The soluble graph and the Engel graph of a finite group

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We can define the related graph $\Gamma_{\mathcal{F}}(G)$ on $G \setminus I_{\mathcal{F}}(G)$, where $I_{\mathcal{F}}(G)$ is the set of isolated vertices in the complement of $\Lambda_{\mathcal{F}}(G)$.

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- If $\mathcal{F} = \mathcal{S}$ is the class of soluble groups, then, by a theorem of Guralnick, Kunyavskii, Plotkin and Shalev, $I_{\mathcal{F}}(G)$ coincides with the soluble radical $R(G)$ and $\Gamma_{\mathcal{F}}(G)$ is the soluble graph of G.

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- In both of these special cases, $I_{\mathcal{F}}(G)$ is a normal subgroup of G.
- If $\mathcal{F} = \mathcal{S}$, then $\Gamma_{\mathcal{F}}(G)$ is connected if and only if $\Gamma_{\mathcal{F}}(G/I_{\mathcal{F}}(G))$ is connected and moreover, the diameters of these two graphs are equal. These attractive properties provide further impetus for studying the soluble graph of a finite group. Various difficulties arise when we switch our focus to one of the other families.

The soluble graph

Definition

Let G be a finite insoluble group with soluble radical $R(G)$. The soluble graph of G, denoted $\Gamma_{\mathcal{S}}(G)$, has vertex set $G \setminus R(G)$, with distinct vertices x and y adjacent if and only if $\langle x, y \rangle$ is a soluble subgroup of G.

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Theorem (Daniele Nemmi, Tim Burness, AL 2021)

Let G be a finite insoluble group. Then $\Gamma_{\mathcal{S}}(G)$ is connected and $\delta_{\mathcal{S}}(G) \leq 5$.

A more detailed statement

Consider the following collections of simple groups:

$$\mathcal{A} = \{A_{11}, A_{12}, \mathrm{L}_5^{\epsilon}(2), \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{M}_{24}, \mathrm{HS}, \mathrm{J}_3\}$$
$$\mathcal{B} = \{A_n, \mathrm{L}_7^{\epsilon}(2), E_6(2), \mathrm{Co}_2, \mathrm{Co}_3, \mathrm{McL}, \mathbb{B}\}$$

where $n \in \{19, 20, 23, 24, 31, 43, 44, 47, 48, 59, 60\}$.

Theorem

Let G be a finite insoluble group.

(i) *If G is not almost simple, then $\delta_{\mathcal{S}}(G) \leq 3$.*

(ii) *If G is almost simple with socle G_0, then $\delta_{\mathcal{S}}(G) \leq 5$. In addition:*

  (a) *If $G_0 = A_n$ and $n \geq 7$, then either $\delta_{\mathcal{S}}(G) = 3$, or $G = A_n$ and $n \in \{p, p+1\}$, where p is a prime with $p \equiv 3 \mod 4$.*

  (b) *If $G \in \mathcal{A} \cup \mathcal{B}$ then $\delta_{\mathcal{S}}(G) \geq 4$, with equality if $G \in \mathcal{A}$.*

  (c) *If $G = G_0$ is not isomorphic to a classical group, then $\delta_{\mathcal{S}}(G) \geq 3$.*

A prime p is a Sophie Germain prime if $2p+1$ is also a prime number; the examples with $p < 200$ are as follows:

$$\{2, 3, 5, 11, 23, 29, 41, 53, 83, 89, 113, 131, 173, 179, 191\}.$$

It is conjectured that there are infinitely many such primes, but this remains a formidable open problem in number theory. Modulo this conjecture, our next result establishes the existence of infinitely many simple groups with $\delta_{\mathcal{S}}(G) \geq 4$.

THEOREM

If $p \geq 5$ is a Sophie Germain prime, then $\delta_{\mathcal{S}}(A_{2p+1}) \in \{4, 5\}$.

Suppose $R(G) = 1$ and observe that the set of involutions in G forms a clique in $\Gamma_{\mathcal{S}}(G)$ since any two involutions generate a dihedral group. Therefore, the bound $\delta_{\mathcal{S}}(G) \leq 5$ will follow if we can show that for all nontrivial $x \in G$ there is a path in $\Gamma_{\mathcal{S}}(G)$ of length at most 2 from x to an involution.

