Sylow subgroups of finite permutation groups

Michael Giudici

Centre for the Mathematics of Symmetry and Computation

THE UNIVERSITY OF WESTERN AUSTRALIA

on joint work with John Bamberg, Alexander Bors, Alice Devillers, Cheryl E. Praeger and Gordon Royle

Ischia Group Theory
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Property $(*)_p$

Let G be a finite group acting on a set Ω, let p be a prime and let $P \in \mathrm{Syl}_p(G)$.

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$Q_3 = Q$ is a Sylow 2-subgroup of $G_3 = Q$ but $Q_1 = 1$ is not a Sylow 2-subgroup of $G_1 = \langle (23) \rangle$.

Thus G has Property $(*)_3$ but not Property $(*)_2$.

Motivation

Given $G \leqslant \mathrm{Sym}(\Omega)$ with $|\Omega| = n$, the Burger-Mozes group $U(G)$ is the largest group of automorphisms of the n-regular tree T_n such that, for all vertices v, the stabiliser of v induces the group G on the set of all neighbours of v.

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Tornier (2018): For $P \in \mathrm{Syl}_p(G)$ and a finite subtree T of T_n, the group $U(P)_{(T)}$ is a local Sylow p-subgroup of $U(G)_{(T)}$ if and only if G has Property $(*)_p$.

(A local Sylow p-subgroup of H is a maximal pro-p-subgroup of a compact open subgroup of H.)

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Tornier (2018):

- $G = A_n$ or S_n acting on n points has Property $(*)_p$, with p dividing $|G|$, if and only if $n_p p > n$.
- If either P has the same orbits as G or $|\Omega| = p^n$, then G has Property $(*)_p$.

Intransitive Groups

Let G have orbits $\Omega_1, \Omega_2, \ldots, \Omega_t$ on Ω.

Then G has Property $(*)_p$ on Ω if and only if G^{Ω_i} has Property $(*)_p$ for all $i \in \{1, 2, \ldots, t\}$.

Some observations for transitive groups

Let G be a group acting transitively on a set Ω of size n and let $P \in \mathrm{Syl}_p(G)$.

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1. G has Property $(*)_p$ if and only if all orbits of P on Ω have the same length, namely n_p.

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Corollaries:

1. G has Property $(*)_p$ if and only if all orbits of P on Ω have the same length, namely n_p.
2. If $pn_p > n$ then G has Property $(*)_p$.

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2. If G acts faithfully on Ω with Property $(*)_p$ such that p divides $|G|$, then p divides n.
3. If H is a transitive subgroup of G and G has Property $(*)_p$ then H has Property $(*)_p$.

   [If $P \in \mathrm{Syl}_p(H)$ then all orbits of P have size at least n_p, but all orbits of a a Sylow p-subgroup of G have size n_p]

Imprimitive Groups

Theorem (BBDGPR): Let G act transitively on a set Ω with system of imprimitivity $\mathcal{B}$. Let $B \in \mathcal{B}$, let G_B^B be the permutation group induced on B by the setwise stabiliser G_B and let $G^{\mathcal{B}}$ be the permutation group induced by G on $\mathcal{B}$.

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2. If G has Property $(*)_p$ then G_B^B has Property $(*)_p$.

Note that if G has Property $(*)_p$ then $G^{\mathcal{B}}$ does not necessarily have Property $(*)_p$:

$G = D_{12}$ acting regularly on itself has property $(*)_2$. However, G has a system of imprimitivity $\mathcal{B}$ of size 3 with $G^{\mathcal{B}} \cong S_3$, which does not have Property $(*)_2$.

Primitive Groups

Theorem (BBDGPR) Let G be a primitive permutation group on Ω and suppose that G has Property $(*)_p$ for some prime p dividing $|\Omega|$. Then one of the following holds:

1. G is an almost simple group;
2. G is of Affine type and $|\Omega| = p^k$;
3. $\Omega = \Delta^k$ for some $k \geqslant 2$ and $G \leqslant H \,\mathrm{wr}\, K$ where H is an almost simple group acting primitively on Δ with Property $(*)_p$ and $K \leqslant S_k$. Moreover, either p is coprime to $|K|$, or p divides $|K|$ and $|\Delta|$ is a power of p.

Moreover, any primitive group in cases (2) and (3) has Property $(*)_p$.

Almost Simple Groups

Problem: Determine all the almost simple primitive permutation groups G of degree n that have Property $(*)_p$ for some prime p dividing n and for which $pn_p < n$ and p divides $|G_\omega|$.

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- The action of $\mathrm{PSL}_2(q)$ for q even acting on $\binom{q}{2}$ points has Property $(*)_2$.
- Apart from this infinite family the only examples of degree less than 4095 are:

Degree	G	p
6	A_5	2
12	M_{11}	3
36	$\mathrm{PSU}_3(3)$	3
36	$\mathrm{P\Gamma U}_3(3)$	3
112	$\mathrm{PSU}_4(3)$	2
135	$\mathrm{PSp}_6(2)$	3

2-transitive groups

Theorem (BBDGPR): Let G be a 2-transitive permutation group of degree n on a set Ω with $\omega \in \Omega$, and let p be a prime dividing n. Then G has Property $(*)_p$ if and only if one of the following holds:

a. $pn_p > n$;
b. p does not divide $|G_\omega|$;

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Theorem (BBDGPR): Let G be a 2-transitive permutation group of degree n on a set Ω with $\omega \in \Omega$, and let p be a prime dividing n. Then G has Property $(*)_p$ if and only if one of the following holds:

a. $pn_p > n$;
b. p does not divide $|G_\omega|$;
c. $G = A_5$ with $n = 6$ and $p = 2$;
d. $G = M_{11}$ with $n = 12$ and $p = 3$;
e. $G = \mathrm{P\Gamma L}_2(8)$ with $n = 28$ and $p = 2$;

All 2-transitive groups with Property $(*)_p$ and $n_p = p$ were determined by Praeger in 1974.