κ-EXISTENTIALLY CLOSED GROUPS AND AUTOMORPHISMS

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Algebraically Closed Groups

Let $W_i(x_j, g_k)$ be a word in indeterminates x_j where $j = 1, \ldots, n$ and $g_k \in G$ for $k = 1, \ldots, m$.

A group G is called **algebraically closed** if for every finite set

$$W_i(x_j, g_k) = 1 \qquad (i = 1, 2 \ldots, l)$$

$$W_i(x_j, g_k) \neq 1 \qquad (i = l + 1, \ldots, s)$$

of equations and in-equations which has a solution in an overgroup $H \geq G$ already has a solution in G.

Existentially Closed groups

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After Scott's paper algebraically closed groups are studied by many group theorists, nowadays they are called as **existentially closed groups.**

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In particular there are existentially closed groups of any given infinite cardinality.

But we will see in coming slides that, this is not true for κ-existentially closed groups when κ is an uncountable singular cardinal.

κ-Existentially Closed Groups

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κ-existentially closed groups are the analogs of existentially closed groups, allowing the number of equations and the number of in-equations to be infinite.

Definition of Existentially Closed groups

Definition. Let κ be an infinite cardinal. A group G with $|G| \geq \kappa$ is called κ-**existentially closed** if every system of less than κ-many equations and in-equations with coefficients in G which has a solution in some overgroup $H \geqslant G$ already has a solution in G.

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$\aleph_0$-existentially closed groups are the groups introduced by W. R. Scott in 1951.

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This is another difference between $\aleph_0$-existentially closed groups and κ-existentially closed groups other than the existence.

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Lemma 2 (Kegel-K, 2018)

If κ is uncountable and G is a κ-existentially closed group, then isomorphic copy of every group A of order $|A| < \kappa$ is contained in G.
Moreover if κ is uncountable, then isomorphic copy of every group of order κ is contained in G.

κ-Existentially Closed Groups

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Proposition 3 (Kegel-K, 2018)

Let G be a group and κ be an uncountable cardinal. Then G is κ-existentially closed if and only if
(i) *G contains an isomorphic copy of every group of cardinality less than κ, and*
(ii) *every isomorphism between two subgroups of G of cardinality less than κ is induced by an inner automorphism of G.*

Existence of Explicit Examples of Existentially Closed Groups

B. H. Neumann in 1937, [2] stated that "However, no algebraically closed group is explicitly known, the existence proof being highly non-constructive. This stem in part from the fact that there is no useful criterion known that tells one what sentences are or are not consistent over a given group".

Existence of Explicit Examples of Existentially Closed Groups

B. H. Neumann in 1937, [2] stated that "However, no algebraically closed group is explicitly known, the existence proof being highly non-constructive. This stem in part from the fact that there is no useful criterion known that tells one what sentences are or are not consistent over a given group".

We gave in [1], explicit examples of existentially closed groups for large cardinals. In particular we answer the more general question, namely; existence of explicit examples of κ-existentially closed groups. Hence we answer Neumann's question in a more general case, positively.

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Let κ be any infinite regular cardinal. We may start with an arbitrary group G_0 of countably infinite order. Embed G_0 into $Sym(G_0) = G_1$ by right regular representation. Then embed G_1 into $Sym(G_1) = G_2$ again by right regular representation, continue like this, for limit ordinals β let $G_\beta = \bigcup_{i<\beta} G_i$. We continue κ steps until we reach the group G_κ. Then the group G_κ is κ-existentially closed.

Existentially Closed Groups

Since every κ-existentially closed group is an $\aleph_0$-existentially closed group, the above examples of κ-existentially closed groups are examples of $\aleph_0$-existentially closed groups.
This answers the B. H. Neumann's question positively.

Existence of κ-existentially closed groups of cardinality $\lambda \geq \kappa$

Corollary 4

(GCH) Let $\lambda \geq \kappa$ be uncountable cardinals. Then there exists a κ-existentially closed group of cardinality λ if and only if $cf(\lambda)$ the cofinality of λ satisfies $cf(\lambda) \geq \kappa$.

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Moreover there exists no κ-existentially closed group of cardinality κ for singular cardinals.

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Recall that there exists $\aleph_0$-existentially closed groups for singular cardinals.

Existentially Closed Groups

By the above Corollary, we determine for which cardinals $\lambda \geq \kappa$, there exists κ-existentially closed groups of cardinality λ .

