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NORMAL CLOSURES OF SUBGROUPS AND RELATED TYPES OF SUBGROUPS

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Let G be a group. We consider the following subgroups, which have a significant impact on the structure of the group, but whose properties (especially in infinite groups) have not yet been studied deep enough.

If H, K are subgroups of a group G, then put $H^K = \langle H^x \mid x \in K \rangle$. If $H \leq K$, then H^K is called ***the normal closure of H in the subgroup K.*** Note that H^K is the least normal subgroup of K including H.

Let G be a group and H be a subgroup of G. Starting from the normal closure, we can construct the following canonical series.

Put $\nu_{0\,G}(H) = G$, $\nu_{1\,G}(H) = H^G$, and define $\nu_{\alpha+1\,G}(H) = H^{\nu_{\alpha\,G}(H)}$ for every ordinal α, and $\nu_{\lambda\,G}(H) = \cap_{\beta<\lambda}\nu_{\beta\,G}(H)$ for all limit ordinals λ.

Thus we construct the ***lower normal closure series***

$$G = \nu_{0\,G}(H) \geq \nu_{1\,G}(H) \geq \ \ldots\ \boldsymbol{\nu}_{\alpha\,G}(H) \geq \boldsymbol{\nu}_{\alpha+1\,G}(H) \geq \ldots \boldsymbol{\nu}_{\gamma\,G}(H) = D$$

of a subgroup H in the group G. By the construction, $\boldsymbol{\nu}_{\alpha+1\,G}(H)$ is a normal subgroup of $\boldsymbol{\nu}_{\alpha\,G}(H)$ for all ordinals $\alpha < \gamma$. The last term D of this series has the property $H^D = D$. The last term $\boldsymbol{\nu}_{\gamma\,G}(H)$ of this series is called the ***lower normal closure of a subgroup H in G.***

Immediately note two opposite situations, which come to the following types of subgroups:

$$\boldsymbol{\nu}_{\gamma\,G}(H) = H \ \text{ or } \ H^G = G.$$

In first situation the subgroup H is called ***descendant in G***.
A subgroup H of a group G is called ***contranormal*** in G if $G = H^G$.

The term «a contranormal subgroup» has been introduced by J.S. Rose in his paper

RJ1968. J.S. Rose. Nilpotent subgroups of finite soluble groups. Math. Zeitschrift – 106(1968), 97 – 112.

Thus we can see that every subgroup H is contranormal in its lower normal closure, and the lower normal closure of H in G is a descendant subgroup.

By the definition, contranormal subgroups are antipodes not only to normal subgroups, they are antipodes to subnormal and descendant subgroups: a contranormal subgroup H of a group G is subnormal (respectively descendant) if and only if H = G.

In this connection it is interesting to consider another type of subgroups, which are antagonistic to normal subgroups. They are the selfnormalizing subgroups.

Recall that a subgroup S of a group G is called ***selfnormalizing*** in G if $N_G(S) = S$.

If H is a selfnormalizing subgroup of G and S is a subgroup including H, then not always S is also selfnormalizing. This fact leads us to the following important types of selfnormalizing subgroups.

A subgroup H is called ***weakly abnormal*** in G if every subgroup including H is selfnormalizing in G. In those connections, it should be pointed out the following characterization of weakly abnormal subgroups

The subgroup S is weakly abnormal in a group G if and only if $x \in S^{\langle x \rangle}$ for each element x of a group G.

This result was obtained in the paper

BBZ1988. Ba M.S., Borevich Z.I. On arrangement of intermediate subgroups. Rings and Linear Groups, Kubanskij Univ., Krasnodar, 1988, 14–41.

A slight strengthening leads us to the following important type of subgroups.

A subgroup S of a group G is called ***abnormal*** in G if $g \in \langle S, S^g \rangle$ for each element g of a group G.

Abnormal subgroups have appeared in the paper

HP1937. Hall P. On the system normalizers of a soluble groups. Proc. London Math. Soc. – 43(1937), 507–528.

The term "an abnormal subgroup" belongs to R. Carter

CR1961. Carter R.W. Nilpotent self – normalizing subgroups of soluble groups. Math. Z. 75(1961), 136–139.

In the paper [**BBZ1988**] the following characterization of abnormal subgroups was obtained.