With this observation in mind, our proof establishes the following result (in the statement, $\delta(x, y)$ denotes the distance in $\Gamma_{\mathcal{S}}(G)$ from x to y).

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Theorem

Let G be a finite insoluble group with $R(G) = 1$ and let $x \in G$ be nontrivial. Then either

(i) *There exists an involution $y \in G$ with $\delta(x, y) \leq 2$; or*

(ii) *G is the Mathieu group M_{23} and $|x| = 23$.*

Let P_n be the path graph with n vertices. A graph Γ is a cograph if it has no induced subgraph isomorphic to the four-vertex path P_4. If Γ is a connected cograph, then the complement of Γ is disconnected.

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The complement of the soluble graph $\Gamma_{\mathcal{S}}(G)$ is the insoluble graph of G: the vertices are once again labelled by the elements of $G \setminus R(G)$, with x and y adjacent if they generate an insoluble group. The insoluble graph of a finite insoluble group is connected with diameter 2 so if G be a finite insoluble group, then $\Gamma_{\mathcal{S}}(G)$ is not a cograph.

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Let G be a finite insoluble group. Then G contains four distinct elements a_1, a_2, a_3, a_4 such that:

- $\langle a_1, a_2\rangle, \langle a_2, a_3\rangle, \langle a_3, a_4\rangle$ are soluble;
- $\langle a_1, a_3\rangle, \langle a_1, a_4\rangle, \langle a_2, a_4\rangle$ are not soluble.

Some related graphs

The soluble graph encodes the pairs of elements that generate a soluble subgroup of G, whereas the commuting graph is concerned with the pairs generating an abelian group. Of course, there are many natural families of groups that lie between soluble and abelian, including the supersoluble, nilpotent, metabelian and metacyclic groups. For each of these families, we can construct a graph associated to G.

Suppose $\mathcal{F} = \mathcal{M}$ is the class of metabelian groups. It is difficult to give an efficient description of the isolated vertices $I_{\mathcal{F}}(G)$, which need not even be a subgroup of G.

AN EXAMPLE

- Let p be a prime number and let G be a Sylow p-subgroup of $\mathrm{GL}_n(p)$, where $n \geq 2$.

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- Therefore, $I_{\mathcal{M}}(G)$ is a subgroup of G if and only if $I_{\mathcal{M}}(G) = G$.

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- G is generated by its abelian normal subgroups. In addition if N is an abelian normal subgroup of G, then $\langle x, y \rangle$ is metabelian for all $x \in N$ and $y \in G$.
- Therefore, $I_{\mathcal{M}}(G)$ is a subgroup of G if and only if $I_{\mathcal{M}}(G) = G$.
- If n is large enough, then G contains 2-generated subgroups that are not metabelian and thus $I_{\mathcal{M}}(G)$ is not a subgroup (indeed, any given p-group embeds in $\mathrm{GL}_n(p)$ for some n).

Example

- Let $G = \mathrm{SL}_2(3) = Q_8{:}C_3$.
- $I_{\mathcal{M}}(G) = Z(G) = C_2$.
- $G/I_{\mathcal{M}}(G) \cong A_4$ is metabelian and $\Gamma_{\mathcal{M}}(G/I_{\mathcal{M}}(G))$ is the null graph with a single vertex.
- $\Gamma_{\mathcal{M}}(G)$ has 22 vertices and 5 connected components: one comprising the 6 elements of order 4 and the remainder corresponding to the 4 elements of order 3 or 6 in each of the four cyclic subgroups of G with order 6.

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Although the previous example shows that $\Gamma_{\mathcal{M}}(G)$ is not connected, in general, we can establish the following result.

Proposition

Let G be a nontrivial finite group with $R(G) = 1$. Then the metabelian graph $\Gamma_{\mathcal{M}}(G)$ is connected and its diameter is at most $2\delta_{\mathcal{S}}(G)$.

A natural question

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For which families $\mathcal{F}$ of finite soluble groups is it true that $\Gamma_{\mathcal{F}}(G)$ is connected for every finite group $G \notin \mathcal{F}$?

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The nilpotent and supersoluble graphs of $G = A_4$ are equal and disconnected. Indeed this graph has 11 vertices and 5 connected components: one comprising the 3 involutions, and four more consisting of an element of order 3 and its inverse.

