Constructing the automorphism group of a finite group

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Cannon and Holt (2003): use structure of $G/O_\infty(G)$ to obtain answer, and then lift results through elementary abelian layers.

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Hard case: G finite p-group.

O'B (1993); Eick, Leedham-Green, O'B (2003).

Lower exponent-p central series

The lower p-central series of a p-group G is defined by
$\mathcal{P}_0(G) = G$
$\mathcal{P}_{i+1}(G) = [\mathcal{P}_i(G), G]\mathcal{P}_i(G)^p$ for $i \geq 0$

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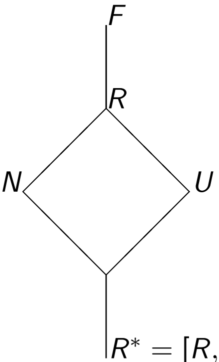
Proceed by induction down the lower p-central series:
$G_1 = G/\mathcal{P}_1(G)$ is elementary abelian of order p^d, and
$Aut(G_1) \cong GL(d, p)$.

The general framework: G_i to G_{i+1}



$R^* = [R, F]R^p$

$G_i := F/R$

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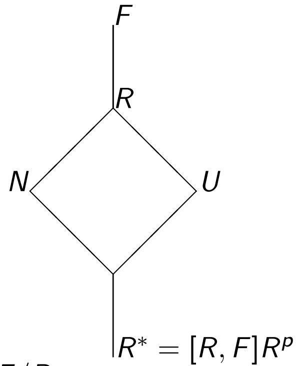

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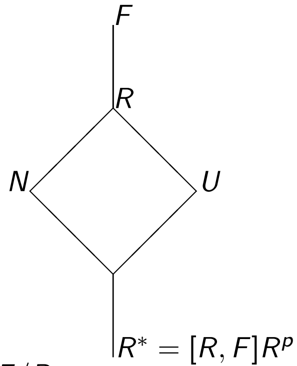

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$M := R/R^*$ the p-multiplicator, characteristic

$N := \mathcal{P}_i(P) \leq M$

$G_{i+1} := P/U$ where U supplements N.

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- $G_{i+1}/\mathcal{P}_i(G_{i+1}) \cong G_i$. Let $T_{i+1} \leq Aut(G_{i+1})$ consisting of those automorphisms which fix G_i and $\mathcal{P}_i(G_{i+1})$ – normal, elementary abelian p-subgroup of $Aut(G_{i+1})$.

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Theorem

$Aut(G_{i+1}) = A_{i+1} T_{i+1}$.

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Main task: compute the stabiliser of U in $Aut(G_i)$, where $Aut(G_i)$ acts as a group of automorphisms on M.

$M = R/[R, F]R^p$ is elementary abelian p-group.

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Standard approach: Construct the orbit of U under action of A_i and use standard orbit-stabiliser algorithm to list generators for the stabiliser.

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Central problem: Orbit is frequently too large to construct – and generating set is too large.

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Difficult cases: p-groups of small class, particularly class 2.

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How to do this? Use characteristic subgroups.

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Brooksbank & O'B (2007): construct a system of equations in matrix algebra which must be satisfied by the stabiliser, solve this system to obtain group of units.

Locating characteristic subgroups

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Generalize the N-series of Lazard: new subgroups are located via correspondences with certain graded Lie rings.

Theorem (Maglione, 2015)

Let $S \leq GL(d, q)$ be the group of upper unitriangular matrices. Adjoint refinements of lower central series of S gives a characteristic series of length $\Theta(d^2)$ with factors of order p or p^2.

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The space of alternating forms of degree d on V is naturally isomorphic with the dual vector space $(\Lambda(V))^\star$.

So can identify U with set of bilinear forms.

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Possible strategy for larger dimensional U:

- Basis of U determines set of bilinear forms.
- Construct intersection I of corresponding symplectic groups.
- Normaliser in $GL(d, p)$ of I contains stabiliser.

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Graded Algebras

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then A is $\mathbb{N}$*-graded*.

An isomorphism between graded algebras that maps each graded component of one algebra to the corresponding component of the other is a *graded isomorphism*.

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Theorem (Brooksbank; O'B; Wilson, 2020)

For each prime p and integer $n > 0$, there is a family of nilpotent matrix Lie algebras of order p^n, containing $p^{O(n^2)}$ non-isomorphic members, for which there is an $O(p^n)$ isomorphism test.

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Stabiliser of U is now limited to corresponding subgroup.

Often extremely effective in proving that only scalars stabilise U.