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Coincidence Reidemeister zeta functions

for nilpotent groups of finite (Prüfer) rank

with A. Fel'shtyn : Pólya - Carlson dichotomy ...

(Inda Math 2022)

- in memory of Bas Edixhoven -

§ 1 Motivation / Origins

- topological fixed pt theory

$\boxed{X \circlearrowleft f}$ cont self-map on connected compact polyhedron (with isolated fixed pts in interior of max simplices)

- Lefschetz-Hopf:

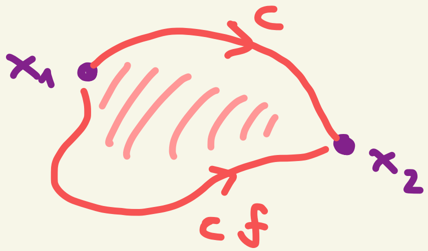
$$0 \neq \boxed{L(f) = \sum_{x_i} \underbrace{I(f, x_i)}_{\in \mathbb{Z} \text{ index ('local degree')}}} \Rightarrow \mathrm{Fix}(f) \neq \emptyset$$

Lefschetz #

$$= \sum_{q} (-1)^q \, \mathrm{Tr}(f_{*q})$$

(think Euler characteristic)

but "$\Leftarrow$" does NOT hold in general



- <u>remedy</u>: Nielsen classes / Nielsen #

$$0 \leq N(f) \leq \# \,\mathrm{Fix}(f)$$

(homotopy invariant)

...

- Covering space approach

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{f}", "\text{lift}"'] \arrow[d, "\eta"'] & \tilde{X} \ni \tilde{x}_0 \arrow[d, "\eta"] \\ X \arrow[r, "f"'] & X \end{tikzcd}$$

$\forall$ deck transf $\alpha : \tilde{X} \to \tilde{X}$

$\exists\, \tilde{\alpha} : \quad \tilde{x}_0(\tilde{f}\tilde{\alpha}) = \tilde{x}_0(\alpha \tilde{f})$

$\rightsquigarrow \quad \varphi \in \mathrm{End}(\pi_1(X)), \ \alpha \mapsto \tilde{\alpha}$

lifting classes

$$\boxed{\mathrm{Fix}(f) = \dot{\bigcup} \left(\mathrm{Fix}(\tilde{f})\right)\eta}$$

$\rightsquigarrow$ Reidemeister classes & GROUPS !

$N(f) \leq R(f) \ldots$

§2 Basic Concepts / Notation / Example

- $\varphi, \psi : G \to G$ group endomorphisms

- $x \sim_{\varphi,\psi} y \;\overset{\text{def}}{\longleftrightarrow}\; \exists g \in G : \; x = (g\varphi)^{-1}\, y\, (g\psi)$

"(φ,ψ)-twisted conjugate"

$[x]_{\varphi,\psi}$ twisted conj class $\equiv$ coincidence Reidemeister class

- $\mathcal{R}(\varphi,\psi) = \{ [x]_{\varphi,\psi} \mid x \in G \}$

$$X \underset{f'}{\overset{f}{\rightrightarrows}} X \quad \dots$$

$R(\varphi,\psi) = \# \mathcal{R}(\varphi,\psi)$

coincidence Reidemeister #

- (φ,ψ) tame $\overset{\text{def}}{\longleftrightarrow}$ $\forall n \in \mathbb{N} : R(\varphi^n, \psi^n) < \infty$

synchronisation pts

$\rightsquigarrow$ <u>coincidence Reidemeister zeta fct</u>

$$\boxed{Z_{\varphi,\psi}(s) = \exp\left(\sum_{n=1}^{\infty} \frac{R(\varphi^n,\psi^n)}{n} s^n\right)} \quad (s \in \mathbb{C})$$

analogy:
Hasse-Weil
Artin-Mazur

special case: $\psi = \mathrm{id}_G$

$\rightsquigarrow$ ordinary Reidemeister zeta fct

[if $\psi \in \mathrm{Aut}(G)$ and $\varphi\psi = \psi\varphi$ $\rightsquigarrow$

$Z_{\varphi,\psi} = Z_{\psi^{-1}\varphi,\mathrm{id}}$ ordinary]

<u>Lemma 0</u>: $G\psi \subseteq Z(G)$ (central!) [exercise!]

