Groups having all elements off a normal subgroup with prime power order

Mark L. Lewis

Kent State University

June 21, 2022

Ischia Group Theory 2022 (Virtual)

Introduction

Throughout this talk, all groups are finite.

Introduction

Throughout this talk, all groups are finite.

Our goal is to characterize the groups that have a proper

Introduction

Throughout this talk, all groups are finite.

Our goal is to characterize the groups that have a proper

normal subgroup where every element outside of the normal

Introduction

Throughout this talk, all groups are finite.

Our goal is to characterize the groups that have a proper normal subgroup where every element outside of the normal subgroup has prime power order.

To understand these groups, it is useful to first understand

To understand these groups, it is useful to first understand

the groups where all elements have prime power order.

To understand these groups, it is useful to first understand

the groups where all elements have prime power order.

Note that in our question, this is the case where the

To understand these groups, it is useful to first understand

the groups where all elements have prime power order.

Note that in our question, this is the case where the

normal subgroup is trivial.

The question of groups where all elements have prime power

The question of groups where all elements have prime power order was first addressed by Higman where he determined the

The question of groups where all elements have prime power order was first addressed by Higman where he determined the solvable groups with this property.

The question of groups where all elements have prime power order was first addressed by Higman where he determined the solvable groups with this property.

Suzuki found the simple groups with this property, and then

The question of groups where all elements have prime power order was first addressed by Higman where he determined the solvable groups with this property.

Suzuki found the simple groups with this property, and then Brandl completed the classification of these groups (with

The question of groups where all elements have prime power order was first addressed by Higman where he determined the solvable groups with this property.

Suzuki found the simple groups with this property, and then Brandl completed the classification of these groups (with one omission).

Higman proved:

Higman proved:

Theorem 1 (Higman).

Let G be a solvable group. Then every element of G has prime power order if and only if one of the following occurs:

1. *G is a p-group for some prime p.*
2. *There exist distinct primes p and q so that G is a $\{p, q\}$-group and either G is a Frobenius group or G is a 2-Frobenius group.*

We next also explicity write down the classification of nonsolvable

We next also explicity write down the classification of nonsolvable groups with all elements have prime power order.

We next also explicity write down the classification of nonsolvable

groups with all elements have prime power order.

We note that Brandl missed the group M_{10} (this is the nonsplit

We next also explicity write down the classification of nonsolvable

groups with all elements have prime power order.

We note that Brandl missed the group M_{10} (this is the nonsplit

extension of $\mathrm{PSL}_2(9)$ by Z_2 which occurs as a point

We next also explicity write down the classification of nonsolvable groups with all elements have prime power order.

We note that Brandl missed the group M_{10} (this is the nonsplit extension of $\mathrm{PSL}_2(9)$ by Z_2 which occurs as a point stabilizer in M_{11}).

Theorem 2 (Brandl).

Let G be a nonsolvable group. Then every element of G has prime power order if and only if one of the following occurs:

1. *G is isomorphic to* $\mathrm{PSL}_2(7)$, $\mathrm{PSL}_2(9)$, $\mathrm{PSL}_2(17)$, $\mathrm{PSL}_3(4)$, *or* M_{10}.
2. *G has a normal subgroup N so that G/N is isomorphic to one of* $\mathrm{PSL}_2(4)$, $\mathrm{PSL}_2(8)$, $\mathrm{Sz}(8)$, *or* $\mathrm{Sz}(32)$ *and either $N = 1$ or N is a nontrivial, elementary abelian 2-group that is isomorphic to a direct sum of natural modules for G/N.*

p-elements

We first address the question of a group G with a normal subgroup

p-elements

We first address the question of a group G with a normal subgroup

N and a prime p so that every element in $G \setminus N$ has p-power order.

p-elements

We first address the question of a group G with a normal subgroup

N and a prime p so that every element in $G \setminus N$ has p-power order.

Obviously, if G is a p-group, then every normal subgroup N will

p-elements

We first address the question of a group G with a normal subgroup N and a prime p so that every element in $G \setminus N$ has p-power order.

Obviously, if G is a p-group, then every normal subgroup N will have this property.

Frobenius complements

To obtain the non p-groups with this property, we need to look at a

Frobenius complements

To obtain the non p-groups with this property, we need to look at a generalization of Frobenius groups that Wielandt studied.

