Finite and profinite groups with automorphisms whose fixed points satisfy Engel-type conditions

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Engel groups

Notation: left-normed simple commutators

$$[a_1, a_2, a_3, \ldots, a_r] = [...[[a_1, a_2], a_3], \ldots, a_r].$$

Recall: a group G is an Engel group if for every $x, g \in G$,

$$[x,\, _n g] := [x, \underbrace{g, g, \ldots, g}_{n}] = 1,$$

where g is repeated sufficiently many times depending on x and g.

Clearly, any locally nilpotent group is an Engel group.

The converse is not true in general,

but is a coveted result for particular classes of groups.

For example, finite Engel groups are nilpotent (Zorn's theorem).

Engel profinite groups

J. Wilson and E. Zelmanov, 1992

Any Engel profinite group is locally nilpotent.

Proof relies on

Zelmanov's Theorem

If a Lie algebra L satisfies a nontrivial identity and is generated by d elements such that each commutator in these generators is ad-nilpotent, then L is nilpotent.

Yu. Medvedev, 2003

Any Engel compact (Hausdorff) group is locally nilpotent.

Generalizations using Engel sinks

Definition

A left Engel sink of an element $g \in G$ is a set $\mathscr{E}(g)$ such that for every $x \in G$,

$$[x, \underbrace{g, g, \ldots, g}_{n}] \in \mathscr{E}(g) \qquad \text{for all } n \geqslant n(x, g).$$

g is a (left) Engel element when $\mathscr{E}(g) = \{1\}$

Engel group when $\mathscr{E}(g) = \{1\}$ for all $g \in G$.

Note ambiguity of notation, as Engel sink may not be unique and a minimal Engel sink may not exist (unless there is a finite one).

Right Engel sinks

Definition

A right Engel sink of an element $g \in G$ is a set $\mathscr{R}(g)$ such that for every $x \in G$,

$$[g, \underbrace{x, x, \ldots, x}_{n}] \in \mathscr{R}(g) \qquad \text{for all } n \geqslant n(x, g).$$

g is a right Engel element when $\mathscr{R}(g) = \{1\}$

Engel group when $\mathscr{R}(g) = \{1\}$ for all $g \in G$.

Again, a right Engel sink may not be unique and a minimal right Engel sink may not exist (unless there is a finite one).

Compact groups with finite or countable Engel sinks

Reported at Ischia-2018 and Ischia-2020-21:

EK & P. Shumyatsky, 2018, 2020

Suppose that G is a compact (Hausdorff) group in which every element has a finite (or countable) left Engel sink. Then G has a finite normal subgroup N such that G/N is locally nilpotent.

EK & P. Shumyatsky, 2021

Suppose that G is a compact (Hausdorff) group in which every element has a finite (or countable) right Engel sink. Then G has a finite normal subgroup N such that G/N is locally nilpotent.

• In both results, G also has a locally nilpotent subgroup of finite index: $C_G(N)$.

• The inverse of a right Engel element is left Engel, but there is no straightforward connection between left and right Engel sinks.

Fixed-point-free automorphisms

Recall: φ is a fixed-point-free automorphism of a group G

if the fixed-point subgroup is trivial: $C_G(\varphi) = 1$.

Thompson, 1959; Higman, 1957

If a (pro)finite group G admits a (coprime) fixed-point-free automorphism φ of prime order p, then G is nilpotent of class $h(p)$.

Higman's function $h(p)$ was explicitly estimated above by Kreknin and Kostrikin.

Almost fixed-point-free automorphisms

In most general terms, studying groups G with an 'almost fixed-point-free automorphism' φ means obtaining results on the structure of G depending on $C_G(\varphi)$.

If $C_G(\varphi)$ is 'small', it is expected that the properties of G are 'close' to the fixed-point-free case.

Example (Fong–Hartley–Meixner–EK)

If a finite group G admits an automorphism φ of prime order p such that $C_G(\varphi)$ is finite of order m, then G has a nilpotent subgroup of class $\leqslant f(p)$ and of finite index $\leqslant g(p, m)$.

Or, if $C_G(\varphi)$ is 'nice' in some sense, it is expected that the properties of G are 'nicely' aligned with the fixed-point-free case.

Right Engel fixed points

C. Acciarri, EK, P. Shumyatsky, 2019

Suppose that a profinite group G admits a coprime automorphism φ of prime order such that every element of $C_G(\varphi)$ is a right Engel element of G. Then G is locally nilpotent.

Immediately G is pronilpotent. Assuming G is finitely generated, need uniform bound for every Sylow p-subgroup.

First a bound depending on p is obtained for every p.

Then a bound independent of p is obtained for all large enough p.

Lie ring methods are used including Zelmanov's theorem, criteria for a pro-p group to be p-adic analytic in terms of the associated Lie algebra (Lazard) and in terms of the rank (Lubotzky–Mann), and Bahturin–Zaicev theorem on Lie algebras with automorphisms with PI fixed-point subalgebra.

Questions: • for compact groups?
• for automorphisms of composite order?

