Commutators, centralizers, and strong conciseness in profinite groups

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If G is a group, the set $\Delta(G)$ of FC-elements of G is a subgroup, and it is called the FC-center of G.

This happens because $C_G(xy) \geq C_G(x) \cap C_G(y)$ for all $x, y \in G$, so if both $C_G(x)$ and $C_G(y)$ have finite index the same holds for $C_G(xy)$.

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A group G is a FC-group if $G = \Delta(G)$.

Shalev, 1994

If G is a profinite FC-group then G' is finite, so G is finite-by-abelian.

Remark

A profinite finite-by-abelian group is abelian-by-finite.

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If G is a profinite group with restricted centralizers, then G is abelian-by-finite.

A word w on n variables is an element of the free group F with free generators $x_1, \dots, x_n$.
Given a group G, we can think of w as a function $w : G^n \mapsto G$.
We denote by G_w the set of w-values and by $w(G)$ the verbal subgroup generated by G_w.
When G is a profinite group we always mean “topologically generated”.

Recall that multilinear commutator words, also known as outer commutator words, are words obtained by nesting commutators but using always different variables.
For example the word $[[x_1, x_2], [x_3, x_4, x_5], x_6]$ is a multilinear commutator.
The lower central words γ_i are examples of multilinear commutator words. They are defined by:
$\gamma_1 = x_1, \quad \gamma_i = [\gamma_{i-1}, x_i] = [x_1, x_2, \dots, x_i]$ for $i \geq 1$.

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are not multilinear commutator words.
Shalev's result can be generalized as follows.

Theorem (Detomi, M., Shumyatsky 2020)

Let w be a multilinear commutator word and G a profinite group in which all centralizers of w-values are either finite or of finite index. Then $w(G)$ is abelian-by-finite.

We are interested. In γ_k-commutators. Recently, Shumyatsky proved the following result:

Theorem (Shumyatsky 2021)

Let $k \geq 1$ and G be a group in which $|x^G| \leq m$ for any γ_k-value $x \in G$. Then G has a nilpotent subgroup of (k, m)-bounded index and (k, m)-bounded class.

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This lead to our first result.

Proposition (Detomi, M., Shumyatsky 2022)

Let k be a positive integer and G a profinite group in which the centralizers of γ_k-commutators are either finite or open. Then G is virtually nilpotent.

Recently, uniform commutators, also called anticoprime commutators, have attracted some interest. They are commutators of the type $[g_1, g_2]$ were the orders of the elements g_1 and g_2 are divisible by the same primes. This definition makes sense also in profinite groups.

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An element g of a profinite group G is a uniform k-step commutator (u_k-commutator for short) if there are elements $g_1, g_2, \dots, g_k \in G$ such that $g = [g_1, g_2, \dots, g_k]$ and $\pi(g_1) = \cdots = \pi(g_k)$, where $\pi(g_i)$ is the set of prime divisors of the order of g_i.

Note that in a (pro-)nilpotent group every γ_k-commutator is a u_k-commutator, because elements of coprime orders commute.
The set of uniform k-step commutators of G will be denoted by $\mathcal{U}_k(G)$.

Let G be a profinite group. Then the set $\mathcal{U}_k$ generates $\gamma_k(G)$.

PROOF. Recall that $[x, y, y] = [y^{-xy}, y]$ is a uniform commutator for any $x, y \in G$, so that $[x,_k y] = [y^{-xy},_{k-1} y] \in \mathcal{U}_k$. Let $N = \langle \mathcal{U}_k \rangle$.
Obviously, $N \leq \gamma_k(G)$.
If $\bar{x} = Nx$ and $\bar{y} = Ny$ are elements of G/N, then $[\bar{x},_k \bar{y}] = 1$. Since finite Engel groups are nilpotent, we deduce that G/N is pronilpotent.
Now every γ_k-commutator in G/N is a u_k-commutator, thus it is trivial.
It follows that $\gamma_k(G) \leq N$. $\square$

THEOREM 1 (DETOMI, M., SHUMYATSKY 2022)

Let G be a profinite group in which the centralizers of uniform k-step commutators are either finite or open. Then G is virtually nilpotent and $\gamma_k(G)$ is virtually abelian.

When $k = 2$ we can be more precise.

THEOREM 2 (DETOMI, M., SHUMYATSKY 2022)

Let G be a profinite group in which the centralizers of uniform commutators are either finite or open. Then G has an open subgroup which is nilpotent of class at most 3.

We do not know if for $k > 2$ there exists a constant C, depending only on k , such that any group G satisfying the hypothesis of Theorem 1 has an open nilpotent subgroup of class at most C.

The proof of the case $k = 2$ depends on a beautiful result recently obtained by Eberhard and Shumyatsky using probabilistic methods, which implies:

THEOREM (EBERHARD, SHUMYATSKY 2021)

If G is a group in which $|x^G| \leq m$ for any commutator $x \in G$, then G has a subgroup H of nilpotency class at most 4 such that $[G : H]$ and $|\gamma_4(H)|$ are both finite and m-bounded.

As an intermediate step in our proof, we study what happens when the centralizers of nontrivial u_k-commutators are finite.

Proposition (Detomi, M., Shumyatsky 2022)

Let G be a profinite group in which the centralizers of nontrivial u_k-commutators are finite. Then G is either finite or nilpotent of class at most $k-1$.

A word w is said to be concise in a class of groups $\mathcal{X}$ if for every $G \in \mathcal{X}$

$$|G_w| < \infty \iff |w(G)| < \infty$$

In the sixties Hall raised the problem whether all words are concise, but in 1989 Ivanov solved the problem in the negative.

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A word w is called boundedly concise in a class of groups $\mathcal{X}$ if for every $G \in \mathcal{X}$

$$|G_w| \leq m \implies |w(G)| \leq f(m, w)$$

for some function f of m and w.

Every word which is concise in the class of all groups is actually boundedly concise.

Proposition (Fernández-Alcober, M. 2010)

If w is a multilinear commutator word and G is a group then

$$|G_w| \leq m \implies |w(G)| \leq f(m)$$

for some function f of m, independently of w.

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Proposition (Detomi, M., Shumyatsky 2022)

If G is a profinite group then

$$|\mathcal{U}_k| \leq m \implies |\gamma_k(G)| \leq f(m)$$

for some function f of m, independently of k.

A word w is called strongly concise if for every profinite group G

$$|G_w| < 2^{\aleph_0} \iff |w(G)| < \infty$$

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PROPOSITION (DETOMI, KLOPSCH, SHUMYATSKY 2020)

Multilinear commutator words are strongly concise.

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If G is a profinite group then

$$|\mathcal{U}_k| < 2^{\aleph_0} \iff |\gamma_k(G)| < \infty$$