Frobenius complements

To obtain the non p-groups with this property, we need to look at a

generalization of Frobenius groups that Wielandt studied.

Recall that a proper, nontrivial subgroup H of a group G is called a

Frobenius complements

To obtain the non p-groups with this property, we need to look at a generalization of Frobenius groups that Wielandt studied.

Recall that a proper, nontrivial subgroup H of a group G is called a *Frobenius complement* if $H \cap H^g = 1$ for all $g \in G \setminus H$.

Frobenius groups

A group G is called a *Frobenius group* if it contains a Frobenius

Frobenius groups

A group G is called a *Frobenius group* if it contains a Frobenius complement H.

Frobenius groups

A group G is called a *Frobenius group* if it contains a Frobenius complement H.

Frobenius' theorem states that if $N = G \setminus \cup_{g \in G} (H \setminus 1)^g$,

Frobenius groups

A group G is called a *Frobenius group* if it contains a Frobenius complement H.

Frobenius' theorem states that if $N = G \setminus \cup_{g \in G} (H \setminus 1)^g$, then N is a normal subgroup of G.

Frobenius groups

A group G is called a *Frobenius group* if it contains a Frobenius complement H.

Frobenius' theorem states that if $N = G \setminus \cup_{g \in G} (H \setminus 1)^g$, then N is a normal subgroup of G.

In addition $G = HN$ and $H \cap N = 1$.

The subgroup N is called the *Frobenius kernel* of G.

The subgroup N is called the *Frobenius kernel* of G.

It is known that $(|N|, |H|) = 1$ and that the Sylow subgroups of

The subgroup N is called the *Frobenius kernel* of G.

It is known that $(|N|, |H|) = 1$ and that the Sylow subgroups of

H are either cyclic or generalized quaternion groups.

The subgroup N is called the *Frobenius kernel* of G.

It is known that $(|N|, |H|) = 1$ and that the Sylow subgroups of

H are either cyclic or generalized quaternion groups.

Also, Thompson proved that N is nilpotent.

Frobenius Wielandt triples

Wielandt studied groups G that have a proper, nontrivial subgroup

Frobenius Wielandt triples

Wielandt studied groups G that have a proper, nontrivial subgroup

H and a normal subgroup L of H that is proper in H for which

Frobenius Wielandt triples

Wielandt studied groups G that have a proper, nontrivial subgroup

H and a normal subgroup L of H that is proper in H for which

$H \cap H^g \leq L$ for all $g \in G \setminus H$.

Frobenius Wielandt triples

Wielandt studied groups G that have a proper, nontrivial subgroup

H and a normal subgroup L of H that is proper in H for which

$H \cap H^g \leq L$ for all $g \in G \setminus H$.

We say that (G, H, L) is a *Frobenius-Wielandt triple*, if L is

Frobenius Wielandt triples

Wielandt studied groups G that have a proper, nontrivial subgroup H and a normal subgroup L of H that is proper in H for which $H \cap H^g \leq L$ for all $g \in G \setminus H$.

We say that (G, H, L) is a *Frobenius-Wielandt triple*, if L is normal in H and $H \cap H^g \leq L$ for all $g \in G \setminus H$.

Observe that H is a Frobenius complement if $L = 1$; so this a

Observe that H is a Frobenius complement if $L = 1$; so this a generalization of Frobenius complements.

Observe that H is a Frobenius complement if $L = 1$; so this a

generalization of Frobenius complements.

Wielandt proved that H and L determine a unique normal

Observe that H is a Frobenius complement if $L = 1$; so this a

generalization of Frobenius complements.

Wielandt proved that H and L determine a unique normal

subgroup N so that $G = NH$ and $N \cap H = L$.

Observe that H is a Frobenius complement if $L = 1$; so this a generalization of Frobenius complements.

Wielandt proved that H and L determine a unique normal subgroup N so that $G = NH$ and $N \cap H = L$.

In fact, $N = G \setminus \cup_{g \in G} (H \setminus L)^g$.

The subgroup N is called the *Frobenius-Wielandt kernel*.

The subgroup N is called the *Frobenius-Wielandt kernel*.

It is not difficult to see when $L = 1$ that N is the usual

The subgroup N is called the *Frobenius-Wielandt kernel*.

It is not difficult to see when $L = 1$ that N is the usual

Frobenius kernel.

The subgroup N is called the *Frobenius-Wielandt kernel*.

