1 Groups of intermediate growth

G - finitely generated group

S - finite system of generators

$|g|$ - length of $g \in G$ with respect to S.

$\boxed{\gamma(n) = \#\{g \in G \mid |g| \leq n\} \quad \text{growth function}}$

$\gamma(n)$
- polynomial ($\Leftrightarrow$ G is virtually nilpotent)
- exponential (like $F_k, k \geq 2$, free metabilian groups, ...)
- intermediate (between polynomial and exponential).

- J. Milnor 1968
- S. Adian 1975, book Burnside Problem.
- R. Grigorchuk 1984 - There are many groups of intermediate growth even in the clas of p-groups, p - prime.

Cantor set

is

$$\Omega_k = \{0,1,\dots,k-1\}^{\mathbb{N}}$$ – space of sequences + topology of coordinate convergence

$$\omega = (\omega_i)_{i=1}^{\infty}, \quad \omega_i \in \{0,1,\dots,k-1\}$$

$\forall p$ **Construction of groups** $G_\omega = \langle a, b_\omega, c_\omega \rangle$

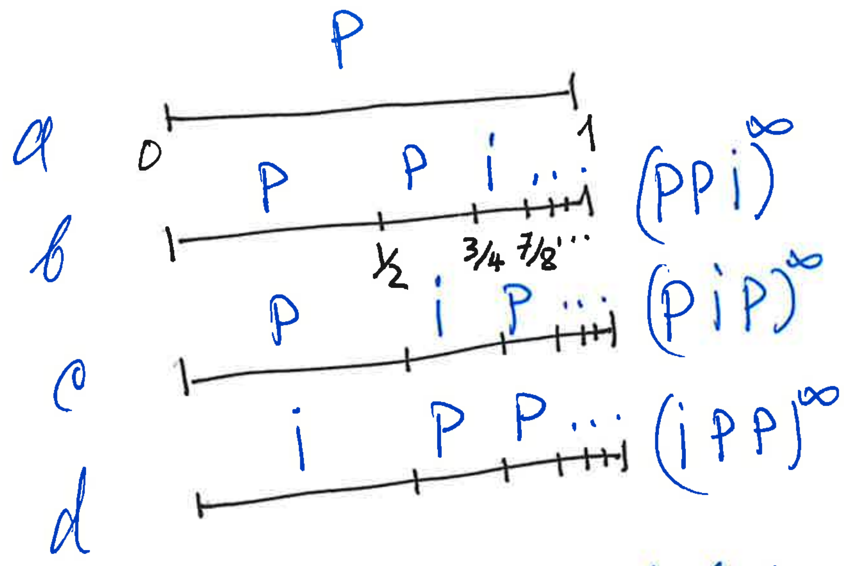
$$\omega \in \Omega_{p+1}$$

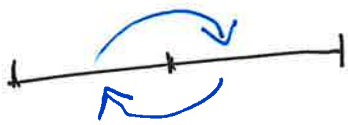
$$p = 2, \quad \Omega = \Omega_3 = \{0,1,2\}^{\mathbb{N}}$$

Two examples:

1) $G_{(012)^\infty} = \langle a, b, c, d \rangle = G_1$

P – permutation





I – identity map

G_1 — intermediate growth, torsion 2-group, branch, just infinite, ...

Let $\beta = 0.7574\ldots$ = log (rational expression of ~~[illegible]~~ root of cubic polynomial).