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Open Question. Let κ be not an inaccessible cardinal. Does there exist an explicit example of κ-existentially closed group of cardinality κ?

Automorphisms of κ-existentially closed groups

Not much known about the structure of the automorphism group of κ-existentially closed groups.

It was proved by Macintyre in [4, Page 56] that every countable, $\aleph_0$-existentially closed group has $2^{\aleph_0}$ automorphisms.

A. Macintyre; On algebraically closed groups, Annals of Math. **96**, (1972) 53–97.

κ-inner Automorphisms of existentially closed groups

Definition. Let G be a group. An automorphism $\varphi \in Aut(G)$ is called *κ-inner* if for every subgroup $X \subseteq G$ with $|X| < \kappa$, there exists an element $g \in G$ such that $\iota_g(x) = \varphi(x)$ for all $x \in X$.

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Let κ-$Inn(G)$ denote the set of all κ-inner automorphisms of G.

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Moreover, the inclusion on right hand side is indeed an equality for κ-existentially closed groups.

Automorphisms of κ-existentially closed groups

Proposition 5

Let κ be uncountable and let G be κ-existentially closed. Then every automorphism of G is κ-inner. i.e. κ-$Inn(G) = Aut(G)$.

Definition. Let κ be a regular uncountable cardinal. A set $C \subset \kappa$ is called closed unbounded subset (**club subset**) of κ if C is unbounded in κ and if it contains all its limit points less than κ.

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We have seen in the explicit examples of κ-existentially closed groups that the construction has κ levels.

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We now introduce the notion of a level preserving automorphism.
An automorphism $\varphi \in Aut(G)$ is said to be preserving the level α if $\varphi(G_\alpha) = G_\alpha$

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An automorphism $\varphi \in Aut(G)$ is said to be *C-level preserving* if

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for all $\alpha \in C$.

Automorphisms of κ-existentially closed groups

Let $C \subseteq \kappa$.

An automorphism $\varphi \in Aut(G)$ is said to be C-*level preserving* if

$$\varphi(G_\alpha) = G_\alpha$$

for all $\alpha \in C$.

Lemma 6

Let G be κ-existentially closed group of cardinality κ where κ is inaccessible. For every $\varphi \in Aut(G)$, we have that

$$Stab(\varphi) = \{\alpha < \kappa : \varphi(G_\alpha) = G_\alpha\}$$

is a club (i.e. closed and unbounded) subset of κ.

We shall denote the set of C-level preserving automorphisms of G by $Aut_C(G)$. We clearly have $Aut_{\emptyset}(G) = Aut(G)$ and $Aut_C(G) \leqslant Aut_D(G)$ whenever $D \subseteq C$.

Corollary 7

Let G be κ-existentially closed group and κ is inaccessible. For every $H \leqslant Aut(G)$ with $|H| < \kappa$, there exists a club set $C \subseteq \kappa$ with $H \leqslant Aut_C(G)$.

Corollary 8

Let κ be inaccessible and let G be the unique κ-existentially closed group of cardinality κ, which is isomorphic to a limit of regular representations of length κ with countable base. Then

$$Aut(G) = \bigcup_{\substack{C \subseteq \kappa \\ C \text{ is club}}} Aut_C(G) = \bigcup_{\alpha < \kappa} Aut_{\{\alpha\}}(G)$$

Open Question. Let G be a κ-existentially closed group of cardinality $\lambda \geq \kappa$ where κ is a regular cardinal. Determine the structure of the $Aut(G)$.

Open Question. Let κ be an infinite cardinal which is not an uncountable inaccessible cardinal. Does there exist explicit example of a κ-existentially closed group of cardinality κ?
In particular explicit examples of countable, $\aleph_0$-existentially closed group is still open.

We prove the following in [3]:

Corollary 9

Let κ be an inaccessible cardinal and let G be the unique κ-existentially closed group of cardinality κ. Then $|Aut(G)| = 2^{\kappa}$.

Our methods give information also the cardinality of automorphism group of limit regular representations of groups of countable base and length κ for uncountable regular cardinal κ.

We also prove that for a κ-existentially closed group of cardinality κ the $|Aut(G)| = 2^{\kappa}$, see [1].

Question What can we say about the cardinality of automorphism groups of κ-existentially closed groups of cardinality $\lambda > \kappa$?

THANK YOU

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