It is not difficult to see when $L = 1$ that N is the usual

Frobenius kernel.

Wielandt also proved that $(|G : H|, |H : L|) = 1$.

It is a natural question to ask how closely related to Frobenius

It is a natural question to ask how closely related to Frobenius

groups are Frobenius-Wielandt triples.

It is a natural question to ask how closely related to Frobenius groups are Frobenius-Wielandt triples.

This question has been addressed by Espuelas.

Among other things he proves that if H splits over N and $|H/N|$

Among other things he proves that if H splits over N and $|H/N|$

is even, then H/N is isormorphic to a Frobenius complement and if

Among other things he proves that if H splits over N and $|H/N|$

is even, then H/N is isormorphic to a Frobenius complement and if

$|N|$ is odd and q is a prime divisor of $|N|$ so that a Sylow

Among other things he proves that if H splits over N and $|H/N|$ is even, then H/N is isormorphic to a Frobenius complement and if $|N|$ is odd and q is a prime divisor of $|N|$ so that a Sylow q-subgroup of N is abelian and is complemented in a Sylow

Among other things he proves that if H splits over N and $|H/N|$ is even, then H/N is isormorphic to a Frobenius complement and if $|N|$ is odd and q is a prime divisor of $|N|$ so that a Sylow q-subgroup of N is abelian and is complemented in a Sylow q-subgroup of H, then the Sylow q-subgroups of H/N are cyclic.

On the other hand, Scopolla has proved that if P is any p-group,

On the other hand, Scopolla has proved that if P is any p-group,

then there is a group G with a Sylow p-subgroup Q and subgroup

On the other hand, Scopolla has proved that if P is any p-group,

then there is a group G with a Sylow p-subgroup Q and subgroup

L normal in Q so that (G, Q, L) is a Frobenius-Wielandt triple

On the other hand, Scopolla has proved that if P is any p-group,

then there is a group G with a Sylow p-subgroup Q and subgroup

L normal in Q so that (G, Q, L) is a Frobenius-Wielandt triple

and $Q/L \cong P$.

We now prove a result regarding Frobenius-Wielandt triples.

We now prove a result regarding Frobenius-Wielandt triples.

Lemma 3.

If (G, H, L) is a Frobenius-Wielandt triple, then either $N_G(L) = H$ or $N_G(L)/L$ is a Frobenius group.

We now prove a result regarding Frobenius-Wielandt triples.

Lemma 3.

If (G, H, L) is a Frobenius-Wielandt triple, then either $N_G(L) = H$ or $N_G(L)/L$ is a Frobenius group.

Proof:

We now prove a result regarding Frobenius-Wielandt triples.

Lemma 3.

If (G, H, L) is a Frobenius-Wielandt triple, then either $N_G(L) = H$ or $N_G(L)/L$ is a Frobenius group.

Proof:

Obviously, we have $H \leq N_G(L)$. Suppose $H < N_G(L)$.

We now prove a result regarding Frobenius-Wielandt triples.

Lemma 3.

If (G, H, L) is a Frobenius-Wielandt triple, then either $N_G(L) = H$ or $N_G(L)/L$ is a Frobenius group.

Proof:

Obviously, we have $H \leq N_G(L)$. Suppose $H < N_G(L)$.

Thus, we know that $H/L \cap H^x/L = (H \cap H^x)/L \leq L/L$ for all

We now prove a result regarding Frobenius-Wielandt triples.

Lemma 3.

If (G, H, L) is a Frobenius-Wielandt triple, then either $N_G(L) = H$ or $N_G(L)/L$ is a Frobenius group.

Proof:

Obviously, we have $H \leq N_G(L)$. Suppose $H < N_G(L)$.

Thus, we know that $H/L \cap H^x/L = (H \cap H^x)/L \leq L/L$ for all

$x \in N_G(L) \setminus H$.

It follows that H/L is a Frobenius complement in $N_G(L)/L$.

It follows that H/L is a Frobenius complement in $N_G(L)/L$.

We conclude that $N_G(L)/L$ is a Frobenius group.

It follows that H/L is a Frobenius complement in $N_G(L)/L$.

We conclude that $N_G(L)/L$ is a Frobenius group.

Corollary 4.

If (G, H, L) is a Frobenius-Wielandt triple and L is normal in G, then G/L is a Frobenius group.

Lemma 5.