Then $\forall \varepsilon > 0$

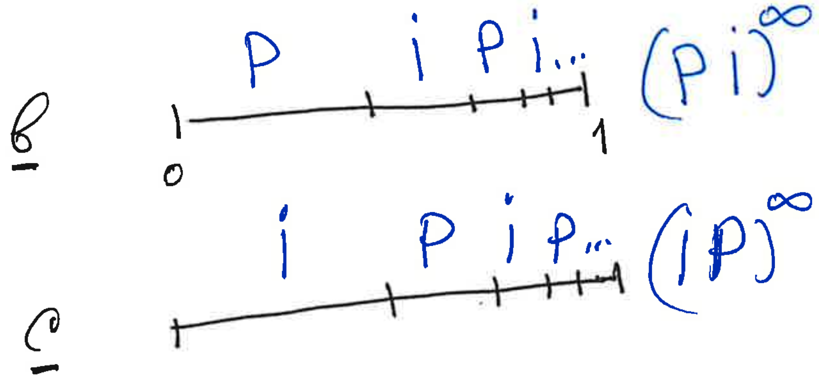
$$e^{n^{\beta-\varepsilon}} < \gamma_1(n) < e^{n^{\beta}}$$

A. Erschler, T. Zheng 2020. } inventiones.

Laurent Bartholdi 1998, R. Muchnik, I. Pak...

Question. Is there a group of intermediate growth with growth smaller than the growth of G_1?

2) $G_{(01)^\infty} = \langle a, b, c \rangle = \boxed{G_2}$ The main group of the talk.



It has many properties of but is not torsion group and has much larger growth

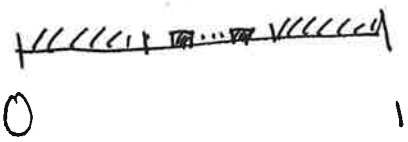
$\forall \varepsilon > 0$

$$e^{\frac{n}{\log^{2+\varepsilon} n}} < \gamma_2(n) < e^{\frac{n}{\log^{1-\varepsilon} n}}$$

Anna Erschler, 2005 Annals of Math.

② Topological Full Groups (TFG's)

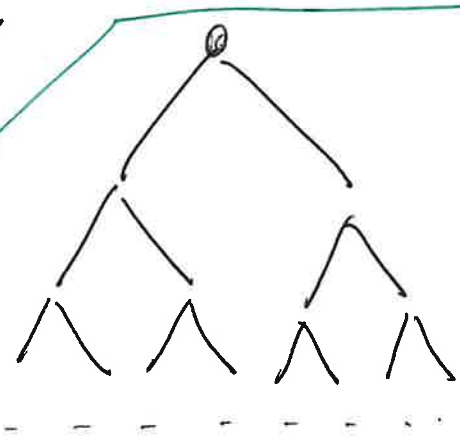
$\boxed{X \text{ — Cantor set}}$



$\Omega_k = \{0,1,\dots\}^{\mathbb{N}}$

$\Lambda_k = \{0,1,\dots\}^{\mathbb{Z}}$

∂T_d



X has many clopens (closed & open subsets)

$\partial T_2 \simeq \{0,1\}^{\mathbb{N}} \simeq$ Cantor set

$Homeo(X)$ - group of homeomorphisms

$\varphi \in Homeo(X)$

φ is minimal if $\forall x \in X$ the orbit $\{\varphi^n(x)\}_{n=-\infty}^{\infty}$ is dense in X.

(φ, X) – minimal Cantor system.

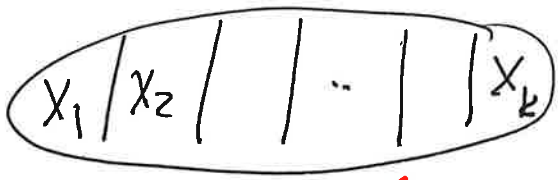
Def. Topological Full Group (TFG) associated with (φ, X) is a subgroup of $Homeo(X)$ consisting of homeomorphisms acting locally by powers of φ.

$[[\varphi]]$, G_φ, $\mathcal{T}_\varphi$ – notations

$\mathcal{T}_\varphi \ni g$, $\quad g(x) = \varphi^{n_g(x)}(x)$, $\quad x \in X$

$n_g : X \longrightarrow \mathbb{Z}$ – continuous function

$X = \bigsqcup_{i=1}^{k} X_i$, $\quad g|_{X_i} = \varphi^{n_{g,i}}$



X

X_i – clopens

introduced by T. Giordano, I. Putnam, Skau, ~ 1999.

