TALK IN THE ISCHIA GROUP THEORY CONFERENCE 2022

On conjugacy classes in groups

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ISRAEL

This talk is about conjugacy classes in groups, finite and infinite. All the results in this talk, which are not attributed to other authors, were obtained jointly by **Patrizia Longobardi**, **Mercede Maj** and myself.

We start with the basic definitions. Let G be a group. Then $G^{\#} = G \setminus \{1\}$ and if x is an element of G, then either $\langle x \rangle = C_G(x)$ or $\langle x \rangle < C_G(x)$. Accordingly, we introduce the following definitions.

Definitions

Let G be a group. Then the element $x \in G^{\#}$ and the conjugacy class x^G of x will be called **deficient** if $\langle x \rangle < C_G(x)$, and they will be called **non-deficient** if $\langle x \rangle = C_G(x)$.

Moreover,

Definitions

let j denote a non-negative integer. We shall say that the group G has **defect** j, denoted by $G \in D(j)$ or by "G is a $D(j)$-group", if **exactly** j non-trivial conjugacy classes of G are deficient.

This paper consists of three parts: Part A, dealing with finite groups , Part B dealing with certain groups with all elements of prime power order and Part C dealing with groups which are either finite or infinite.

Part A

The aim of Part A is to classify all finite groups which belong to $D(0)$ (i.e. finite groups with no deficient classes) and to $D(1)$ (i.e. finite groups with exactly one deficient class).

In Part A we proved the following two theorems. In their statement, and elsewhere, the phrase "$G = A \rtimes B$ is a Frobenius group" means that G is a finite Frobenius group with the kernel A and a complement B. It is well known that $|A| \equiv 1 \pmod{|B|}$.

Moreover, C_n denotes the cyclic group of order n, D_n denotes the dihedral group of order n, Q_8 denotes the quaternion group and if p is a prime and s is a positive integer, then E_{p^s} denotes the elementary abelian group of order p^s.

Our first result is the following theorem.

Theorem 1

Let $G \neq \{1\}$ be a finite group. Then $G \in D(0)$ if and only if either $G = C_p$ for some prime p or $G = C_p \rtimes C_q$ is a Frobenius group, with p and q distinct primes.

For the proof of Theorem 1, we used the following two preliminary results. The first one is the following proposition.

Proposition 3

The finite group G satisfies $G \in D(0)$ if and only if the following two statements hold:

1. All elements of $G^{\#}$ are of prime order, and
2. If p is a prime dividing $|G|$, then a Sylow p-subgroup of G is of prime order.

Indeed, if $C_G(x) = \langle x \rangle$ for all $x \in G^{\#}$, then clearly all elements of $G^{\#}$ are of prime order and for each prime p dividing $|G|$, the Sylow p-subgroups of G are of prime order. The converse is also easy.

The second preliminary result is the following theorem of Cheng Kai Nah, M. Deaconescu, Lang Mong Lung and Shi Wujie (see [1]).

Theorem 4

Let $G \neq \{1\}$ be a finite group and suppose that all elements of $G^{\#}$ are of prime order. Moreover, let p and q denote distinct primes. Then either G is a p-group of exponent p, or $G = P \rtimes C_q$ is a Frobenius group, where P is the Sylow p-subgroup of G of exponent p, or $G = A_5$.

Our second result in Part A is the following theorem.

Theorem 2

Let G be a finite group. Then $G \in D(1)$ if and only if one of the following conditions holds:

1. G is a solvable group and either $G \in \{C_4, Q_8, D_{18}\}$ or G is a Frobenius group in $\{E_{2^s} \rtimes C_q,\ C_r \rtimes C_4\}$, where $q = 2^s - 1$ is a Mersenne prime and r is a prime satisfying $r \equiv 1 \pmod 4$.
2. G is a non-solvable group and either $G = A_5$ or $G = PSL(2,7)$.

It is easy to see that if G is a finite group in $D(1)$ (i.e. G has exactly one deficient class), then each element of $G^{\#}$ is either of prime order or of prime-squared order.