Let N be a normal subgroup of a group G. Suppose H is a subgroup of G so that $G = HN$. Then every element of $G \setminus N$ is conjugate to an element in H if and only if $(G, H, H \cap N)$ is a Frobenius-Wielandt triple.

Lemma 5.

Let N be a normal subgroup of a group G. Suppose H is a subgroup of G so that $G = HN$. Then every element of $G \setminus N$ is conjugate to an element in H if and only if $(G, H, H \cap N)$ is a Frobenius-Wielandt triple.

We next show that quotients that are Frobenius groups yield

Lemma 5.

Let N be a normal subgroup of a group G. Suppose H is a subgroup of G so that $G = HN$. Then every element of $G \setminus N$ is conjugate to an element in H if and only if $(G, H, H \cap N)$ is a Frobenius-Wielandt triple.

We next show that quotients that are Frobenius groups yield

Frobenius-Weilandt triples.

Corollary 6.

Let N be a normal subgroup of a group G. If G/N is a Frobenius group with Frobenius complement H/N, then (G, H, N) is a Frobenius-Wielandt triple.

Corollary 6.

Let N be a normal subgroup of a group G. If G/N is a Frobenius group with Frobenius complement H/N, then (G, H, N) is a Frobenius-Wielandt triple.

Using Frobenius-Wielandt triples, we can determine the groups G

Corollary 6.

Let N be a normal subgroup of a group G. If G/N is a Frobenius group with Frobenius complement H/N, then (G, H, N) is a Frobenius-Wielandt triple.

Using Frobenius-Wielandt triples, we can determine the groups G

and primes p with a normal subgroup N so that every element of

Corollary 6.

Let N be a normal subgroup of a group G. If G/N is a Frobenius group with Frobenius complement H/N, then (G, H, N) is a Frobenius-Wielandt triple.

Using Frobenius-Wielandt triples, we can determine the groups G and primes p with a normal subgroup N so that every element of $G \setminus N$ has p-power order.

Theorem 7.

Let G be a group, let N be a normal subgroup, and let p be a prime. If P is a Sylow p-subgroup of G, then every element of $G \setminus N$ has p-power order if and only if either (1) $G = P$ or (2) $G = PN$ and $(G, P, P \cap N)$ is a Frobenius-Wielandt triple.

Proof:

Theorem 7.

Let G be a group, let N be a normal subgroup, and let p be a prime. If P is a Sylow p-subgroup of G, then every element of $G \setminus N$ has p-power order if and only if either (1) $G = P$ or (2) $G = PN$ and $(G, P, P \cap N)$ is a Frobenius-Wielandt triple.

Proof:

Suppose first that every element of $G \setminus N$ has p-power order.

Theorem 7.

Let G be a group, let N be a normal subgroup, and let p be a prime. If P is a Sylow p-subgroup of G, then every element of $G \setminus N$ has p-power order if and only if either (1) $G = P$ or (2) $G = PN$ and $(G, P, P \cap N)$ is a Frobenius-Wielandt triple.

Proof:

Suppose first that every element of $G \setminus N$ has p-power order.

If G is a p-group, then the result is obvious.

Thus, we assume that G is not a p-group.

Thus, we assume that G is not a p-group.

Then $G = PN$.

Thus, we assume that G is not a p-group.

Then $G = PN$.

Notice that all the elements of $G \setminus N$

Thus, we assume that G is not a p-group.

Then $G = PN$.

Notice that all the elements of $G \setminus N$

are conjugate to an element of P.

By Lemma 5, $(G, P, P \cap N)$ is a

By Lemma 5, $(G, P, P \cap N)$ is a

Frobenius-Wielandt triple.

By Lemma 5, $(G, P, P \cap N)$ is a

Frobenius-Wielandt triple.

Conversely, if G is a p-group, then obviously

By Lemma 5, $(G, P, P \cap N)$ is a

Frobenius-Wielandt triple.

Conversely, if G is a p-group, then obviously

every element in $G \setminus N$ has p-power order.

Thus, suppose that $G = PN$ and $(G, P, P \cap N)$ is

Thus, suppose that $G = PN$ and $(G, P, P \cap N)$ is

a Frobenius-Wielandt triple.

Thus, suppose that $G = PN$ and $(G, P, P \cap N)$ is

a Frobenius-Wielandt triple.

Then by Lemma 5 every element in $G \setminus N$

Thus, suppose that $G = PN$ and $(G, P, P \cap N)$ is

a Frobenius-Wielandt triple.