$\mathcal{T}_\varphi$ is a complete invariant of minimal Cantor system up to flip conjugasy:

$$(\varphi, X) \underset{\text{or}}{\sim} (\psi, Y)$$
$$\sim (\psi^{-1}, Y)$$

Some properties of $\mathcal{T}_\varphi$:

1) countable, commutator subgroup
2) $\mathcal{T}_\varphi'$ is simple and in most cases finitely generated. (Matui)
3) amenable, not elementary amenable (K. Juschenko, N. Monod ≈ 2015, confirming conjectur of R. Grigorchuk and K. Medynets)
4) Factorization $\mathcal{T}_\varphi = A\,B$ into a product of two locally finite groups A and B.

5) Local embeddability into finite groups (property LEF).

6) $\mathcal{Y}_\varphi$ is never finitely presented. infinite presentations are found.

7) The universal theory of $\mathcal{Y}_\varphi'$ coincides with the universal theory of the class of finite groups and hence is undecidable (M. Sapir)

4), 5), 6) is due to Medynets and Gri.

Recall that factorization in product of locally finite groups was studed in Ukraine (the schools of N. Chernikov and L. Kaloujnin: L. Kurdachenko, I. Subbotin, V. Suschanskii, ...).

General Problem about TFG's: Study properties of TFG's and its subgroups from algebraic, geometric and dynamical points of view.

Theorem (N. Matte-Bon, based on the work of Yaroslav Vorobets "Notes on the Schreier graphs of the Grigorchuk group".).

Groups G_ω of intermediate growth can be embedded into TFG's of minimal Cantor systems.

Hence $\mathcal{Y}_\varphi$ may contain groups of intermediate growth, infinite f.g. torsion groups, branch groups, just-infinite groups etc.

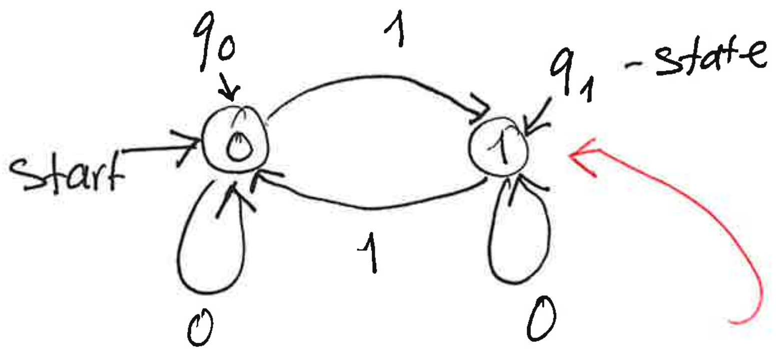
Problem. Study maximal subgroups of TFG's.

A progress has been made recently in a joint work of Yaroslav Vorobets and Gri...

③ Automatically generated sequences and intermediate growth.

Thue - Morse sequence
1906 1921

Prouhet
1851



Can be defined by automaton

or by a substitution

$$\theta : \begin{cases} 0 \to 01 \\ 1 \to 10 \end{cases}$$

$$0 \xrightarrow{\theta} 01 \xrightarrow{\theta} 0110 \xrightarrow{\theta} 01101001 \xrightarrow{\theta} \cdots\cdots$$

$\theta_\infty = \lim_{n\to\infty} \theta^n(0)$ – Thue - Morse sequence

– fixed point of θ.