Let G be a group and A be a subgroup of G. Then A is abnormal in G if and only if it satisfies the following conditions:

(i) if S is a subgroup of G including A, then S is selfnormalizing;
(ii) if S, K are two conjugate subgroups of G including A, then S = K,

If G is a finite soluble group then condition (ii) could be omitted. In other words, in a finite soluble group every weakly abnormal subgroup is abnormal. We show now the most general extension of this result. But first we recall that a group G is called an ***Ñ – group*** if G satisfies the following condition:

If M, L are subgroup of G such that M is maximal in L, then M is a normal subgroup of L.

Let G be a group, having ascending series of normal subgroups whose factors are Ñ – groups, and A be a subgroup of G. Then A is abnormal in G if and only if every subgroup S including A is selfnormalizing. In particular, in radical group every weakly abnormal subgroup is abnormal.

This result was obtained in the paper

KKS2011. Kirichenko V.V., Kurdachenko L.A., Subbotin I.Ya. Some related to pronormality subgroup families and properties of groups. Algebra and Discrete Mathematics 2011, Volume 11, number 1, 75 – 108

We note that every locally nilpotent group is an Ñ – group, but converse is not true, as was demonstrated in the paper

WJ1977. Wilson J.S. On periodic generalized nilpotent groups. Bulletin London Math. Soc. 9 (1977), 81-85.

Similarly to contranormality, the abnormal subgroups are antipodes to normal subgroups: an abnormal subgroup H of a group G is normal if and only if H = G. However, the following type of subgroups includes the abnormal and normal subgroups.

The subgroup H of a group G is said to be **pronormal** in G if for every element g of a group G the subgroups H and H^g are conjugate in $< H, H^g >$.

A similar situation we have for contranormality. We consider the property, which combine the properties "to be a contranormal subgroup" and "to be a normal subgroup".

A subgroup H of a group G is called **conormal** in G if H is contranormal in H^G.

Clearly, every contranormal subgroup is conormal, and every normal subgroup is conormal (if H is normal in G, then $H = H^G$ so that $H = H^H$).

We remark that every subgroup, satisfying the Frattini property, is conormal. Indeed, let G be a group and suppose that its subgroup S satisfies the Frattini property. Put $K = S^G$. The equality $G = K \mathbf{N}_G(S)$ implies that $K = S^K$, so that S is a conormal subgroups. In particular, every pronormal subgroup is conormal.

But these two types of subgroups do not coincide, and the influences of these subgroups on a structure of groups are different. For example, let $G = D \rtimes \langle b \rangle$ where D is a divisible abelian 2 – subgroup and $d^b = d^{-1}$ for each element $d \in D$. It is not hard to see that the subgroup $\langle b \rangle$ is contranormal, so that it is conormal. But $\langle b \rangle$ cannot be pronormal, because pronormal subgroups in locally nilpotent group are normal.

We note that every maximal subgroup of a group G is conormal in G, the subgroup generated by conormal subgroups of G is conormal in G. On the other hand, the intersection of conormal subgroups can be not conormal.

Conormal subgroups have the following interesting property.

Let G be a group and C be a conormal subgroup of G. If C is subnormal in G, then C is normal in G.

As a corollary we obtain

Let G be a group. Every subgroup of G is conormal in G if and only if the relation "to be normal subgroup" is transitive for a group G..

It is well known that in general the relation "to be normal subgroup" in groups is not transitive. The groups in which this relation is transitive is called **T – groups.**

It should be noted that T – groups have been investigating for a long period of time. If we consider the relation "to be normal subgroup" not only in a group, but in its subgroups, then we come to the following concept.

A subgroup H of a group G is called ***transitively normal*** in G, if H is normal in every subgroup S such that H is subnormal in S.

If H is a transitively normal subgroup of G, then clearly $\mathbf{N}_G(\mathbf{N}_G(H)) = \mathbf{N}_G(H)$.

At once we noted the following connection with the conormal subgroups.

Let G be a group and C be a conormal subgroup of G. If C is conormal in every subgroup H, including C, then C is transitively normal in G.

A group G is said to be a **$\overline{T}$ – group** if every subgroup of G is a T – group.
Clearly, G is a $\overline{T}$ – group if and only if every subgroup of G is transitively normal in G.

The structure of finite soluble T – groups has been described by W. Gaschütz in the paper

GW1957. Gaschûtz W. Gruppen in denen das Normalreilersein transitiv ist. J. Reine.Angew. Math. 198 (1957), 87 – 92.

In particular, he proved that every finite soluble T – group is a $\overline{T}$ – group.

The infinite soluble T – groups have been considered very neatly in the fundamental paper of D.J.S. Robinson

RD1968. Robinson D.J.S. Groups in which normality is a transitive relation. Proc. Cambridge Phil. Soc. (1964), 60, 21 – 38.