Then by Lemma 5 every element in $G \setminus N$

is conjugate to an element in P.

Thus, suppose that $G = PN$ and $(G, P, P \cap N)$ is

a Frobenius-Wielandt triple.

Then by Lemma 5 every element in $G \setminus N$

is conjugate to an element in P.

Thus, every element in $G \setminus N$ has p-power order.

We note that N need not be solvable in Theorem 7.

We note that N need not be solvable in Theorem 7.

Let G be the group M_{10} and take N to be the normal

We note that N need not be solvable in Theorem 7.

Let G be the group M_{10} and take N to be the normal

subgroup isomorphic to $\mathrm{PSL}(2,9) \cong A_6$.

We note that N need not be solvable in Theorem 7.

Let G be the group M_{10} and take N to be the normal

subgroup isomorphic to $\mathrm{PSL}(2,9) \cong A_6$.

One can see that all of the elements in $G \setminus N$ have order

We note that N need not be solvable in Theorem 7.

Let G be the group M_{10} and take N to be the normal

subgroup isomorphic to $\mathrm{PSL}(2,9) \cong A_6$.

One can see that all of the elements in $G \setminus N$ have order

4 or 8, and thus, have 2-power order.

At this time, this essentially is the only example we know of where

At this time, this essentially is the only example we know of where

N is not solvable.

At this time, this essentially is the only example we know of where

N is not solvable.

It would be interesting to study the question:

Suppose G is a group, N is a normal subgroup, p is a prime, P

Suppose G is a group, N is a normal subgroup, p is a prime, P

is a Sylow p-subgroup so that that $(G, P, P \cap N)$ is a

Suppose G is a group, N is a normal subgroup, p is a prime, P

is a Sylow p-subgroup so that that $(G, P, P \cap N)$ is a

Frobenius-Wielandt triple, $O_p(G) = 1$, $G = NP$, and G is

Suppose G is a group, N is a normal subgroup, p is a prime, P is a Sylow p-subgroup so that that $(G, P, P \cap N)$ is a Frobenius-Wielandt triple, $O_p(G) = 1$, $G = NP$, and G is nonsolvable.

Suppose G is a group, N is a normal subgroup, p is a prime, P is a Sylow p-subgroup so that that $(G, P, P \cap N)$ is a Frobenius-Wielandt triple, $O_p(G) = 1$, $G = NP$, and G is nonsolvable.

Is this enough to imply that $G \cong M_{10}$?

Suppose G is a group, N is a normal subgroup, p is a prime, P is a Sylow p-subgroup so that that $(G, P, P \cap N)$ is a Frobenius-Wielandt triple, $O_p(G) = 1$, $G = NP$, and G is nonsolvable.

Is this enough to imply that $G \cong M_{10}$?

Or do other examples exist?

Prime powers with more than one prime

The next obvious question is what can occur when $G \setminus N$

Prime powers with more than one prime

The next obvious question is what can occur when $G \setminus N$

contains all elements that have prime power but not for the same

Prime powers with more than one prime

The next obvious question is what can occur when $G \setminus N$ contains all elements that have prime power but not for the same prime.

Prime powers with more than one prime

The next obvious question is what can occur when $G \setminus N$ contains all elements that have prime power but not for the same prime.

It turns out that the answer depends whether there are two primes

Prime powers with more than one prime

The next obvious question is what can occur when $G \setminus N$ contains all elements that have prime power but not for the same prime.

It turns out that the answer depends whether there are two primes or more than two primes.

Recall that G is a 2-Frobenius group if there exist normal

Recall that G is a 2-Frobenius group if there exist normal

subgroups $K < L < G$ so that G/K and L are Frobenius groups

Recall that G is a 2-Frobenius group if there exist normal

subgroups $K < L < G$ so that G/K and L are Frobenius groups

with Frobenius kernels L/K and K, respectively.

Recall that G is a 2-Frobenius group if there exist normal

subgroups $K < L < G$ so that G/K and L are Frobenius groups

with Frobenius kernels L/K and K, respectively.

We first address the case with two primes.

Theorem 8.

Let G be a group and let N be a normal subgroup of G. Suppose that all elements of $G \setminus N$ have prime power order and that two distinct primes p and q divide the orders of such elements. Then the following are true: G is a $\{p, q\}$-group for distinct primes p and q and either G/N is either a Frobenius group or a 2-Frobenius group.