$\{x, y\}$

Period - doubling

$\sigma : \begin{cases} 0 \to 01 \\ 1 \to 00 \end{cases}$ equivalently $\sigma : \begin{cases} x \to xy \\ y \to xx \end{cases}$

change of alphabet

$\hat{\sigma} : \begin{cases} x \to y \\ y \to x \\ z \to zyz \end{cases}$! "relative of σ"

$\eta : \begin{cases} a \to aca \; ! \\ b \to d \\ c \to b \\ d \to c \end{cases}$ Lysenok's substitution Used by him to get a presentation of $G_{(012)^\infty}$

$$G_1 = G_{(012)^\infty} = \langle a, b, c, d \mid 1 = a^2 = b^2 = c^2 = d^2 = bcd = \eta^k((ad)^4) = \eta^k((adacac)^4),\ k = 0, 1, 2, \dots \rangle$$

- finite L-presentation. A similar presentation

can be find for $G_{(01)^\infty} =: G_2$ using $\hat{\sigma}$.

A sequence $\xi = (\xi_n)_{n=0}^{\infty}$ is automatic

$\Leftrightarrow$ it can be determined by automaton

$\Leftrightarrow$ — " — by a substitution

$\Leftrightarrow$ power series $\sum_{n=0}^{\infty} \xi_n t^n$ is algebraic

over $\mathbb{F}_q(t)$, $q = p^n$, p - prime. $|A| = q$ alphabet

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There is a number of connections of automatic sequences with Group Theory.

Gri., Y. Leonov, V. Nekrashevych, V. Sushchansky.

"Self-similar groups, automatic sequences, and unitriangular representations". Bull. Math. Sci. (2016), N 6, 231-285.

Next: Automatic sequences determine minimal subshifts $\longrightarrow$ minimal Cantor systems

$\longrightarrow$ TFG's

④ Subshifts and the results.

A - finite alphabet

$\Omega = \Omega_A = A^{\mathbb{Z}}$ - space of bi-infinite sequences

$T: \Omega \to \Omega$ - shift map

$$\boxed{(T\omega)_n = \omega_{n+1}}, \qquad \omega = (\omega_n)_{n=-\infty}^{\infty}$$

(T, Ω) - full shift

(T, Λ) - subshift if $\Lambda \subset \Omega$ is a T-invariant closed subset.

We are interested in the case when (T, Λ) is a minimal system ($\Rightarrow \Lambda \sim$ Cantor).

Such examples come from substitutions

$\theta, \sigma, \hat{\sigma}, \eta, \dots,$ primitive substitutions, ...

Notation: $\Lambda_\theta, \Lambda_\sigma, \dots$

$$\boxed{\Lambda_\theta = \{\omega \in \{0,1\}^{\mathbb{Z}} \mid \text{a word } w \text{ appears in } \omega \iff \text{it appears in } \theta_\infty = \lim_{n\to\infty} \theta^n(0)\}}$$

Comparison

Groups	Subshifts
Subgroup	subshift $(T, \Lambda_1) \subset (T, \Lambda_2)$ $\Lambda_1 \subset \Lambda_2$
Factor group vs Extension	Factor Extension- sion $\Lambda_1 \xrightarrow{\pi} \Lambda_2$ factor T- equivariant continuous surjection
Isomorphism	Topological Conjugacy.

topological conjugacy

$\boxed{\text{Proposition}}$ $(T_\sigma, \Lambda_\sigma) \sim (T_{\hat\sigma}, \Lambda_{\hat\sigma})$

$$\sigma: \begin{cases} x \to xy \\ y \to xx \end{cases} \qquad \hat\sigma: \begin{cases} x \to y \\ y \to x \\ z \to zyz \end{cases}$$

period doubling

$\boxed{\text{Proposition}}$ Period doubling system $(T_\sigma, \Lambda_\sigma)$ is a 2:1 factor of Thue-Morse system

$(T_\theta, \Lambda_\theta)$

$\boxed{\text{Proposition}}$ TFG of a factor system canonically embeds into a TFG of a cover system.

$$(T, \Lambda_1) \xrightarrow{\pi} (S, \Lambda_2)$$

$$\Rightarrow \mathcal{Y}_S \xhookrightarrow{\pi_*} \mathcal{Y}_T.$$

condition (*).

