On characterization of a finite group by its Gruenberg-Kegel graph

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Agreements

We use the term "group" while meaning "finite group".

We use the term "graph" while meaning "undirected graph without loops and multiple edges".

A clique (resp. coclique,) with n vertices is called n-clique (resp. n-coclique).

A Frobenius group is a group G containing a proper non-trivial subgroup H such that $\forall g \in G \setminus H : H \cap H^g = 1$; H is called a Frobenius complement of G. Any Frobenius group can be represented as a semidirect product $G = F \rtimes H$, where $F = \{1\} \cup (G \setminus \cup_{g \in G} H^g)$ is a non-trivial normal subgroup of G which is called the Frobenius kernel of G.

Example 1. Let $F = F_q$ be a finite field, $|F_q^*| = q - 1$, F_q^* acts on F_q^+ as follows: $x : y \mapsto xy$. Then the semidirect product $G = F_q^+ \rtimes F_q^*$ with respect to this action is a Frobenius group with Frobenius kernel F_q^+ and Frobenius complement F_q^*.

A 2–Frobenius group is a group G which can be represented in the form $G = ABC$, where A and AB are normal subgroups of G; AB and BC are Frobenius groups with kernels A and B and complements B and C, respectively. Any 2–Frobenius group is solvable.

Definitions

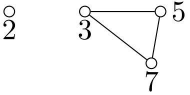
Let Ω be a finite set of positive integers.

Define $\pi(\Omega)$ to be the set of all prime divisors of integers from Ω.

Example 2. If $\Omega = \{2, 15, 21, 35\}$, then $\pi(\Omega) = \{2, 3, 5, 7\}$.

A graph $\Gamma(\Omega)$ whose vertex set is $\pi(\Omega)$ and two distinct vertices p and q are adjacent if and only if pq divides some element from Ω is called the prime graph of Ω.

Example 3. If $\Omega = \{2, 15, 21, 35\}$, then $\Gamma(\Omega)$ is the following:



Definitions

Let G be a group.

The spectrum $\omega(G)$ is the set of all element orders of G.

The prime spectrum $\pi(G)$ is the set of all prime elements of $\omega(G)$ (equivalently, the set of all prime divisors of $|G|$).

A graph $\Gamma(G)$ whose vertex set is $\pi(G)$ and two distinct vertices p and q are adjacent if and only if $pq \in \omega(G)$ is called the Gruenberg–Kegel graph or GK-graph or the prime graph of G.

It is clear that $\Gamma(G) = \Gamma(\omega(G))$.

Exercise. If p and q are primes then $pq \in \omega(G)$ IFF there exist $x, y \in G$ such that $|x| = p$, $|y| = q$ and $xy = yx$.

Groups whose Gruenberg–Kegel graphs are cocliques

Theorem (Many authors; summarized by P. Cameron and N.M., 2021+). Let G be a group. The following statements are equivalent:

(i) each element order of G is a prime power;

(ii) $\Gamma(G)$ is a coclique;

(iii) G is a group from the List A.

Groups whose Gruenberg–Kegel graphs are cocliques

A group G belongs to the List A if and only if one of the following statements hold:

(1) $|\pi(G)| = 1$ and G is a p-group;

(2) $|\pi(G)| = 2$ and G is a (solvable) Frobenius group or 2-Frobenius group;

(3) $|\pi(G)| = 3$ and $G \in \{A_6, PSL_2(7), PSL_2(17), M_{10}\}$;

(4) $|\pi(G)| = 3$, $G/O_2(G)$ is $PSL_2(2^n)$ for $n \in \{2, 3\}$, and if $O_2(G) \neq \{1\}$, then $O_2(G)$ is the direct product of minimal normal subgroups of G, each of which is of order 2^{2n} and as a $G/O_2(G)$-module is isomorphic to the natural $GF(2^n)SL_2(2^n)$-module.

(5) $|\pi(G)| = 4$ and $G \cong PSL_3(4)$.

(6) $|\pi(G)| = 4$, $G/O_2(G)$ is $Sz(2^n)$ for $n \in \{3, 5\}$, and if $O_2(G) \neq \{1\}$, then $O_2(G)$ is the direct product of minimal normal subgroups of G, each of which is of order 2^{4n} and as a $G/O_2(G)$-module is isomorphic to the natural $GF(2^n)Sz(2^n)$-module of dimension 4.