Theorem 8.

Let G be a group and let N be a normal subgroup of G. Suppose that all elements of $G \setminus N$ have prime power order and that two distinct primes p and q divide the orders of such elements. Then the following are true: G is a $\{p, q\}$-group for distinct primes p and q and either G/N is either a Frobenius group or a 2-Frobenius group.

In this case that G/N is solvable.

Theorem 8.

Let G be a group and let N be a normal subgroup of G. Suppose that all elements of $G \setminus N$ have prime power order and that two distinct primes p and q divide the orders of such elements. Then the following are true: G is a $\{p, q\}$-group for distinct primes p and q and either G/N is either a Frobenius group or a 2-Frobenius group.

In this case that G/N is solvable.

When G/N is solvable we obtain iterated Frobenius-Wielandt triples.

Theorem 9.

Let G be a group and let N be a normal subgroup so that G/N is solvable. Then all elements in $G \setminus N$ have prime power order if and only if one of the following occur:

1. *G is a p-group for some prime p.*
2. *There is a prime p and a Sylow p-subgroup P so that $G = NP$ and $(G, P, P \cap N)$ is a Frobenius-Wielandt triple.*
3. *There are primes p and q and Sylow p- and q-subgroups P and Q respectively, and a normal subgroup M in G so that*
    a. *$M = NQ$ and $G = MP$.*
    b. *$(G, P, P \cap M)$ is a Frobenius-Wielandt triple.*
    c. *Either $M = Q$ or $(M, Q, Q \cap N)$ is a Frobenius-Wielandt triple.*

Theorem (Continued).

4. *There are primes p and q and Sylow p- and q-subgroups P and Q respectively, and normal subgroups M and K in G so that*
   a. $K = N(K \cap P)$, $M = KQ$, *and* $G = MP$.
   b. $(G, P, P \cap M)$ *and* $(M, Q, Q \cap K)$ *are Frobenius-Wielandt triples.*
   c. *Either* $K \leq P$ *or* $(K, P \cap K, P \cap N)$ *is a Frobenius-Wielandt triple.*

Theorem (Continued).

4. *There are primes p and q and Sylow p- and q-subgroups P and Q respectively, and normal subgroups M and K in G so that*
   a. $K = N(K \cap P)$, $M = KQ$, *and* $G = MP$.
   b. $(G, P, P \cap M)$ *and* $(M, Q, Q \cap K)$ *are Frobenius-Wielandt triples.*
   c. *Either* $K \leq P$ *or* $(K, P \cap K, P \cap N)$ *is a Frobenius-Wielandt triple.*

We now obtain restrictions on groups with iterated

We now obtain restrictions on groups with iterated

Frobenius-Wielandt triples.

We now obtain restrictions on groups with iterated

Frobenius-Wielandt triples.

Theorem 10.

Let G be a group. Let p and q be primes so that P and Q are Sylow p and q-subgroups, respectively and M and N are normal subgroups so that $G = MP$ and $M = NQ$. Assume also that $(G, P, P \cap M)$ and $(M, Q, Q \cap N)$ are Frobenius-Wielandt triples.

Theorem (Continued).

Then the following are true:

1. $N_G(Q)$ *is a Frobenius group with Frobenius kernel* Q.
2. G/N *is a Frobenius group with Frobenius kernel* M/N.
3. *If* P *is chosen so that* $P \cap N_G(Q) = N_P(Q)$ *is a Sylow* p*-subgroup of* $N_G(Q)$, *then* $N_P(Q)$ *Frobenius complement of* $N_G(Q)$ *and* $P = (N \cap P) \rtimes N_P(Q)$.
4. *If* $O^p(N) < N$, *then* $G/O^p(N)$ *is a 2-Frobenius group.*
5. *Either* $N_G(Q \cap N) = N_G(Q)$ *or* $N_G(Q \cap N)/(Q \cap N)$ *is a 2-Frobenius group.*
6. G *is a* $\{p, q\}$*-group.*

The following Corollary is Theorem 8.

The following Corollary is Theorem 8.

Corollary 11.

Suppose G has a normal subgroup N so that every element in $G \setminus N$ has prime power order and the orders of these elements are divisible by the distinct primes p and q. Then G/N is either a Frobenius or a 2-Frobenius group and G is a $\{p, q\}$-group.

