4", "x1^1*x2^1*D4", "x1^2*D3" ]
7
index 4, grades [ [ 4, 6 ], [ 3, 4 ], [ 2, 2 ], [ 1, 1 ] ]
[ "x1^5*D5", "x1^3*x2^1*D5", "x1^2*x3^1*D5", "x1^1*x2^2*D5", "x1^1*x4^1*D5", "x2^1*x3^1*D5", "x1^4*D4",
  "x1^2*x2^1*D4", "x1^1*x3^1*D4", "x2^2*D4", "x1^3*D3", "x1^1*x2^1*D3", "x1^2*D2" ]
13
index 5, grades [ [ 4, 1 ] ]
[ "x1^6*D5" ]
1
index 6, grades [ [ 4, 2 ], [ 3, 1 ] ]
[ "x1^7*D5", "x1^4*x2^1*D5", "x1^5*D4" ]
3
index 7, grades [ [ 4, 4 ], [ 3, 2 ], [ 2, 1 ] ]
[ "x1^8*D5", "x1^5*x2^1*D5", "x1^3*x3^1*D5", "x1^2*x2^2*D5", "x1^6*D4", "x1^3*x2^1*D4", "x1^4*D3" ]
7
index 8, grades [ [ 4, 7 ], [ 3, 4 ], [ 2, 2 ], [ 1, 1 ] ]
[ "x1^9*D5", "x1^6*x2^1*D5", "x1^4*x3^1*D5", "x1^3*x2^2*D5", "x1^2*x4^1*D5", "x1^1*x2^1*x3^1*D5", "x2^3*D5",
  "x1^7*D4", "x1^4*x2^1*D4", "x1^2*x3^1*D4", "x1^1*x2^2*D4", "x1^5*D3", "x1^2*x2^1*D3", "x1^3*D2" ]
14
```

1, 3, 7, 14, 26, 45, 75, ...

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0 1 3 6 2 7 13 20 12 21 11 22 10 23 OEIS THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

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1,3,7,14,26,45,75 Search Hints

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Search: **seq:1,3,7,14,26,45,75**

Displaying 1-1 of 1 result found. page 1

Sort: relevance | references | number | modified | created Format: long | short | data

A014153 Expansion of 1/((1-x)^2*Product_{k>=1} (1-x^k)). +30 29

1, 3, 7, 14, 26, 45, 75, 120, 187, 284, 423, 618, 890, 1263, 1771, 2455, 3370, 4582, 6179, 8266, 10980, 14486, 18994, 24757, 32095, 41391, 53123, 67865, 86325, 109350, 137979, 173450, 217270, 271233, 337506, 418662, 517795, 638565, 785350, 963320, 1178628 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS Number of partitions of n with three kinds of 1. E.g., a(2)=7 because we have 2, 1+1, 1+1', 1+1", 1'+1', 1'+1", 1"+1". - Emeric Deutsch, Mar 22 2005

Partial sums of the partial sums of the partition numbers A000041. Partial sums of A000070. Euler transform of 3,1,1,1,...

- When $m = 0$ the Lie ring L_n has infinite rank as a $\mathbb{Z}$-module and is not nilpotent.

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- Let $\{a_n\}_{n=0}^{\infty}$ be the sequence whose term a_n is equal to the number of partitions of n. Also let $b_n = \sum_{i=0}^{n} a_i$, for $n \geq 0$ and $c_n = \sum_{i=0}^{n} b_n$.

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- We set

$$r_i := ((i-1) \bmod n-1)+1 \qquad \mathrm{wd}(x^{\Lambda}) := \mathrm{wt}(\Lambda) - \deg(x^{\Lambda}) + n - k$$

$$h_i := \left\lfloor \frac{i-1}{n-1} \right\rfloor + 1$$

Modular Idealizer CHAin

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$$h_i := \left\lfloor \frac{i-1}{n-1} \right\rfloor + 1$$

Theorem (M.I.CHA. $m = 0$)

The element $x^\Lambda \partial_k$ belongs to $I_n^i \setminus I_n^{i-1}$ iff $n-k \leq \mathrm{wd}(x^\Lambda \partial_k) < r_i$ and $i = h_i \,\mathrm{wd}(x^\Lambda \partial_k) + \deg(x^\Lambda) - 1$.

- When $m = 0$ the Lie ring L_n has infinite rank as a $\mathbb{Z}$-module and is not nilpotent.
- Let $\{a_n\}_{n=0}^{\infty}$ be the sequence whose term a_n is equal to the number of partitions of n. Also let $b_n = \sum_{i=0}^{n} a_i$, for $n \geq 0$ and $c_n = \sum_{i=0}^{n} b_n$.
- We set

$$r_i := ((i-1) \bmod n - 1) + 1 \qquad \mathrm{wd}(x^\Lambda) := \mathrm{wt}(\Lambda) - \deg(x^\Lambda) + n - k$$

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Theorem (M.I.CHA. $m = 0$)

Let $m = 0$. If $i > (n-4)(n-1)$ then I^i/I^{i-1} is a free $\mathbb{Z}$-module of rank c_{r_i-1}. In particular the rank of I^i/I^{i-1} is a definitely periodic sequence.

Modular Idealizer CHAin

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- For $n \geq 3$ it is possible to see that $x_2^3 \partial_3 \notin I^i$ for all $i \geq 0$. In particular

$$L_n \neq \cup_i I^i$$

Open problems and proposals for future research

- When $m = 2$ through the rigid commutators machinery it's possible to see that the modular ideal chain in L_n and the normalizer chain in Σ_n have the same sequence of indices. What can be said when $m = p$ is an odd prime? Is the sequence of indices of the idealizer and normalizer chain the same in L_n and in the Sylow p-subgroup of $\mathrm{Sym}(p^n)$?

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- Study the case wen the normalizer chain start from a regular subgroup that is not necessarily elementary abelian.

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Thank you!