Example 4. Non-isomorphic groups having the same spectrum:

$\omega(S_5) = \omega(S_6) = \{1, 2, 3, 4, 5, 6\}$.

Example 5. Groups having distinct spectra but the same Gruenberg-Kegel graph:

$\omega(A_5) = \{1, 2, 3, 5\}$;

$\omega(A_6) = \{1, 2, 3, 4, 5\}$;

$\Gamma(A_5) = \Gamma(A_6)$:

Moreover, if $\omega(G) = \omega(A_5)$, then $G \cong A_5$ (W. Shi, 1985).

Definitions

We say that the group G is

- recognizable by its spectrum (Gruenberg–Kegel graph, respectively) if for each group H, $\omega(G) = \omega(H)$ ($\Gamma(G) = \Gamma(H)$, respectively) if and only if $G \cong H$;
- k-recognizable by spectrum (Gruenberg–Kegel graph, respectively), where k is a non-negative natural number, if there are exactly k pairwise non-isomorphic groups with the same spectrum (Gruenberg–Kegel graph, respectively) as G;
- almost recognizable by spectrum (Gruenberg–Kegel graph, respectively) if it is k-recognizable by spectrum (Gruenberg–Kegel graph, respectively) for some non-negative natural number k;
- unrecognizable by spectrum (Gruenberg–Kegel graph, respectively), if there are infinitely many pairwise non-isomorphic groups with the same spectrum (Gruenberg–Kegel graph, respectively) as G.

Theorem (V. D. Mazurov, W. Shi, 2012). Let G be a finite group. The following statements are equivalent:

(1) there exist infinitely many groups H such that $\omega(G) = \omega(H)$;

(2) there exists a finite group H with non-trivial solvable radical such that $\omega(G) = \omega(H)$.

On the Classification of Finite Simple Groups

Recall, a non-trivial group G is simple if it doesn't contain nontrivial proper normal subgroups.

A group G is almost simple with socle S, if

$$S \cong Inn(S) \trianglelefteq G \leq Aut(S),$$

where S is a non-abelian simple group. (Notation: $S = Soc(G)$.)

Simple groups were classified in 1980. In accordance with Classification of Finite Simple Groups (CFSG), non-abelian simple groups are contained in the following list:

Alternating groups: A_n for $n \geq 5$;

Classical groups: $PSL_n(q) = A_{n-1}(q)$, $PSU_n(q) = {}^2A_{n-1}(q)$, $PSp_{2n}(q) = C_n(q)$, $P\Omega_{2n+1}(q) = B_n(q)$, $P\Omega^+_{2n}(q) = D_n(q)$, $P\Omega^-_{2n}(q) = {}^2D_n(q)$;

Exceptional groups of Lie type: $E_8(q)$, $E_7(q)$, $E_6(q)$, ${}^2E_6(q)$, ${}^3D_{2n}(q)$, $F_4(q)$, ${}^2F_4(q)'$, $G_2(q)$, ${}^2G_2(q)$ (q is a power of 3), ${}^2B_2(q)$ (q is a power of 2);

26 sporadic groups.

Theorem (Many authors, still in progress). Let L be one of the following nonabelian simple groups:

(i) a sporadic group other than J_2;

(ii) an alternating group A_n, where $n \notin \{6, 10\}$;

(iii) an exceptional group of Lie type other than ${}^3D_4(2)$;

(iv) classical group of a rather large dimension.

Then every finite group H such that $\omega(H) = \omega(L)$ is isomorphic to some group G with $L \trianglelefteq G \leq Aut(L)$. In particular, there are only finitely many pairwise non-isomorphic finite groups H such that $\omega(H) = \omega(L)$.

Theorem (Many authors; finished by I. B. Gorshkov, A. N. Grishkov, 2016). If $n \neq 10$, then the recognition problem (by spectrum) is solved for $G = Sym(n)$. Moreover, if $n > 45$ and H is a finite group such that $\omega(H) = \omega(G)$, then $H \cong G$.

