Commutator width of Chevalley groups and model theoretic applications

Boris Kunyavskĭ, Bar-Ilan University, Israel
Eugene Plotkin, Bar-Ilan University, Israel

ISCHIA Group Theory 2022
Ischia, Italy

June 23, 2022

"Bounded generation and commutator width of Chevalley groups: function case"
arXiv:2204.10951 [math.GR], (2022) 54 pp.

joint work of B.Kunyavskii, E.Plotkin, N.Vavilov

Special thanks go to Inna Capdeboscq

Motivation
1. Kac-Moody groups.

- **Øre problem**: Commutator width of a finite simple group is 1?

Positive answer

For fields K, $|K| \geq 8$, — Ellers and Gordeev

Over small fields $\mathbb{F}_q$, $q = 2, 3, 4, 5, 7$, — Liebeck, O'Brien, Shalev, and Pham Huu Tiep.

However:

There exist finitely generated simple groups of infinite commutator width (A. Muranov, small-cancellation methods).

There exist finitely presented simple groups of infinite commutator width (Caprace-Fujiwara). In particular, these are all irreducible (non-spherical and non-affine) Kac–Moody groups over finite fields.

Question

Is the commutator width of affine Kac-Moody groups over finite fields finite?

Implies

Question

Is the commutator width of Chevalley groups over Laurent polynomials $G(\Phi, \mathbb{F}_q[t, t^{-1}])$ finite?

2. Model theory

Theorem (D.Segal-K.Tent, 2020)

Let $G(\Phi, R)$ be a Chevalley group, rk$G \geq 2$. Let R be integral domain. Then R and $G(\Phi, R)$ are bi-interpretable provided either

- *$G(\Phi, R)$ is adjoint*
- *$G(\Phi, R)$ has finite elementary width*

Similar approach is developed in two recent papers by A.Myasnikov-M.Sohrabi.

Question

Is the commutator width of Chevalley groups over polynomials $G(\Phi, \mathbb{F}_q[t])$ and Laurent polynomials $G(\Phi, \mathbb{F}_q[t, t^{-1}])$ finite?

Notation

- Φ — reduced irreducible root system *usually* of rank ≥ 2;
- R — commutative ring with 1;
- $G(\Phi, R)$ — **Chevalley group** of type Φ over R;
- $T(\Phi, R)$ — split maximal torus in $G = G(\Phi, R)$;
- $x_\alpha(\xi)$, where $\alpha \in \Phi$, $\xi \in R$, — **elementary unipotents** with respect to $T(\Phi, R)$;
- $E(\Phi, R)$ — **elementary subgroup** in $G(\Phi, R)$:

$$E(\Phi, R) = \left\langle x_\alpha(\xi),\ \alpha \in \Phi,\ \xi \in R \right\rangle.$$

Let G be a group and X be a set of its generators. Usually one considers symmetric sets, for which $X^{-1} = X$.

- The **length** $l_X(g)$ of an element $g \in G$ with respect to X is the minimal k such that g can be expressed as the product

$$g = x_1 \dots x_k, \qquad x_i \in X.$$

- The **width** $w_X(G)$ of G with respect to X is the supremum of $l_X(g)$ over all $g \in G$.

We say that a group G has **bounded generation** with respect to X if the width $w_X(G)$ is finite.

Generating set X for $E(\Phi, R)$:

- The set of *elementary* root unipotents

$$\Omega = \{x_\alpha(\xi) \mid \alpha \in \Phi, \xi \in R\}$$

relative to the choice of a split maximal torus T;

- The set of commutators

$$C = \{[x, y] = xyx^{-1}y^{-1} \mid x \in G(\Phi, R),\ y \in E(\Phi, R)\}.$$

It is a classical theorem due to Suslin, Kopeiko and Taddei that for $\mathrm{rk}(R) \geq 2$ one indeed has $C \subseteq E(\Phi, R)$.

The width $w_\Omega(E(\Phi, R))$ is usually denoted $w_E(G(\Phi, R))$ and is called the **elementary width** of $G(\Phi, R)$.

Clearly, $w_E(G(\Phi, R))$ is the smallest such L that

$$E(\Phi, R) = E^L(\Phi, R).$$

Here, $E^L(\Phi, R)$ is the *subset* of $E(\Phi, R)$, consisting of products of $\leq L$ root unipotents.