A subgroup S of a group G is said to be **polynormal** in G if for every subgroup H including S, S is contranormal in S^H [**BBZ1988**]. In other words, S is a polynormal subgroup if C is conormal in each subgroup including S.

As a corollary of above result we obtain

A group G is a $\overline{T}$ – group if and only if every subgroup of G is polynormal in G.

In this connection it is interesting to consider the subgroup S of a group G, which is contranormal in every subgroup H including S. In this case suppose that H ≠ $N_G(H)$ = K. Since H is normal in K, the inclusion S ≤ H implies that $S^K \leq H$, in particular, $S^K \neq K$. This contradiction shows that H is selfnormalizing. Thus we obtain

Let G be a group. A subgroup S is contranormal in every subgroup H including S if and only if S is weakly abnormal in G. In particular, if G is a group having ascending series of normal subgroups whose factors are Ñ – groups, then the subgroup S is abnormal if and only if S is contranormal in every subgroup H including S.

For the finite groups we have the following criteria of nilpotency.

Finite group G is nilpotent if and only if G does not include proper contranormal subgroups.

Finite group G is nilpotent if and only if every conormal subgroup of G is normal.

Finite group G is nilpotent if and only if every polynormal subgroup of G is normal.

In this connection it is natural to consider the structure

of infinite groups which does not include proper contranormal subgroups,
of infinite groups whose conormal subgroups are normal,
of infinite groups whose polynormal subgroups are normal.

Note that in this situation we cannot speak of the nilpotency of these group. Moreover, they could be very far from being nilpotent.

Indeed, the groups whose subgroups are subnormal do not include proper contranormal subgroups. Such groups are locally nilpotent but can be not nilpotent. In this connection, it is suitable to recall the example constructed by H. Heineken and A. Mohamed in the paper

HM 1968. Heineken H., Mohamed A. A group with trivial centre satisfying the normalizer condition. J. Algebra 10(3) (1968), 368 – 376.

This is a p – group H with the following properties:

H includes a normal elementary abelian p – subgroup A such that H/A is a Prüfer p – group; every proper subgroup of H is subnormal in G, ζ(H) = <1>.

We say that a group G is **contranormal – free** if G does not include proper contranormal subgroups.

Some types of contranormal – free groups have been studied in the following papers

KS2003. Kurdachenko L.A., Subbotin I.Ya. Pronormality, contranormality and generalized nilpotency in infinite groups. Publicacions Matemàtiques, 2003, 47, number 2, 389 – 414

KOS2002. Kurdachenko L.A, Otal J., Subbotin I.Ya. On some criteria of nilpotency, Comm. Algebra, 30 (8) (2002) 3755 – 3776.

KOS2009. Kurdachenko L.A., Otal J. and Subbotin I.Ya. Criteria of nilpotency and influence of contranormal subgroups on the structure of infinite groups. C Turkish J. Math. – 33 (2009), 227 – 237.

KOS2010. Kurdachenko L.A., Otal J. and Subbotin I.Ya. On influence of contranormal subgroups on the structure of infinite groups. Communications in Algebra – 37 (2010), 4542 – 4557.

WB2020. Wehrfritz B.A.F. Groups with no proper contranormal subgroups. Publicacions Matemàtiques, 2020, 64, 183 – 194

DKS2021. Dixon M.R., Kurdachenko L.A., Subbotin I.Ya. On the structure of some contranormal – free groups – Comm. Algebra, 49 (11) (2021) 4940 – 4046.

KLM2022. A. Kurdachenko L.A., Longobardi P., Maj M., On the structure of some locally nilpotent groups without contranormal subgroups, J. Group Theory, 25 (2022) 75 – 90.

KLM2022[A]. A. Kurdachenko L.A., Longobardi P., Maj M., On some contranormal – free groups, 2022

Consider some established and some newer results about contranormal – free groups.

A periodic group G is said to be ***Sylow – nilpotent*** if G is locally nilpotent and a Sylow p – subgroup of G is nilpotent for each prime p.

It is not hard to see that every Sylow – nilpotent group does not include proper contranormal subgroups.

The following results have been obtained in [**KS2003**].

Let G be a Chernikov group. If G is contranormal – free, then G is nilpotent.

Let G be a locally finite group, every Sylow p – subgroup of which is Chernikov for every prime p. If G is contranormal – free, then G is Sylow – nilpotent.

For the periodic contranormal – free groups the following results have been obtained in the paper [**KLM2022**]

Let G be a group, H be a locally nilpotent normal subgroups of G such that G/H is hyperfinite. If G is contranormal – free, then G is locally nilpotent.