Theorem (V. D. Mazurov, 1997). If $G = Sz(2^7) \times Sz(2^7)$ and H is a finite group such that $\omega(H) = \omega(G)$, then $H \cong G$.

Theorem (I. B. Gorshkov, N. M., 2021). If $G = J_4 \times J_4$ and H is a finite group such that $\omega(H) = \omega(G)$, then $H \cong G$.

Theorem (I. B. Gorshkov, 2020+). If $m > 5$,

$$G = PSL_{2^m}(2) \times PSL_{2^m}(2) \times PSL_{2^m}(2),$$

and H is a finite group such that $\omega(H) = \omega(G)$, then $H \cong G$.

Theorem 1 (P. Cameron, N.M., 2022).[1] Let G be a finite group. The following statements are equivalent:

(1) there exist infinitely many groups H such that $\Gamma(G) = \Gamma(H)$;

(2) there exists a finite group H with non-trivial solvable radical such that $\Gamma(G) = \Gamma(H)$.

Note that (2) $\Rightarrow$ (1) follows form the Mazurov–Shi theorem.

[1] The work was supported by the Russian Science Foundation (project 19-71-10067).

(1) $\Rightarrow$ (2). Let G be a group such that there exist infinitely many groups H with $\Gamma(G) = \Gamma(H)$. Assume that for each group H with $\Gamma(G) = \Gamma(H)$, the solvable radical of H is trivial. The vertex 2 is non-adjacent to at least one odd vertex in $\Gamma(G)$, otherwise $\Gamma(G) = \Gamma(C_2 \times G)$, a contradiction.

Proposition 1 (A. V. Vasil'ev, 2005) Let G be a non-solvable group such that 2 is non-adjacent to at least one odd prime in $\Gamma(G)$. Then there exists a simple non-abelian group S such that

$$S \trianglelefteq G/S(G) \leq Aut(S),$$

where $S(G)$ is the solvable radical of G.

Thus, by Proposition 1, $Soc(H)$ is a non-abelian simple group such that $\pi(Soc(H)) \subseteq \pi(G)$. The following proposition gives a contradiction.

Proposition 2. Let π be a finite set of primes. The number of non-abelian simple groups S with $\pi(S) \subseteq \pi$ is finite, and is at most $O(|\pi|^3)$.

Proof of Proposition 2.

Proposition 2. Let π be a finite set of primes. The number of non-abelian simple groups S with $\pi(S) \subseteq \pi$ is finite, and is at most $O(|\pi|^3)$.

Proof. In accordance with CFSG, we consider cases.

Case S sporadic: There are clearly at most 26 such groups.

Case S alternating: The alternating group A_m has order divisible by all primes less than m. So, if $\pi(A_m) \subseteq \pi$, then m does not exceed the $(|\pi|+1)$st prime number $p_{|\pi|+1}$. By the Prime number theorem, $p_{|\pi|+1}$ is roughly $|\pi| \log |\pi|$. So the number of alternating groups does not exceed this number.

To be continue...

Proof of Proposition 2.

Zsigmondy's theorem. Let q and n be natural numbers, $q \geq 2$. There exists a prime r that divides $q^n - 1$ and doesn't divide $q^i - 1$ for $1 \leq i \leq n-1$, except for the following two cases: $q = 2$ and $n = 6$; $q = 2^k - 1$ for some prime k and $n = 2$.

Lemma 1. Let π be a finite set of primes, and $S = G_n(q)$, where $q = p^l$, be a simple group of Lie type of Lie rank n such that $\pi(S) \subseteq \pi$. Then the following statements hold:

(1) there are at most $|\pi|$ choices for p;

(2) there are at most $|\pi| + 1$ choices for l;

(3) $d(l) \leq |\pi| + 1$, where $d(l)$ is the number of pairwise distinct divisors of l;

(4) If S is a classical group, then $n \leq 2|\pi| + 3$.

Proof of Proposition 2.

Proposition 2. Let π be a finite set of primes. The number of non-abelian simple groups S with $\pi(S) \subseteq \pi$ is finite, and is at most $O(|\pi|^3)$.

