Generalizing the Chermak-Delgado measure

Preliminary report

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Joint Work

This is joint work with:

- Elizabeth Wilcox (SUNY-Oswego)

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- Elizabeth Wilcox (SUNY-Oswego)

Later joined by

- William Cocke (Augusta University)
- Arturo Magidin (University of Louisiana at Lafayette)

The CD Measure and Lattice

Definition

Let G be a finite group. The Chermak-Delgado measure on G is the function that associates to every subgroup H the number $m_G(H) = |H|\,|C_G(H)|$.

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Let G be a finite group. The Chermak-Delgado lattice of G, $CD(G)$, is the lattice of subgroups H with

$$m_G(H) = \max\{m_G(K) \mid K \leq G\}.$$

Some properties of the CD Lattice

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- *If* $H \in CD(G)$, *then* $C_G(H) \in CD(G)$, *and* $H = C_G(C_G(H))$.

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Corollary

Let G be a finite group. Then G has a characteristic abelian subgroup M such that $[G : M] \leq [G : A]^2$ *for all abelian subgroups A of G.*

Goal

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The proof that the collection of subgroups of maximal measure forms a sublattice uses some properties of the centralizer. The key ones can be distilled to two.

The two properties

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Let G be a finite group, and $\mathrm{Sub}(G)$ the lattice of subgroups of G. Let $M\colon \mathrm{Sub}(G) \to \mathrm{Sub}(G)$ be a function such that for all $H, K \in \mathrm{Sub}(G)$:

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- *$M(H) \cap M(K) \leq M(\langle H, K\rangle)$.*

If we define the measure $m(H) = |H|\,|M(H)|$, then the set of subgroups with maximal measure form a sublattice of $\mathrm{Sub}(G)$.

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This is a well-known property of the centralizer.

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If H and K have maximum measure, then $m(H \cap K)m(\langle H, K \rangle) \leq m(H)m(K)$, giving equality.

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If H and K have maximum measure, then $m(H \cap K)m(\langle H, K\rangle) \leq m(H)m(K)$, giving equality.

Therefore, if H and K have maximum measure, then so do $H \cap K$ and $\langle H, K\rangle$. $\square$

Consequences of the equality

Moreover, we have equality in all steps. So

$$\begin{aligned} m(H\cap K) &= |H\cap K||M(H\cap K)| \\ &\geq |H\cap K||\langle M(H), M(K)\rangle| \\ &\geq |H\cap K|\,|M(H)M(K)| \\ &= \left(\frac{|H|\,|K|}{|HK|}\right)\left(\frac{|M(H)|\,|M(K)|}{|M(H)\cap M(K)|}\right) \\ &\geq \left(\frac{|H|\,|K|}{|\langle H, K\rangle|}\right)\left(\frac{|M(H)|\,|M(K)|}{|M(\langle H, K\rangle)|}\right) \\ &= \frac{m(H)m(K)}{m(\langle H, K\rangle)}. \end{aligned}$$

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So $HK = \langle H, K\rangle$.

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Marginal subgroups

Given $H \leq G$, the centralizer of H in G is the set of all $c \in G$ such that

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Definition

Let $w(x_1, \ldots, x_n)$ be a group word. The left ith marginal subgroup of G is the collection of all $x \in G$ such that

$$w(g_1, \ldots, g_n) = w(g_1, \ldots, g_{i-1}, xg_i, g_{i+1}, \ldots, g_n)$$

for all $g_1, \ldots, g_n \in G$.

Relative marginals

Definition

Let $w(x, y)$ be a 2-variable word. We can use $w(x, y)$ to define a family of functions M from $\mathrm{Sub}(G)$ to $\mathrm{Sub}(G)$, what we are calling relative marginals by:

- ${}^*w_1(H) = \{x \in G \mid w(g, h) = w(xg, h), \text{ with } g \in G, h \in H\}$.

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(These were the examples that inspired this investigation)

But some do not...

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Using marginal $^*w_1$, we have the nine types of subgroups of G, their relative margin and measure are:

H	$^*w_1(H)$	$m(H)$	H	$^*w_1(H)$	$m(H)$
1	G	60	K_4	G	240
C_2	G	120	S_3	G	360
C_3	G	180	D_{10}	C_5	50
C_5	C_5	25	A_4	G	720
G	1	60			

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So the subgroups of maximum measure are the copies of A_4.
They do not form a lattice.

Another family that always works

Proposition (Cocke)

Let $w(x, y) = [x^i, y]$, $i \geq 1$. Then $M(H) = {}^*w_1(H)$ and $M(H) = w_1^*(H)$ both define CD-like lattices.

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Proof. Since $[a, bc] = [a, c][a, b]^c$, if $x \in {}^*w_1(H) \cap {}^*w_2(K)$, then

$$w(xg, hk) = [(xg)^i, hk] = [(xg)^i, k][(xg)^i, h]^k$$

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Proof. Since $[a, bc] = [a, c][a, b]^c$, if $x \in {}^*w_1(H) \cap {}^*w_2(K)$, then

$$\begin{aligned} w(xg, hk) &= [(xg)^i, hk] = [(xg)^i, k][(xg)^i, h]^k \\ &= (w(xg, k))^h w(xg, h) \\ &= w(g, k)w(g, h)^k \end{aligned}$$

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Inductively, $w(xg, y) = w(g, y)$ for every $y \in \langle H, K \rangle$, proving the inclusion. □

Some work for certain classes of groups

Proposition

Let $w(x, y) = [x, y, y]$. Then $M(H) = {}^*w_2(H)$ and $M(H) = w_2^*(H)$ define CD-like lattices whenever G is nilpotent of class at most 3.

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Proof. Because G is nilpotent of class at most 3, any weight 3 commutator is linear in each component. Therefore,

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and it follows that if $x \in {}^*w_2(H) \cap {}^*w_2(K)$, then

$$[hk,xg,xg] = [h,xg,xg][k,xg,xg] = [h,g,g][k,g,g] = [hk,g,g].$$

Etc.

Once it works

For measures defined by the marginals, since the measure is invariant under automorphisms of G, we obtain:

Theorem

Let G be a finite group, and let M be a relative marginal associated to w which satisfies

$$M(H) \cap M(K) \leq M(\langle H, K \rangle).$$

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Let $m(H) = |H|\,|M(H)|$ be the measure associated to M. Then the collection $CD_M(G)$ of subgroups of G of maximal measure is a sublattice of $\mathrm{Sub}(G)$, and the smallest element is a characteristic subgroup of G that contains the corresponding marginal subgroup of G.

Questions we are exploring

- For which words do the relative margins work for every group?

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- If H is in the CD lattice, then so is $C_G(H)$. Is there a related result using marginals?

Thank you

Thank you for your attention.