As corollaries we obtain

Let G be a periodic group, H be a normal locally nilpotent subgroups of G such that G/H is nilpotent. If G is contranormal – free, then G is locally nilpotent.

Let G be a periodic group and H be a normal locally nilpotent subgroup such that G/H is a Chernikov group. If G is contranormal – free, then G is locally nilpotent.

Let G be a locally finite group and H be a normal locally nilpotent subgroup such that the Sylow p – subgroups of G/H are Chernikov for all prime p. If G is contranormal – free, then G is locally nilpotent.

Let G be a hyperfinite group. If G is contranormal – free, then G is hypercentral.

Let G be a periodic group, H be a normal nilpotent subgroups of G such that G/H is nilpotent and $\Pi(H) \cap \Pi(G/H) = \varnothing$. If G is contranormal – free, then G is nilpotent.

If G/H is a Chernikov group, then we obtain assertion (iii) of the paper [**WB2020**].

Let A be a torsion – free abelian normal subgroup of a group G. We say that A is ***rationally irreducible with respect to G*** or that A is ***G – rationally irreducible*** if for every nontrivial G – invariant subgroup B of A the factor – group A/B is periodic.

Let G be a group and A be a normal nilpotent subgroup of G such that G/A is a Chernikov π – group. Suppose that A has a finite series of G – invariant subgroups

$$A = A_0 \geq A_1 \geq \ldots \geq A_j \geq A_{j+1} \geq \ldots \geq A_t = \langle 1 \rangle$$

every factor A_j / A_{j+1} of which satisfies one of the following conditions:

A_j / A_{j+1} is torsion – free and G – rationally irreducible;
A_j / A_{j+1} is a periodic π' – group;
A_j / A_{j+1} is a Chernikov π – group, $0 \leq j \leq t - 1$.

If G is contranormal – free, then G is nilpotent.

This result is a quite broad generalization of the main result of the above mentioned paper of B.A.F. Wehrfritz [**WB2020**].

In the paper [**WB2020**] B.A.F. Wehrfritz proved in the following result

Let G be a nilpotent – by – finite group. If G is contranormal – free, then G is nilpotent.

In this connection is interesting to consider nilpotent – by – (finitely generated groups). In the paper [**DKS2021**] the following results have been obtained

Let G be a group and H be a nilpotent normal subgroup of G such that G/H is finitely generated and soluble – by – finite. If G is contranormal – free, then G is hypercentral.

In the paper [**KLM2022[A]**] the following results have been obtained

Let G be a group and H be a nilpotent normal subgroup of G such that G/H is finitely generated and soluble – by – finite. Suppose that H is periodic and the set Π(H) is finite. If G is contranormal – free, then G is nilpotent.

The groups whose conormal subgroups are normal have been considered in the paper

DKS2022. Dixon M.R., Kurdachenko L.A., Subbotin I.Ya. On conormal subgroups – International Journal of Algebra and Computation, 32, no 2, 327 – 345

In this paper the following results have been obtained

Let G be a group and let A,C be normal subgroups of G such that A is a periodic nilpotent subgroup. Suppose that Π(A) is finite, A ≤ C and C/A is a finitely generated nilpotent group. If every conormal subgroup of G is normal, then C is nilpotent.

The following corollaries naturally follow from this theorem.

Let G be a group and let A,C be normal subgroups of G such that A is a periodic nilpotent subgroup. Suppose that A ≤ C and C/A is a finitely generated nilpotent group. If every conormal subgroup of G is normal, then C is hypercentral and the upper central series of C has length at most ω + t for some positive integer t.

Let G be a group and let A be a periodic normal nilpotent subgroup of G such that Π(A) is finite and G/A is abelian. If every conormal subgroup of G is normal, then G is a Fitting group.

Let G be a group and let A be a periodic normal nilpotent subgroup of G such that G/A is abelian. If every conormal subgroup of G is normal, then G is generated by its normal hypercentral subgroups. In particular, G is locally nilpotent.

We say that a group G has a normal covering by finitely generated soluble – by – finite subgroups if the normal closure of every finite subset of G is a finitely generated soluble-by-finite group.

Let G be a group and let A be a periodic normal nilpotent subgroup of G such that G/A has a normal covering by finitely generated soluble-by-finite subgroups. If every conormal subgroup of G is normal, then G is generated by its normal hypercentral subgroups. In particular, if Π(A) is finite, then G is a Fitting group.

Thank you