The width $w_C(E(\Phi, R))$ is often called the **commutator width** of $G(\Phi, R)$ itself.

State of art

1. Fields, 0-dimensional rings

- **Bruhat decomposition**

$$w_E(G(\Phi, K)) \leq 2N + 4l.$$

where $N = |\Phi^+|$ is the number of positive roots, and $l = \mathrm{rk}(\Phi)$ is the rank of Φ.

- **Øre problem**:

$$w_C(G_{\mathrm{ad}}(\Phi, R)) = 1, \qquad \text{while} \qquad w_C(G_{\mathrm{sc}}(\Phi, R)) \leq 2.$$

- R — local ring. The same bound

$$w_E(G(\Phi, R)) \leq 2N + 4l.$$

as over a field.

- R — semilocal ring. Similar bound, about 1.5 times worse:

$$w_E(G(\Phi, R)) \leq 3N + 4l.$$

Follows from **Gauß decomposition** $G = DU^+U^-U^+$.

- It can be derived that

$*$ $w_C(E(\Phi, R)) \leq 3$ for $\Phi = \mathrm{A}_l$ and $\mathbf{F}_4$;

$*$ $w_C(E(\Phi, R)) \leq 4$ for all other types, apart, maybe from E_6;

$*$ $w_C(E(\Phi, R)) \leq 5$ for $G(\mathrm{E}_6, R)$.

Smolensky, 2019.

• The elementary width of rank ≥ 2 groups over a *Euclidean ring* can be infinite.

van der Kallen has proven that $SL(3, \mathbb{C}[t])$ — and thus all $SL(n, \mathbb{C}[t]$ — have INFINITE ELEMENTARY WIDTH.

Dennis and Vaserstein noticed that $SL(3, \mathbb{C}[t])$ has INFINITE COMMUTATOR WIDTH.

Now from the work of Stepanov and others we know these results are equivalent.

Bounded generation is a rare phenomenon,and, in fact, an it arithmetic question.

Dedekind rings of arithmetic type

Borderline case are 1-dimensional rings.

K is a a global field, i.e. a finite extension of $\mathbb{Q}$ in charactersistic 0, or a finite extension of $\mathbb{F}_q(t)$, $q = p^m$, in positive characteristic p.

S is finite set of valuations of K, containing all Archimedean ones in the number case.

$R = \mathcal{O}_S$ — S-integers in K = **Dedekind rings of arithmetic type**.

When K is of characteristic 0 — **number case**.

When K is of characteristic $p > 0$ — **function case**.

Number case, rank ≥ 2 groups.

• Carter and Keller, 1983–1984, proved that $\mathrm{SL}(n, R)$, $n \geq 3$, are boundedly generated.

• Tavgen, 1990, the same for all Chevalley groups of rank ≥ 2.

With good bounds depending on the type of Φ and the class number of R alone.

• Carter, Keller, Paige, 1985 — model theoretic proof, redeveloped and illuminated by Morris.

SL(2, R), for a Dedekind ring $R = \mathcal{O}_S$, with infinite multiplicative group.

• Cooke and Weinberger, 1975 —excellent bounds, **conditional**, modulo the Generalised Riemann Hypothesis.

• Liehl, 1981 — explicit unconditional bounds in some cases, grossly exaggerated.

• Vsemirnov [and Sury], 2012 — SL $\left(2, \mathbb{Z}\left[\frac{1}{p}\right]\right)$, the bound $w_E(\mathrm{SL}(2,R)) = 5$ *unconditionally*.

• Morgan, Rapinchuk and Sury, 2018:

$$w_E(SL(2,R)) \leq 9.$$

• Jordan and Zaytman, S contains at least one real or non-Archimedean valuation 2020:

$$w_E(SL(2,R)) \leq 8,$$

Dedekind rings of arithmetic type: function case

Note:

• The group $\mathrm{SL}(2, \mathbb{F}_q[t])$ is not even finitely generated.

• The groups $\mathrm{SL}(2, \mathbb{F}_q[t, t^{-1}])$ and $\mathrm{SL}(3, \mathbb{F}_q[t])$ are finitely generated but not finitely presented.