Proof. …*Case S of Lie type*: These groups fall into six families $A_n(p^l)$, $B_n(p^l)$, $C_n(p^l)$, $D_n(p^l)$, ${}^2A_n(p^l)$ and ${}^2D_n(p^l)$ parametrised by rank (one parameter n) and field order (two parameters p and l), and ten families $E_6(p^l)$, $E_7(p^l)$, $E_8(p^l)$, $F_4(p^l)$, $G_2(p^l)$, ${}^2E_6(p^l)$, ${}^3D_4(p^l)$, ${}^2F_4(2^l)$, ${}^2B_2(2^l)$ and ${}^2G_2(3^l)$ parametrised by field order (two parameters p and l and for the last three, only one parameter l).
By Lemma 1, there are at most at most $|\pi|$ choices for characteristic p (except for one-parameter families), at most $|\pi|+1$ choices for l, and for classical groups at most $2|\pi|+3$ choices for their ranks. Thus, there are at most $O(|\pi|)$ groups in each of the one-parameter families, $O(|\pi|^2)$ groups in each of the two-parameter families, and at most $O(|\pi|^3)$ groups in each of the three-parameter families.

Theorem (I. Gorshkov and N. M., 2018). Let G be an almost simple group. Then the following conditions are equivalent:

(1) $\Gamma(G)$ doesn't contain a 3-coclique;

(2) $\Gamma(G)$ is isomorphic to the Gruenberg–Kegel graph of a solvable group;

(3) $\Gamma(G)$ is equal to the Gruenberg–Kegel graph of an appropriative solvable group.

Theorem (I. Gorshkov and N. M., 2018). All the almost simple groups whose Gruenberg–Kegel graphs coincide with Gruenberg–Kegel graphs of solvable groups were described.

Theorem 2 (P. Cameron, N.M., 2022).[2] Let G be a group such that G is k-recognizable by Gruenberg–Kegel graph for some non-negative integer k. Then the following conditions hold:

(1) G is almost simple;

(2) each group H with $\Gamma(H) = \Gamma(G)$ is almost simple;

(3) 2 is non-adjacent to at least one odd prime in $\Gamma(G)$;

(4) $\Gamma(G)$ contains at least 3 pairwise non-adjacent vertices.

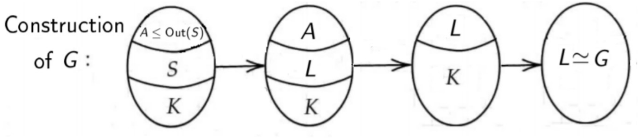
Problem 1. Let G be an almost simple group. Decide whether G is recognizable, k-recognizable for some integer $k > 1$, or unrecognizable by its Gruenberg–Kegel graph.

[2] The work was supported by the Russian Science Foundation (project 19-71-10067).

Recognition of simple groups by GK-graph

Let L be a non-abelian simple group.

Let G be a group with $\Gamma(G) = \Gamma(L)$.



Recognition of simple groups by GK-graph

- Sporadic simple groups.

  M. Hagie, 2003; A V. Zavarnitsine, 2006; A. S. Kondrat'ev, 2019 and 2021; M. Lee and T. Popiel, 2021+

- $G_2(7)$ and ${}^2G_2(q)$ for each q are recognizable by Gruenberg–Kegel graph, $PSL_3(7)$ is 2-recognizable.

  A V. Zavarnitsine, 2006

- $PSL_2(q)$, where q is a prime power.

  M. Hagie, 2003; B. Khosravi×3, 2007 (2 papers); A. Khosravi and B. Khosravi, 2008; B. Khosravi, 2008;

- $PSL_{16}(2)$ is recognizable by Gruenberg–Kegel graph. (Note that $\Gamma(PSL_{16}(2))$ is connected.)

  B. Khosravi×3, 2008
  finished by A V. Zavarnitsine, 2010

Recognition of simple groups by GK-graph

- ${}^2D_n(3)$ for odd $n \geq 5$ is recognizable by Gruenberg–Kegel graph.

  A. Babai and B. Khosravi, 2011
  M. F. Ghasemabadi, A. Iranmanesh, N. Ahanjideh, 2012

- $D_n(3)$ for even $n \geq 6$ is recognizable by Gruenberg–Kegel graph.