The metacyclic graph

Proposition

Let G be a non-abelian finite simple group. Then either

(i) *The metacyclic graph $\Gamma_{\mathcal{C}}(G)$ is connected; or*

(ii) $G = \mathrm{L}_2(3^f)$ *and* $f \geq 3$ *is odd.*

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Let $G = \mathrm{L}_2(27)$. Then $\Gamma_{\mathcal{C}}(G)$ has 28 connected components, each containing 3 elements (one such component for each Sylow 3-subgroup of G), plus an additional connected component comprising the remaining 743 elements in G.

ENGEL GRAPH

The notion of commuting graph can be also generalized in a different direction. Let w be a word in the free group F_2 of rank 2. We may define a direct graph $\Lambda_w(G)$ on the elements of G, where there is an edge $x \mapsto y$ if and only if $w(x, y) = 1$.

Engel graph

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$$I_w(G) = \{g \in G \mid w(x, g) = w(g, x) = 1 \text{ for all } x \in G\}.$$

Noticing that $I_w(G)$ is the set of universal vertices of $\Lambda_w(G)$, we define the related graph $\Gamma_w(G)$ on $G \setminus I_w(G)$.

We will use the symbols $\Lambda_{\mathrm{eng,n}}(G)$, $\Gamma_{\mathrm{eng,n}}(G)$, to denote the graphs $\Lambda_w(G)$ and $\Gamma_w(G)$ when $w = [x,_n y]$ is the n-Engel word.

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Notice that $\Gamma_1(G)$ coincides with the commuting graph of G.

We consider the graph $\Lambda_{\mathrm{eng}}(G) = \cup_{n>0}\Lambda_{\mathrm{eng,n}}(G)$, i.e. the graph whose vertices are the elements of G and where there is an edge $x \mapsto y$ if and only if $[x,_n y] = 1$ for some $0 < n \in \mathbb{N}$.

The elements of G that are connected to every other vertex (both as starting and ending vertex of some edge) are the elements of $Z_\infty(G)$. So it makes sense to consider the more restrictive graph $\Gamma_{\mathrm{eng}}(G)$ (the Engel graph of G), which is only defined on the non-hypercentral elements of G.

Since $\Gamma_{\text{eng}}(G)$ is a direct graph we may consider the strong connectivity and the weak connectivity. A directed graph is strongly connected if it contains a directed path from x to y, and from y to x, for every pair of vertices (x, y), and is weakly connected if the undirected underlying graph obtained by replacing all directed edges of the graph with undirected edges is a connected graph.

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Theorem (E. Detomi, D. Nemmi, AL 2021)

If G is a finite group, then $\Gamma_{\mathrm{eng}}(G)$ is weakly connected and its undirected diameter is at most 10.

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The nilpotent graph and the Engel graph of G have the same vertex-set, $G \setminus Z_\infty(G)$. As all elements of a nilpotent group are (left and right) Engel, if g_1 and g_2 are adjacent in the nilpotent graph, then $g_1 \mapsto g_2$ and $g_2 \mapsto g_1$ in the Engel graph. So if the nilpotent graph of G is connected, then the Engel graph $\Gamma_{\mathrm{eng}}(G)$ is strongly connected. The converse is false (see for example S_4 and A_7).

It can be easily proved that $\Gamma_{\rm eng}(G)$ is strongly connected if and only if $\Gamma_{\rm eng}(G/Z_\infty(G))$ is strongly connected and that the Engel graph of a Frobenius group is not strongly connected.

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Theorem (E. Detomi, D. Nemmi, AL 2021)

Suppose that $G/Z_\infty(G)$ is not an almost simple group. Then $\Gamma_{\mathrm{eng}}(G)$ is strongly connected if and only if $G/Z_\infty(G)$ is not a Frobenius group.

In the case where G is soluble and $G/Z_\infty(G)$ is not a Frobenius group, we can also bound the diameter of $\Gamma(G)$.

Theorem

Suppose that G is soluble and $G/Z_\infty(G)$ is not a Frobenius group. Then $\mathrm{diam}(\Gamma_{\mathrm{eng}}(G)) \leq 4$. Furthermore, there exists a soluble group G such that $\mathrm{diam}(\Gamma_{\mathrm{eng}}(G)) = 4$.