$\Rightarrow$ $H_{\varphi,\psi} = \{(g\varphi)^{-1}(g\psi) \mid g \in G\} \leq G$

and $R(\varphi,\psi) = H_{\varphi,\psi} \backslash G$ (cosets), $R(\varphi,\psi) = |G : H_{\varphi,\psi}|$

Example 1 : $G = \mathbb{Z}$ (additive) inf cyclic

$\varphi : x \mapsto d_\varphi x \qquad \psi : x \mapsto d_\psi x$ with $d_\varphi, d_\psi \in \mathbb{Z}$

Lem 0 $\rightsquigarrow$

$$\boxed{R(\varphi^n, \psi^n) = \begin{cases} |d_\psi{}^n - d_\varphi{}^n| & \text{if } d_\varphi{}^n \neq d_\psi{}^n \\ \infty & \text{othw} \end{cases}}$$

hence : (φ, ψ) tame iff $|d_\varphi| \neq |d_\psi|$

and, in this case,

$$\boxed{Z_{\varphi,\psi}(s) = \frac{1 - d_2 s}{1 - d_1 s}}$$

where

$d_1 = \max\{|d_\varphi|, |d_\psi|\}$

$d_2 = d_\varphi d_\psi / d_1$

AIM : generalise (?!) to torsion-free nilp grps of finite rank

§ 3 Some selected basic ingredients

Thm (Roman'kov 2011 + routine generalisation)

G torsion-free class-c nilpotent of finite rank

$\overline{\gamma}_* = \overline{\gamma}_*(G)$ isolated LCS of G

in place of LCS

fully invariant!

G

torsion part ($\overline{\gamma}_i(G)$ — $\gamma_i(G)$

$\varphi_i, \psi_i : \overline{\gamma}_i / \overline{\gamma}_{i+1}$ ↺

torsion-free abelian $\hookrightarrow \mathbb{Q}^{d_i}$

Supp $R(\varphi, \psi) < \infty$

$\Rightarrow \exists$ bij $\mathcal{R}(\varphi, \psi) \longrightarrow \prod_{i=1}^{c} \mathcal{R}(\varphi_i, \psi_i)$

(reduction to abelian case)

hence $R(\varphi, \psi) = \prod R(\varphi_i, \psi_i)$

- <u>profinite completion</u> (widely used up to recently)

$$\iota : G \to \hat{G} = \varprojlim_{N \trianglelefteq_f G} G/N \qquad \text{(functor!)}$$

$\rightsquigarrow$ $\hat{\varphi}, \hat{\psi} : \hat{G} \circlearrowleft$

and

$$\boxed{\begin{aligned} \mathcal{R}(\varphi,\psi) &\longrightarrow \mathcal{R}(\hat{\varphi},\hat{\psi}) \\ [x]_{\varphi,\psi} &\mapsto [x\iota]_{\hat{\varphi},\hat{\psi}} \end{aligned}} \qquad (*)$$

<u>Lem</u> (exercise!):

- each $[x]_{\hat{\varphi},\hat{\psi}} \subseteq \hat{G}$ compact, hence <u>closed</u>
- $\forall x \in G$: $[x\iota]_{\hat{\varphi},\hat{\psi}} = cl([x]_{\varphi,\psi}\,\iota)$ closure
- $R(\varphi,\psi) < \infty \Rightarrow (*)$ surjective
- G abelian & $R(\varphi,\psi) < \infty \Rightarrow (*)$ <u>bijective</u> !

Proposition [stated in literature, but 'incomplete' proof]

G almost abelian & $R(\varphi,\psi) < \infty$

$\Rightarrow$ $\mathcal{R}(\varphi,\psi) \to \mathcal{R}(\hat{\varphi},\hat{\psi})$ <u>bijective</u>

uses ... and : G fg almost-$\mathcal{P}$, where $\mathcal{P}$ inherited by fi subgps

$\to$ G has <u>fully invariant</u> fi subgp with $\mathcal{P}$

<u>sketch of proof</u>: $d = d(G)$ min # gen

wlog $A \trianglelefteq G$ with $\mathcal{P}$, $|G:A| < \infty$

$W = \{ w(x_1,\dots,x_d) \mid w \text{ law in } G/A \}$

d-gen rel free gp in variety def by laws of G/A is finite

$\to$ verbal subgp $B = W(G)$ does it //

§ 4 Adelic formula

- reduction to torsion-free grps

G ab of finite rank with tame (φ, ψ)

$$(G, \varphi, \psi) \rightsquigarrow (G/T \times \underbrace{T}_{=\mathrm{Tor}(G)}, \tilde{\varphi} \times \varphi_0, \tilde{\psi} \times \psi_0)$$

induced

$$\rightsquigarrow (G/T \times \underbrace{T_1}_{\text{finite!}}, \tilde{\varphi} \times \varphi_1, \tilde{\psi} \times \psi_1)$$

st
$$\boxed{R(\varphi^n, \psi^n) = R(\tilde{\varphi}^n, \tilde{\psi}^n) \cdot \underbrace{R(\varphi_1^n, \psi_1^n)}_{\text{well understood}}}$$

$$Z_{\varphi,\psi} = Z_{\tilde{\varphi},\tilde{\psi}} * Z_{\varphi_1,\psi_1}$$

'additive convolution'

Thm (in abelian version $\rightsquigarrow$ nilpotent via ...)