Using Theorem 8, we are to prove following theorem.

Using Theorem 8, we are to prove following theorem.

Theorem 12.

Let G be a group with a normal subgroup N so that G/N is not solvable. Then all elements in $G \setminus N$ have prime power order if and only if all elements in G have prime power order.

Sketch of Proof:

Sketch of Proof:

If every element in G has prime power order, then every element

Sketch of Proof:

If every element in G has prime power order, then every element in $G \setminus N$ has prime power order.

We assume the converse.

We assume the converse.

We assume that every element in $G \setminus N$ has prime power order.

We assume the converse.

We assume that every element in $G \setminus N$ has prime power order.

This implies that G/N is nonsolvable and all elements in G/N have

We assume the converse.

We assume that every element in $G \setminus N$ has prime power order.

This implies that G/N is nonsolvable and all elements in G/N have prime power order.

Hence, G/N is one of the groups listed in Theorem 2.

Hence, G/N is one of the groups listed in Theorem 2.

We claim for each of those groups that there exist distinct primes

Hence, G/N is one of the groups listed in Theorem 2.

We claim for each of those groups that there exist distinct primes

p_1 and p_2 so that G/N has a Frobenius $\{2, p_i\}$-subgroup for each i.

Hence, G/N is one of the groups listed in Theorem 2.

We claim for each of those groups that there exist distinct primes

p_1 and p_2 so that G/N has a Frobenius $\{2, p_i\}$-subgroup for each i.

Let F_i/N be a Frobenius $\{2, p_i\}$- subgroup of G/N.

Notice that F_i/N is solvable, and every element in $F_i \setminus N$ is

Notice that F_i/N is solvable, and every element in $F_i \setminus N$ is

an element in $G \setminus N$; so every element in $F_i \setminus N$ has

Notice that F_i/N is solvable, and every element in $F_i \setminus N$ is an element in $G \setminus N$; so every element in $F_i \setminus N$ has prime power order.

Notice that F_i/N is solvable, and every element in $F_i \setminus N$ is an element in $G \setminus N$; so every element in $F_i \setminus N$ has prime power order.

By Corollary 11, we see that F_i is a $\{2, p_i\}$-group.

Notice that F_i/N is solvable, and every element in $F_i \setminus N$ is an element in $G \setminus N$; so every element in $F_i \setminus N$ has prime power order.

By Corollary 11, we see that F_i is a $\{2, p_i\}$-group.

This implies that N is a $\{2, p_i\}$-subgroup for $i = 1, 2$.

The only way this can occur is if N is a 2-group.

The only way this can occur is if N is a 2-group.

Now, we know every element in N has 2-power order and every

The only way this can occur is if N is a 2-group.

Now, we know every element in N has 2-power order and every

element in $G \setminus N$ has prime power order; so we may conclude

The only way this can occur is if N is a 2-group.

Now, we know every element in N has 2-power order and every element in $G \setminus N$ has prime power order; so we may conclude that every element of G has prime power order.

To complete the classification, we provide the result when

To complete the classification, we provide the result when

the primes in $G \setminus N$ have prime power orders for at

To complete the classification, we provide the result when the primes in $G \setminus N$ have prime power orders for at least three primes.

To complete the classification, we provide the result when

the primes in $G \setminus N$ have prime power orders for at

least three primes.

Theorem 13.

Let G be a group and let N be a normal subgroup of G. Suppose that all elements of $G \setminus N$ have prime power orders and that at least three distinct primes divide the orders of such elements. Then all elements in G have prime power order. In fact, G is one of the groups listed in Theorem 2.

Proof:

Proof:

We know that all the elements of $G \setminus N$ have prime power order,

Proof:

We know that all the elements of $G \setminus N$ have prime power order,

so all the elements of G/N have prime power order.

Proof:

We know that all the elements of $G \setminus N$ have prime power order,

so all the elements of G/N have prime power order.

Since three primes divide the orders of these elements, we know

Proof:

We know that all the elements of $G \setminus N$ have prime power order,

so all the elements of G/N have prime power order.

Since three primes divide the orders of these elements, we know

G/N is not solvable by Theorem 1.

Applying Theorem 12, we see that every element in G has

Applying Theorem 12, we see that every element in G has

prime power order.

Applying Theorem 12, we see that every element in G has

prime power order.

Therefore, G appears in the list in Theorem 2.

Thank You!

Questions?