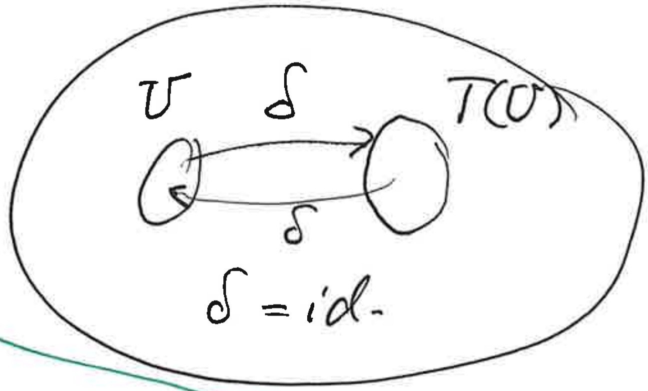
Def. involutions δ_U:

$U \subset \Lambda$ * $U \cap T(U) = \emptyset$, $\delta_U: \begin{cases} \omega \to T\omega \text{ if } \omega \in U \\ \omega \to T^{-1}\omega \text{ if } \omega \in T(U) \\ \omega \to \omega \text{ otherwise} \end{cases}$

clopen (closed & open)

$\boxed{\delta_U^2 = id.}$

Λ



$u \in A$

$$[\,.u] = \{ w \in \Lambda \mid \omega_{1} = u \}$$

Cylinder set.

For $\hat{\sigma}$ $A = \{x, y, z\}$ we have three set $[\,.x]$, $[\,.y]$, $[\,.z]$. ~~a~~nd they satisfy (*).

$\boxed{\text{Theorem}}$ (Y. Vorobets, Gri) The group $G_{(01)^\infty}$ of intermediate growth canonically embeds into TFG $\mathcal{Y}_\theta$ associated with Thue-Morse system. More precisely

$$G_{(01)^\infty} \overset{\varphi}{\hookrightarrow} \mathcal{Y}_{\hat{\sigma}} \simeq \mathcal{Y}_{PD} \overset{\psi}{\hookrightarrow} \mathcal{Y}_\theta$$

$$\boxed{\varphi : \begin{cases} a \to \delta_{[\,.z]} \\ b \to \delta_{[\,.y]} \\ c \to \delta_{[\,.x]} \end{cases}}$$

$\psi = p_*$

$$\begin{tikzcd} \Lambda_\theta \arrow[d, "2:1"] \arrow[dr, "p"] & \\ \Lambda_\sigma & \Lambda_{\hat{\sigma}} \end{tikzcd}$$

$\Lambda_\sigma \xrightarrow{\sim} \Lambda_{\hat{\sigma}}$

conjugacy.

5 Schreier Dynamical Systems

The proof is based on the use of Schreir Dynamics.

T_2 binary rooted tree

$$G = G_\omega \leq \mathrm{Aut}(T_2)$$

$(G, \partial T_2)$ – topological system

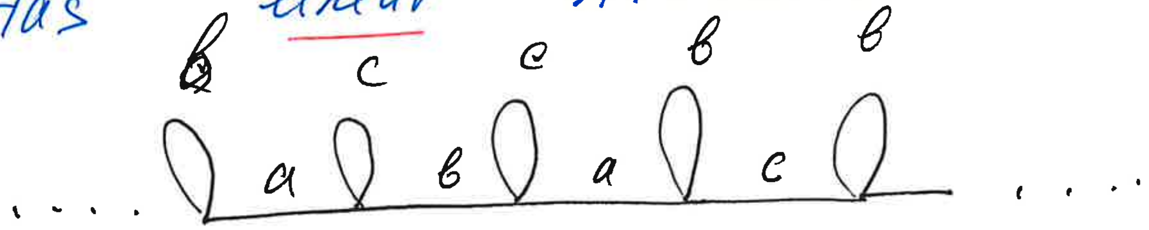
$\partial T_2 = \{0,1\}^{\mathbb{N}} \approx$ Cantor set

Boundary

$\xi \in \partial T_2$

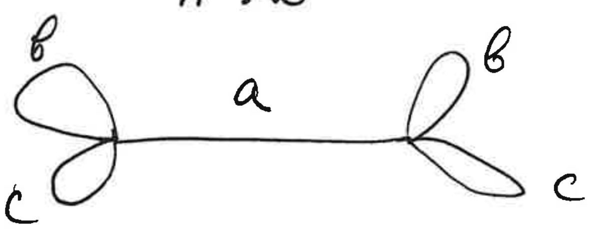
(ξ, Γ_ξ) – graph of action on the orbit $G \cdot \xi$.