G torsion-free abelian, $d = \mathrm{rk}(G) < \infty$

wlog $\mathbb{Z}^d \leq G \leq \mathbb{Q}^d$ $\leftarrow$ div hull of G

(φ, ψ) tame $\rightsquigarrow$ $(\varphi_{\mathbb{Q}}, \psi_{\mathbb{Q}})$ on div hull

supp: $\varphi_{\mathbb{Q}}, \psi_{\mathbb{Q}}$ simultaneously triangularisable

over $L = \mathbb{Q}(\underbrace{\xi_1, \dots, \xi_d}_{\text{ev of } \varphi_{\mathbb{Q}}}, \underbrace{\eta_1, \dots, \eta_d}_{\text{ev of } \psi_{\mathbb{Q}}})$

$\forall p \in \mathbb{P}$ choose: $L \hookrightarrow \overline{\mathbb{Q}_p}$, ext of $|\cdot|_p$ to L

$L \hookrightarrow \mathbb{C}$, $|\cdot|_\infty$ usual abs value

$\Rightarrow$ ... [pto]

$\Rightarrow \quad \exists \quad I(p) \subseteq \{1, \dots, d\}, \; p \in \mathbb{P} \quad$ st

1) $\forall p \in \mathbb{P}: \quad \prod_{i \in I(p)} (X - \xi_i), \; \prod_{i} (X - \eta_i) \in \mathbb{Z}_p[X]$

in part: $|\xi_i|_p, |\eta_i|_p \leq 1$

2) $\forall n \in \mathbb{N}:$

$$R(\varphi^n, \psi^n) = \prod_{p \in \mathbb{P}} \prod_{i \in I(p)} |\xi_i^n - \eta_i^n|_p^{-1}$$

adelic

$$= \prod_{i=1}^{d} |\xi_i^n - \eta_i^n|_\infty \cdot \prod_{p \in \mathbb{P}} \prod_{i \notin I(p)} \overset{(*)}{\underbrace{|\xi_i^n - \eta_i^n|_p}_{=1 \text{ for almost all } p}}$$

!

idea of proof:

pass to profinite / pro-p completion

example

$$G = \mathbb{Z} \oplus \mathbb{Z}[1/3] \oplus \mathbb{Z}[1/15]$$

$$\hat{G}_p = \begin{cases} \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p \\ \mathbb{Z}_p \oplus 0 \oplus 0 \\ \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus 0 \end{cases} \qquad \mathbb{Z}_p \otimes_{\mathbb{Z}} G = \begin{cases} \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p & p \neq 3,5 \\ \mathbb{Z}_p \oplus \mathbb{Q}_p \oplus \mathbb{Q}_p & p = 3 \\ \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Q}_p & p = 5 \end{cases}$$

$I(p) = \{1, 2, 3\}$ $p \neq 3,5$; $I(3) = \{1\}$; $I(5) = \{1, 2\}$

$$\begin{array}{ccc} \hat{G}_p & \xrightarrow{\hat{\varphi}_p} & \hat{G}_p \\ \sigma \downarrow & & \downarrow \sigma \\ \mathbb{Z}_p \otimes G / \text{max div subm} & \xrightarrow{\overline{\varphi_{\mathbb{Z}_p}}} & \mathbb{Z}_p \otimes G / \text{max div subm} \end{array}$$

$\rightarrow$ $\hat{\varphi}_p$ has ev ξ_i, $i \in I(p)$

& similar for ψ

apply lem 0. //

The adelic formula recovers & generalises

periodic point counts for ergodic finite entropy
endomorphisms of fin dim'l compact
abelian gps (Pontryagin duality)

by R Miles (2008), who used different method
(techniques from commutative algebra)

$\rightsquigarrow$ Pólya–Carlson duality (conjectured for RS fct)
[Bell, Miles, Ward 2014]

<u>Cor</u> : (†) additional technical condition regarding (*)

$\rightsquigarrow$ $Z_{\varphi,\mu}$ <u>either</u> rational

<u>or</u> has natural boundary at radius of
convergence

(†) eg if G has no elmts of infinite p-height for almost all p & $|\xi_i|_\infty \neq |\eta_i|_\infty$ $1 \leq i \leq d$

§ 5 Final example

$G = \mathbb{Z}^2$ $\varphi = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\psi = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

not simultaneously triangularisable

sic!

(φ, ψ) tame and ...

$$\boxed{Z_{\varphi,\psi}(s) = \exp\left(\frac{s}{(1-s)^2} \right)}$$

neither rational

nor nat boundary at rad of conv

$\leadsto$ need to allow for new outcomes !