• Queen, 1975 — under some additional assumptions on R — which hold, for instance, for Laurent polynomial rings $\mathbb{F}_q[t, t^{-1}]$ with coefficients in a finite field — one has

$$w_E(\mathrm{SL}(2, R)) = 5,$$

• Nica, 2018, — bounded elementary generation of $\mathrm{SL}(n, \mathbb{F}_q[t])$, $n \geq 3$.

• Trost, 2021, — the ring of integers R of an arbitrary global function field K, with a bound of the form $L(d, q) \cdot |\Phi|$, where the factor L depends on q and of the degree d of K.

Results

Theorem (A)

Let $G(\Phi, R)$ be a simply connected Chevalley group of type Φ, $\mathrm{rk}(\Phi) \geq 2$. Then

$$w_E(G(\Phi, \mathbb{F}_q[t])) < \infty.$$

This result relies, in particular, on reduction to $A_1 \to C_2$, $C_2 \to C_3$, $B_2 \to B_3$. $A_1 \to G_2$ cases and the following theorems:

Theorem

The elementary width of $\mathrm{Sp}(4, \mathbb{F}_q[t])$ *is finite and, moreover,*

$$w_E\big(\mathrm{Sp}(4, \mathbb{F}_q[t]\big) \leq 79.$$

Theorem

The elementary width of $\mathrm{Sp}(6, \mathbb{F}_q[x])$ *is finite and, moreover,*

$$w_E\big(\mathrm{Sp}(6, \mathbb{F}_q[t]\big) \leq 72.$$

Theorem

The elementary width of $\mathrm{SO}(7, \mathbb{F}_q[x])$ *is finite and, moreover,*

$$w_E\big(\mathrm{SO}(7, \mathbb{F}_q[t]\big) \leq 65.$$

Few words about the ideas of the proof

- – Surjective stability of K_1-functor

Embedding $\Delta \subset \Phi$ implies $\varphi : K_1(\Delta, R) \to K_1(\Phi, R)$. Surjectivity of φ means $G(\Phi, R) = G(\Delta, R)E(\Phi, R)$, and this reduction is bounded.

- – Tavgen's rank reduction theorem
- – Arithmetic considerations adopted to function case like

Theorem

(Kornblum–Artin), Let a, b be relatively prime polynomials in $\mathcal{O} = \mathbb{F}_q[t]$, $\deg a > 0$. *Then there are infinitely many monic irreducible polynomials* b' *congruent to b modulo* $a\mathcal{O}$. *Moreover, such* b' *can be of arbitrary degree N, provided N is sufficiently large.*

Theorem (B)

Let $G(\Phi, R)$ be a simply connected Chevalley group of type Φ, $\mathrm{rk}(\Phi) \geq 2$ over $R = \mathbb{F}_q[t]$, Then $G(\Phi, R)$ is of finite commutator width.

Applications

Kac-Moody groups

The case $R = \mathbb{F}_q[t, t^{-1}]$ is much better than $R = \mathbb{F}_q[t]$. This yields the positive result for affine Kac-Moody groups:

Theorem (C)

The commutator width of an affine elementary untwisted Kac–Moody group $\widetilde{E}_{sc}(A, \mathbb{F}_q)$ over a finite field $\mathbb{F}_q$ is $\leq L'$, where

- $L' = 5$ *for* $\Phi = \mathrm{F}_4$ *and* $\Phi = \mathrm{A}_l$, $l = 2k+1$, $k = 0, 1, \ldots$;
- $L' = 6$ *for* $\Phi = \mathrm{A}_l$, $l = 2k$, $k = 1, 2, \ldots$, $\Phi = \mathrm{B}_l, \mathrm{C}_l, \mathrm{D}_l$, *for* $l \geq 3$ *or* $\Phi = \mathrm{E}_7, \mathrm{E}_8$, *or, finally,* $\Phi = \mathrm{C}_2, \mathrm{G}_2$ *under the additional assumption that* 1 *is the sum of two units in* R *(which is automatically the case provided* $q \neq 2$*)*;
- $L' = 7$ *for* $\Phi = \mathrm{E}_6$.

Model Theory

• – A model M of the theory T is called a **prime model** of T if it elementarily embeds in any model of T.