Has "linear" structure.



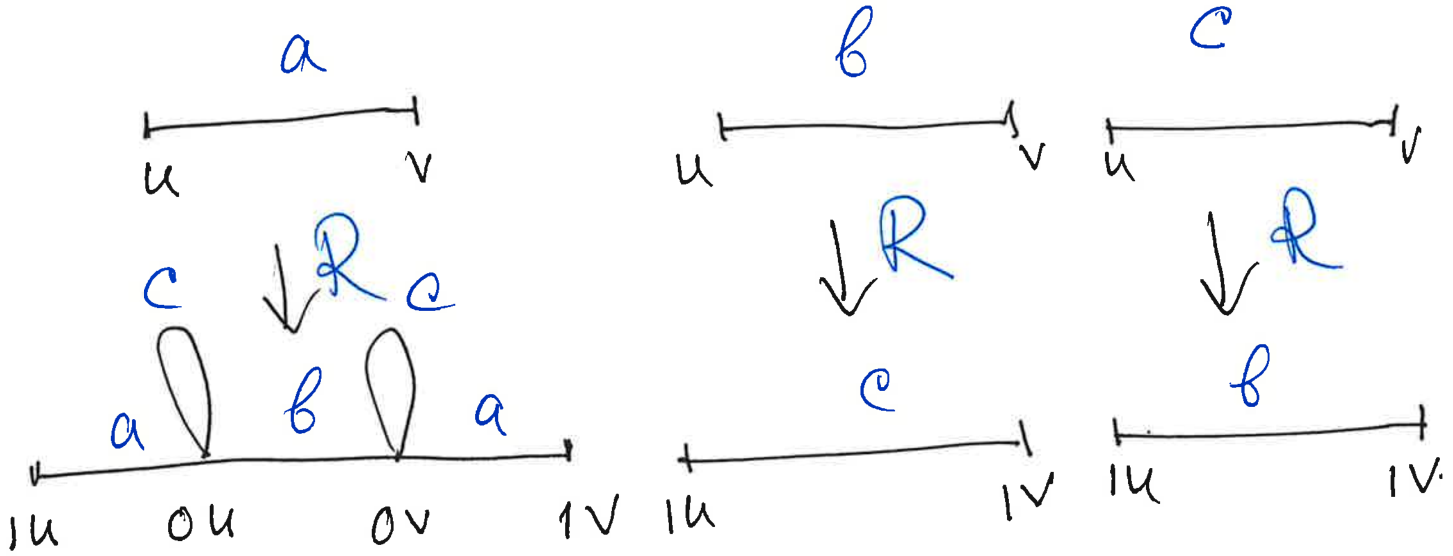
$$(\xi, \Gamma_\xi) = \lim_{n\to\infty} (\xi_n, \Gamma_n) \quad \text{where}$$

Γ_1



and $\Gamma_{n+1} = R(\Gamma_n)$

R is a graph substitution



R is analogouse of $\hat{G}$: $\begin{cases} x \to y \\ y \to x \\ z \to zyz \end{cases}$

Related thing:

- Almost 1:1 extensions
- Uniformly recurrent Schreier graphs
- Uniformly recurrent subgroups
- Regular vs singular points
- Group of germs. (G, X).
- Stability map $X \ni x \longrightarrow G_x = \mathrm{Stab}_G(x) \in \mathrm{Sub}(X)$
- Totally non-free actions, Vorobets condition, ...