Fixed points with finite right Engel sinks

Theorem 1 (EK, P. Shumyatsky, 2022)

Let G be a profinite group admitting a coprime automorphism φ of prime order. If every fixed point of φ has a finite right Engel sink, then G has an open locally nilpotent subgroup.

Unlike the case where all elements of G have finite right Engel sinks, here it is not possible to guarantee a finite normal subgroup with locally nilpotent quotient.

An important tool in the proof is a strengthened version of Neumann's theorem on BFC-groups by Acciarri and Shumyatsky.

Fixed points with finite left Engel sinks

Theorem 2 (EK, P. Shumyatsky, 2022)

Let G be a profinite group admitting a coprime automorphism φ of prime order p. If every fixed point of φ has a finite left Engel sink, then G has an open subgroup that is an extension of a pronilpotent group by a nilpotent group of class $h(p)$, where $h(p)$ is Higman's function depending only on p.

Weaker conclusion than for finite right Engel sinks of fixed points

(in contrast to the cases on p. 7 where all elements of G have finite/countable left/right Engel sinks).

Quantitative analogue for finite groups: left sinks

Theorem 3 (EK, P. Shumyatsky, 2022)

Let G be a finite group admitting an automorphism φ of prime order coprime to $|G|$. Let m be a positive integer such that every fixed point $g \in C_G(\varphi)$ has a left Engel sink $\mathscr{E}(g)$ of cardinality at most m. Then G has a metanilpotent normal subgroup of index bounded in terms of m only.

Reduction to soluble groups uses a recent useful theorem of Guralnick and Tracey.

Quantitative analogue for finite groups: right sinks

The conclusion for right Engel sinks is stronger.

Theorem 4 (EK, P. Shumyatsky, 2022)

Let G be a finite group admitting an automorphism φ of prime order coprime to $|G|$. Let m be a positive integer such that every fixed point $g \in C_G(\varphi)$ has a right Engel sink $\mathscr{R}(g)$ of cardinality at most m. Then G has a nilpotent normal subgroup of index bounded in terms of m only.

An extension to non-cyclic groups of automorphisms

Theorem 5 (EK, P. Shumyatsky, 2022)

Let A be a soluble group of coprime automorphisms of a finite group G, and let m be a positive integer such that each element of $C_G(A)$ has a left (or each ... right) Engel sink of size $\leqslant m$. Then $|G : F_{2\alpha(|A|)+1}(G)|$ is $(|A|, m)$-bounded.

The soluble radical of G has m-bounded index by a proposition following from Guralnick–Tracey theorem;
thus we can assume that G is soluble.

Then $|\gamma_\infty(C_G(A))|$ is m-bounded by quantitative versions of p. 7.

Hence the Fitting height of $C_G(A)$ is m-bounded. Then the Fitting height of G is $(|A|, m)$-bounded by Thompson's theorem.

Next, $C_{F_{i+1}/F_i}(A)$ is small for every i, as in Theorems 3 and 4.

As a result, $|C_{G/F(G)}(A)|$ is $(|A|, m)$-bounded, and the Hartley–Isaacs theorem completes the proof.

Non-cyclic group of automorphisms of order q^2

Well-known fact: if a finite group G admits a non-cyclic abelian group of automorphisms A of coprime order, then

$G = \langle C_G(a) \mid a \in A^{\#} \rangle$, where $A^{\#} = A \setminus \{1\}$.

Therefore many properties of G can be derived from the properties of the centralizers $C_G(a)$.

C. Acciarri, P. Shumyasky, D. da Silveira, 2018

Suppose that A is an elementary abelian q-group of order q^2 acting by coprime automorphisms on a profinite group G. If every element of $C_G(a)$ is left Engel in G for every $a \in A^{\#}$, then G is locally nilpotent.

Kind of 'automorphism' extension of the Wilson–Zelmanov theorem saying that a profinite Engel group is locally nilpotent.

Some further results by C. Acciarri, P. Shumyasky, D. da Silveira, et al.

Automorphism extension for countable Engel sinks

Theorem 6 (EK, P. Shumyasky, 2022)

Suppose that A is an elementary abelian q-group of order q^2 acting by coprime automorphisms on a profinite group G. If for each $a \in A^{\#}$ every element of $C_G(a)$ has a countable left Engel sink in G, then G has a finite normal subgroup N such that G/N is locally nilpotent.

First the case of pro-p groups is considered, using various Lie ring methods, including Zelmanov's theorem on Lie algebras satisfying a polynomial identity and generated by elements all of whose products are ad-nilpotent, Bahturin–Zaitsev theorem on polynomial identities of Lie algebras with automorphisms.

This provides reduction to uniformly powerful pro-p groups.

More on the proof

For uniformly powerful pro-p groups a different Lie algebra over p-adic integers is used via the Baker–Campbell–Hausdorff formula. It turns out that in a uniformly powerful pro-p group elements with countable Engel sinks are in fact Engel elements.

Then the desired result is derived for the case of pronilpotent groups.

In the general case, an open locally nilpotent subgroup is found, and the proof proceeds by induction on its index.