Since G is not in $D(0)$, one of its Sylow subgroups, say the Sylow p-subgroup P, must be of order $\geq p^2$ and of exponent either p or p^2. Each element of order p in G is deficient, so all elements of G of order p must be conjugate in G. The other Sylow subgroups of G are of prime order.

It is easy to see that if $G = P$, then $p = 2$ and G is either C_4 or Q_8. If G is solvable and $G > P$, then the theorem of Graham Higman in [2] can be used. He showed, in particular, that a finite solvable group with all elements of prime power order is divisible by at most two primes. A detailed investigation of various cases yields the solvable part of Theorem 2.

Concerning the non-solvable case, it is easy to check that A_5 and $PSL(2,7)$ are $D(1)$-groups. Conversely, suppose that G is a non-solvable $D(1)$-group. Then by Feit-Thompson G is of even order. If e is an involution in G, then $C_G(e)$ is a 2-group, since all elements in $G^{\#}$ are of prime power order. Such groups are called CIT-groups, and it follows that G is a non-solvable CIT-group. These groups were investigated by Michio Suzuki in his papers [3], [4] and [5]. Using these results, we proved the non-solvable case of Theorem 2.

Part B

Before moving to Part C of this talk, which deals with $D(j)$-groups that are either finite or infinite. we shall present some results which we obtained in our paper [6]. These results are of independent interest and they will be very helpful in Part C.

In Part B, a group G is either finite or infinite. If G is a periodic group, then $\pi(G)$ will denote the set of all primes dividing the order of some element of G.

In the paper [7] of A.L.Delgado and Y.Wu, groups with each element of prime power order were called CP-groups. Such groups are of course periodic. We shall deal with CP-groups, which satisfy some boundedness condition, as defined below.

Definitions

A group G will be called a BCP-group if each element of G is of prime power order and for each $p \in \pi(G)$ there exists a positive integer u_p such that each p-element of G is of order $p^i \leq p^{u_p}$.

Definitions

A group G will be called a BSP-group if each element of G is of prime power order and for each $p \in \pi(G)$ there exists a positive integer v_p such that each finite p-subgroup of G is of order $p^j \leq p^{v_p}$.

The BCP-groups and the BSP-groups are clearly periodic. Notice that each BSP-group is a BCP-group and each BCP-group is a CP-group. Moreover, the BCP-property and the CP-property are inherited by subgroups and quotient groups, and hence by sections. The BSP- property is inherited by subgroups.

First we present our results concerning BCP-groups.

It is well known that finitely generated groups have only a finite number of subgroups of a *given* finite index. In particular, each such group has only a finite number of normal subgroups of a *given* finite index. We proved, using the Zelmanov positive solution of the Restricted Burnside Problem (see [8] and [9]) that finitely generated BCP-groups have only a finite number of normal subgroups of an *arbitrary* finite index.

Theorem 5

Let G be a finitely generated BCP-group. Then G has only a finite number of normal subgroups of finite index.

This basic result is used in the proofs of the next theorems dealing with BCP-groups.

Recall that a group G is residually finite if for each non-trivial element $g \in G$ there exists a normal subgroup $M(g)$ of G such that $g \notin M(g)$ and $G/M(g)$ is finite.

As a corollary of Theorem 5 we obtain the following result.

Theorem 6

Let G be a finitely generated residually finite BCP-group. Then G is a finite group.

It is well known that the residual finite property is inherited by subgroups. Hence Theorem 6 implies the following result.

Theorem 7

Let G be a residually finite BCP-group. Then G is a locally finite group.

Recall that a group G is locally graded if each non-trivial finitely generated subgroup of G has a proper normal subgroup of finite index. Applying the above results, we proved the following theorem concerning locally graded BCP-groups.

Theorem 8

Let G be a locally graded BCP-group. Then G is a locally finite group.

As a corollary we get the following theorem.

Theorem 9

Let G be a finitely generated locally graded BCP-group. Then G is a finite group.

Now we move to the properties of *BSP*-groups. Recall that a group G is a *BSP*-group if each element of G is of prime power order and for each $p \in \pi(G)$ there exists a positive integer v_p such that each *finite p-subgroup* of G is of order $p^j \leq p^{v_p}$. This property is stronger than the *BCP*-property, which requires only that each *p-element* of G is of order $p^i \leq p^{u_p}$. Consequently, our results concerning the *BSP*-groups are stronger than those obtained for the *BCP*-groups.