• – A model M of T is **atomic** if every type realized in M is principal.

• – A model M is **homogeneous** if for every two tuples $\bar{a} = (a_1, \ldots, a_n)$, $\bar{b} = (b_1, \ldots, b_n)$ in M^n that realize the same types in M there is an automorphism of M that takes $\bar{a}$ to $\bar{b}$.

• – A finitely generated group G is called **quasi-finite axiomatizable, or QFA** if the elementary theory $Th(G)$ is determined by a single formula φ, that is every finitely-generated group $H \in \mathcal{C}$ which satisfies φ is isomorphic to G.

Theorem

The groups $G = G(\Phi, \mathbb{F}_q[t])$, $\mathrm{rk}(\Phi) > 2$, *and* $G = G(\Phi, \mathbb{F}_q[t, t^{-1}])$, $\mathrm{rk}(\Phi) > 1$, *are QFA, first order rigid, prime, atomic, homogeneous. All their finitely generated subgroups are definable, and even uniformly definable. Their elementary theories are undecidable.*

The proof follows from Theorem A, Segal–Tent result, and the philosophy of rich groups introduced by Kharlampovich–Myasnikov–Sohrabi.

CONJECTURES

Conjecture

Let $E = E(\Phi, R)$ be a Chevalley group over an arbitrary commutative ring R, and $\mathrm{rank}(E) \geq 2$. Then $E = E(\Phi, R)$ is bi-interpretable with R.

This conjecture would be regarded as a far generalization of Segal-Tent's result:

Theorem (D.Segal-K.Tent, 2020)

Let $G(\Phi, R)$ be a Chevalley group, $rkG \geq 2$. Let R be integral domain. Then R and $G(\Phi, R)$ are bi-interpretable provided either

- *$G(\Phi, R)$ is adjoint*
- *$G(\Phi, R)$ has finite elementary width*

It is also genetically tied with the result of Bunina:

Theorem (Bunina, 2019)

Let $G_1 = G_\pi(\Phi, R)$ and $G_2 = G_\mu(\Psi, S)$ be two elementarily equivalent Chevalley groups. Here Φ, Ψ denote the root systems of rank $\geqslant 1$, R and S are commutative rings, and π, μ are weight lattices. Suppose that $2 \in R^*, S^*$ for A_2, B_l, C_l, and F_4, and $2, 3 \in R^*, S^*$ for G_2. Then root systems of Φ and Ψ coincide, while the rings are elementarily equivalent.

and with conjecture

Conjecture (Avni-Meiri)

Let Δ be an irreducible lattice in a semisimple group $\prod_\nu G(K_\nu)$. Then Δ is bi-interpretable with the ring of integers if and only if $\mathrm{rank}_S G \geq 2$.

Given a group G fix a word $w = w(x_1, x_2, \ldots, x_n)$ and consider the word map

$$w \colon G^n \to G. \tag{1}$$

The map w is the evaluation map: one substitutes n-tuples of elements of the group G instead of the variables and computes the value. Assume the span $\langle w(G) \rangle = G$, where span means the verbal subgroup generated by $w(H)$ is G
Let $w(G)$ be the value set of the map w in the algebra H. Denote by $w(H)^k$ the set of elements $g \in H$ of the form $g = h_1 h_2 \cdots h_k$ where $h_i \in w(H)$, $i = 1, \ldots, k$.

Definition

The smallest k such that $w(H)^k = H$ is called the w-width of the group H. Denote it by $wd_w(H)$.

Conjecture

Let R be an arbitrary commutative ring with 1, $G = G(\Phi, R)$ be a Chevalley group, $rk(\Phi) > 1$, $w(x_1, \ldots, x_n) : G^n \to G$ be such that $\langle w(G) \rangle = G$, Then, $G(\Phi, R)$ is boundedly elementary generated if and only if it is boundedly verbally generated, that is $wd_w(H) < \infty$

Proposition

Bounded elementary width implies bounded verbal width.

Proof relies on decomposition of unipotents formulas by Vavilov-Stepanov.

In the same spirit

Theorem

A Chevalley group $G(\Phi, R)$, $rk(\Phi) > 1$ is strongly boundedly generated if and only if $G(\Phi, R)$ is boundedly elementary generated.

Modulo E_8 case.