  M. F. Ghasemabadi and N. Ahanjideh, 2012

- $B_n(3)$ and $C_n(3)$, where $n \geq 5$ is odd, are 2-recognizable by Gruenberg–Kegel graph.

  Z. Momen and B. Khosravi, 2012
  M. F. Ghasemabadi, A. Iranmanesh, N. Ahanjideh, 2013

- $D_n(5)$ for odd $n \geq 5$ is recognizable by Gruenberg–Kegel graph.

  Z. Akhlaghi, M. Khatami, B. Khosravi, 2013;
  A. Babai and B. Khosravi, 2014

Recognition of simple groups by GK-graph

- Simple groups S such that $|\pi(S)| \in \{3, 4\}$.

  A. S. Kondrat'ev and I. V. Khramtsov, 2010 and 2012 (2 papers)

- $E_8(q)$ for $q \equiv 0, 1, 4 \pmod 5$ is almost recognizable by Gruenberg–Kegel graph.

  A. V. Zavarnitsine, 2013

- $E_7(2)$, $E_7(3)$, ${}^2E_6(2)$, $E_6(2)$, ${}^2E_6(3)$, and $E_6(3)$ are recognizable by Gruenberg–Kegel graph.

  A. S. Kondrat'ev, 2015 and 2019 (2 papers);
  W. Guo, A. S. Kondrat'ev, N. M., 2021;
  A. Khramova, N. M., V. V. Panshin, and A. M. Staroletov, in frame of GMW MCA, 2021

- Alternating and Symmetric groups.

  B. Khosravi and A. Z. Moghanjoghi, 2007;
  A. M. Staroletov, 2017;
  I. B. Gorshkov and A. M. Staroletov, 2019

Let Γ be a simple graph whose vertices are labeled by pairwise distinct primes. We call Γ a labeled graph. We denote the set of labels of Γ by $\pi(\Gamma)$.

Theorem 3 (P. Cameron, N.M., 2022).[3] There exists a function $F(x) = O(x^7)$ such that for each labeled graph Γ the following conditions are equivalent:

(1) there exist infinitely many groups H such that $\Gamma(H) = \Gamma$;

(2) there exist more then $F(|V(\Gamma)|)$ groups H such that $\Gamma(H) = \Gamma$, where $V(\Gamma)$ is the set of the vertices of Γ.

Theorem (A. V. Zavarnitsine, 2006). For any natural number k there is a group $G = G(k)$ such that the number of finite groups H with $\omega(G) = \omega(H)$ is finite and is exactly k. Thus, Theorem 3 can not be generalized.

[3] The work was supported by the Russian Science Foundation (project 19-71-10067).

Theorem 3 (P. Cameron, N.M., 2022). There exists a function $F(x) = O(x^7)$ such that for each labeled graph Γ the following conditions are equivalent:

(1) there exist infinitely many groups H such that $\Gamma(H) = \Gamma$;

(2) there exist more then $F(|V(\Gamma)|)$ groups H such that $\Gamma(H) = \Gamma$, where $V(\Gamma)$ is the set of the vertices of Γ.

Problem 2. Improve an upper bound for the number of almost simple groups with the same Gruenberg–Kegel graph.

Theorem (A. V. Zavarnitsine, 2006). There is no a constant k such that for any almost simple group G the number of pairwise non-isomorphic almost simple groups H such that $\Gamma(G) = \Gamma(H)$ is at most k.

Theorem (M. A Grechkoseeva, A. V. Vasil'ev, 2022). In Theorem 3, the upper bound can be improved to $F(x) = O(x^5)$.

Note that if G is simple, then A. V. Vasil'ev has conjectured that there are at most 4 simple groups H with $\Gamma(G) = \Gamma(H)$, see Problem 16.26 in "Kourovka Notebook".

Let G and H be simple groups. Describe all the cases when $\Gamma(G) = \Gamma(H)$.

- G is sporadic (M. Hagie, 2003).
- $G \cong A_n$ (M. Zvezdina, 2013).
- G and H are groups of Lie type and $char(G) = char(H)$ (M. Zinovieva, 2014).
- G and H are groups of Lie type and $char(G) \neq char(H)$ (M. Zinovieva, 2015+, in progress).

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Thank you!