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It is interesting to note that if $Z_\infty(G) = 1$ and G is not almost simple, then $\Gamma_{\mathrm{eng}}(G)$ is strongly connected if and only if $\Gamma_{\mathrm{eng},2}(G)$ is strongly connected. This is no more true if the assumption $Z_\infty(G) = 1$ is removed. For example $\Gamma_{\mathrm{eng},3}(\mathrm{GL}(2,3))$ is strongly connected, but $\Gamma_{\mathrm{eng},2}(\mathrm{GL}(2,3))$ is not.

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THEOREM (P. SPIGA AL 2022)

Let G be a finite non-abelian almost simple group. Then $\Gamma_{\mathrm{eng}}(G)$ is strongly connected if and only if none of the following cases occurs:

1. $G = \mathrm{PSL}(2, q)$ *with q even;*
2. $G = \mathrm{PSL}(2, q)$ *with* $q \equiv 5 \mod 8$;
3. $G = \mathrm{Sz}(q)$.
4. $G = \mathrm{Aut}(\mathrm{Sz}(2^e))$ *and e is an odd prime.*

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- When $\Gamma_{\mathrm{eng}}(G)$ is strongly connected, an interesting question is which is the smallest $n \in \mathbb{N}$ such that the subgraph $\Gamma_{\mathrm{eng,n}}(G)$ is already strongly connected.

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- When $\Gamma_{\mathrm{eng}}(G)$ is strongly connected, an interesting question is which is the smallest $n \in \mathbb{N}$ such that the subgraph $\Gamma_{\mathrm{eng,n}}(G)$ is already strongly connected.
- We prove that if $\Gamma_{\mathrm{eng}}(G)$ is strongly connected then $\Gamma_{\mathrm{eng},3}(G)$ is strongly connected, except when $G = \mathrm{PSL}_2(q)$ and $q \equiv 3 \pmod 4$.

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- When $\Gamma_{\mathrm{eng}}(G)$ is strongly connected, an interesting question is which is the smallest $n \in \mathbb{N}$ such that the subgraph $\Gamma_{\mathrm{eng,n}}(G)$ is already strongly connected.
- We prove that if $\Gamma_{\mathrm{eng}}(G)$ is strongly connected then $\Gamma_{\mathrm{eng,3}}(G)$ is strongly connected, except when $G = \mathrm{PSL}_2(q)$ and $q \equiv 3 \pmod 4$. In this case, $\Gamma_{\mathrm{eng,n}}(G)$ is strongly connected if and only if $n > a$ where 2^a is the largest power of 2 dividing $(q+1)/2$.

MAIN STEP IN THE PROOF

- Let $\pi_2(G)$ be the connected component of the prime graph of G containing 2. All the elements of G whose order is divisible by at least one prime in $\pi_2(G)$ belongs to the same strong component of $\Gamma_{\mathrm{eng}}(G)$.
- In order to prove that $\Gamma_{\mathrm{eng},2}(G)$ is strongly connected we deal with the following problem: given $1 \neq x \in G$ with $\pi(x) \cap \pi_2(G) = \emptyset$, find $1 \neq y \in G$ with $\pi(y) \cap \pi_2(G) \neq \emptyset$ and $[x, y, y] = 1$.

AN EASY EXAMPLE: $G = A_n, n \geq 8$.

There is a connected component of the commuting graph of G containing all the elements of G except, possibly, the p-cycles with p a prime and $n \in \{p, p+1, p+2\}$.

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Let $\sigma = (1, 2, \ldots, p)$, with $p > 5$.

If $\tau = (1, 3, 5)$, then $\tau \cdot \sigma \cdot \tau^{-1} \cdot \sigma^{-1} = (1, 3, 5)(2, p, 4)$, so $[\sigma, \tau, \tau] = 1$.

A MORE DIFFICULT EXAMPLE: $G = {}^2G_2(q)$

If $G = {}^2G_2(q)$ then $\pi(q + \sqrt{3q} + 1) \cap \pi_2(G) = \emptyset$.

- Let $\mathcal{C}$ be the collection of the elements of G whose order divides $q + \sqrt{3q} + 1$.
- Let ι be an involution in G and $H = C_G(\iota)$.
- We prove, using character theory, that $H\mathcal{C} = G \setminus H$.
- Hence $\iota^{\mathcal{C}}$ consists of all the involution in G different from ι.
- In particular there exists $g \in \mathcal{C}$ with $[\iota^g, \iota] = 1$ and therefore $[g, \iota, \iota] = 1$.