The basic result concerning the BSP-groups is the following theorem.

Theorem 10

If G is a locally finite BSP-group, then G is a finite group.

This theorem does not hold for BCP-group, since if p is a prime, then an infinite abelian p-group of finite exponent is a locally finite BCP-group.

Theorem 10 yields the following strengthening of Theorem 8 for *BSP*-groups.

Theorem 11

Let G be a locally graded *BSP*-group. Then G is a finite group.

Proof: By Theorem 8 applied to *BSP*-groups, G is a locally finite *BSP*-group. Hence, by Theorem 10, G is a finite group, as required $\blacksquare$.

Finally, we proved also the following theorem.

Theorem 12

Let G be a BSP-group and suppose that $2 \in \pi(G)$. Then G is a finite group.

Here we proved that the centralizer of each involution of G is finite, which implies by a theorem of V.P. Shunkow in [10] that G is locally finite. But then G is finite by Theorem 10.

This is the end of Part B. We move now to Part C, which is the last part of this talk.

Part C

This part deals with finite or infinite $D(0)$-groups and $D(1)$-groups. Groups which are either finite or infinite will be called virtually infinite.

There exist infinite $D(j)$-groups. In particular, the Tarski infinite p-groups, whose proper non-trivial subgroups are all of order p, are infinite periodic non-locally-finite $D(0)$-groups. Our aim in Part C is to find properties of virtually infinite $D(0)$-groups and $D(1)$-groups, which force these groups to be either finite or locally finite.

First we shall deal with virtually infinite $D(0)$-groups.

Infinite $D(0)$-groups were first studied by Delizia, Jezernik, Moravec and Nicotera in their paper [11] (see also [12] and [13]). In [11] they noticed that the following theorems hold.

Theorem 13

Let G be a locally finite $D(0)$-group. Then G is a finite group.

Theorem 14

Let G be a locally graded $D(0)$-group. Then G is a finite group.

We also proved the following theorem.

Theorem 15

Let G be a $D(0)$-group and suppose that $2 \in \pi(G)$. Then G is a finite group.

We shall show now how these three theorems follow from our results in Part B.

Let G be a virtually infinite $D(0)$-group. Since $C_G(g) = \langle g \rangle$ for each $g \in G^{\#}$, it follows that G is a periodic group, since if $x \in G$ is of infinite order, then

$$\langle x^2 \rangle < \langle x \rangle \leq C_G(x^2)$$

in contradiction to the definition of a $D(0)$-group.

Hence G is periodic, and it follows, like in the finite case, that each element of $G^{\#}$ is of prime order. Moreover, if $p \in \pi(G)$, then each p-subgroup of G is of order p. Hence G is a BSP-group and it follows by Theorems 10 , 11 and 12, dealing with BSP-groups, that Theorems 13, 14 and 15, dealing with $D(0)$-groups, hold.

Finally, we shall deal with virtually infinite $D(1)$-groups. If a $D(1)$-group is periodic, than it can be shown, that all elements of G are of prime power order and the power is at most 2. Hence a periodic $D(1)$-group is a BCP-group. Moreover, it can be shown that a residually finite $D(1)$-group is periodic, so it is also a BCP-group. Hence Part B yields the following results.

Theorem 7 yields

Proposition 16

If $G \in D(1)$ is a residually finite group, then G is locally finite.

and Theorem 8 yields

Proposition 17

If $G \in D(1)$ is a periodic locally graded group, then G is locally finite.

But $D(1)$-groups have properties, which are not necessarily possessed by BCP-groups. By applying these properties we proved the following three theorems.

Theorem 18

If $G \in D(1)$ is a locally finite group, then G is finite.

Theorem 19

If $G \in D(1)$ is a residually finite group, then G is finite.

Theorem 20

If $G \in D(1)$ is a periodic locally graded group, then G is finite.

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The lecture is now complete.

THANK YOU for your ATTENTION!