Finitely Generated Metabelian Groups Arising from Integer Polynomials

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Finitely generated metabelian groups

There is a close relation between finitely generated metabelian groups and commutative algebra, which was first observed by Philip Hall in the 1950's.

Here we describe a method for constructing certain finitely generated metabelian groups from integer polynomials.

Groups of type G_f

Let $\langle x \rangle$ be infinite cyclic and put $R = Z\langle x \rangle$. Let $f \in R$ be a non-constant, non-unit. Write (f) for the ideal generated by f. Define

$$A_f = R/(f),$$

which is a finitely generated commutative ring. Multiplication by x yields a group automorphism of R and hence of A_f, say τ. Let $T = \langle t \rangle$ be another infinite cyclic group and form the semidirect product in which $t \mapsto \tau$

$$G_f = T \ltimes A_f.$$

Remarks

1. There is no loss in assuming that f is a polynomial.

2. We are interested in how properties of the group G_f can be recognized from the form of the polynomial f.

3. Groups of type G_f occur as sections in many finitely generated metabelian groups of finite rank, so they are widespread.

Aims of the investigation

(i) Universal properties of the groups G_f.
(ii) Structure of the torsion subgroup.
(iii) Structural properties of G_f.
(iv) The centre, Fitting subgroup, Frattini subgroup.
(v) Residual properties.
(vi) Finite presentability of G_f.
(vii) The Schur multiplier $M(G_f)$.
(viii) The isomorphism problem for the groups G_f.

Example

Define $f = 2 - 3x + 8x^2$. Then our results provide the following information about the group G_f.

1. G_f is finitely generated metabelian of finite rank.
2. It is torsion-free
3. It has trivial centre and trivial Frattini subgroup.
4. It is not finitely presented, but its Schur multiplier has order 6.
5. It is a residually finite p-group if and only if $p = 7$.

Theorem 1. *Let* $f = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \in \mathbb{Z}[x]$ *where* $n > 0, a_0, a_n \neq 0$.
Then:

(i) G_f *is a finitely generated metabelian group and* A_f *has torsion-free rank* n.

(ii) *The elements of finite order form a subgroup* S *of* A_f.

(iii) S *has finite exponent equal to a* π*-number and* A_f/S *is a torsion-free abelian* π*-minimax group where* π *is the set of primes dividing* a_0a_n.

The subgroup A_f

A major obstacle to understanding the group G_f is the structure of the abelian group A_f/S. This is a torsion-free abelian group of finite rank. These are very hard to classify.

Example

Let $f = 3x^2 + x + 2$. Here A_f is torsion-free abelian of rank 2. Also it is directly indecomposable and it is not divisible by any prime.

Let $f \in \mathbb{Z}[x]$. The elements of finite order in G_f lie in A_f. Let $c = c(f)$ be the *content* of f, i.e., the gcd of the coefficients. Thus $f = ch$ where $h \in \mathbb{Z}[x]$ is *primitive*, i.e., $c(h) = 1$. Note that $h + (f) \in A_f$, has order c.

Theorem 2. *Let $f \in \mathbb{Z}[x]$ be non-constant with $f(0) \neq 0$ and let S be the torsion subgroup of G_f. Then S_p is a direct sum of cyclic groups $\mathbb{Z}_{p^{r_p}}$ where p^{r_p} is the largest power of p dividing $c(f)$. If $r_p > 0$, then S_p has infinite rank.*

Corollary 1. *G_f is torsion-free if and only if f is primitive.*

Structural properties of G_f

Theorem 3. *Let* $f = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[x]$ *where* $n > 0, a_0, a_n \neq 0$. *Then*

(i) G_f *is polycyclic if and only if* $a_0 = \pm 1$ *and* $a_n = \pm 1$;

(ii) G_f *is supersoluble if and only if* $f = (x-1)^r (x+1)^{n-r}$ *where* $0 \leq r \leq n$;

(iii) G_f *is nilpotent if and only if* $f = (x-1)^n$: *then the nilpotent class is* n;

(iv) G_f *is abelian if and only if* $f = x - 1$.

The centre

There are two possibilities for the centre of G_f:

(i) $Z(G_f) = A^T$, the set of T-fixed points in A_f;

(ii) $Z(G_f) = \langle t^m \rangle A^T$ where $m > 0$ is least such that t^m centralizes A_f.

In the second case, x^m centralizes A_f, so f divides $x^m - 1$. Thus f equals $\pm$ a product of distinct cyclotomic polynomials of orders dividing m, including Φ_m. (Note that $\mathbb{Z}[x]$ is a UFD).

The centre is usually trivial.

Theorem 4. *Let* $f \in \mathbb{Z}[x]$ *be non-constant with* $f(0) \neq 0$*. Then* $Z(G_f) \neq 1$ *if and only if* one *of the following holds:*

(i) f *equals* $\pm$ *a product of distinct cyclotomic polynomials;*

(ii) $f(1) = 0$*, i.e.,* $x - 1$ *divides* f*.*

Note that $\phi(G_f) \leq A_f$ since G_f/A_f is infinite cyclic. By standard arguments

$$\phi(G_f) = \bigcap_M M \cap A_f,$$

where the intersection is over all the maximal subgroups M of G that do not contain A_f. These $M \cap A_f$ are the maximal ideals of A_f. Since A_f is a finitely generated commutative ring, it follows from known results in commutative algebra that

$$\phi(G_f) = \mathrm{Jac}(A_f) = \mathrm{Nil}(A_f).$$

Theorem 5. *Let $f \in \mathbb{Z}[x]$ be non-constant with $f(0) \neq 0$. Write $f = c(f)f_1^{e_1}f_2^{e_2}\cdots f_r^{e_r}$ where the f_i are non-associate primitive irreducible polynomials in $\mathbb{Z}[x]$ and $e_i > 0$. In addition write $c(f) = p_1^{d_1}p_2^{d_2}\cdots p_s^{d_r}$ where the p_j are distinct primes and $d_j > 0$. Then*

$$\phi(G_f) = (h)/(f),$$

where $h = p_1 p_2 \cdots p_s f_1 f_2 \cdots f_r$. Moreover, $(h)/(f) \overset{T}{\simeq} A_k$ where $k = p_1^{d_1-1}p_2^{d_2-1}\cdots p_s^{d_s-1}f_1^{e_1-1}f_2^{e_2-1}\cdots f_r^{e_r-1}$.

The proof uses the fact that $\mathbb{Z}[x]$ is a UFD with the primes and non-associate primitive irreducible polynomials as the complete set of irreducibles.

Corollary 2. *The Frattini subgroup of G_f is trivial if and only if f is square free, i.e., it is not divisible by the square of a prime or an irreducible polynomial.*

Residual nilpotence

What conditions on f will ensure that the group G_f is residually nilpotent, i.e., $\bigcap_{i=1,2,\dots} \gamma_i(G_f) = 1$?

If $f(1)$, i.e., the sum of the coefficients of f, is not 0, then

$$(A_f)_T \simeq \mathbb{Z}[x]/(x-1)+(f) \simeq \mathbb{Z}/f(1) \simeq \mathbb{Z}_{|f(1)|},$$

while $(A_f)_T \simeq \mathbb{Z}$ if $f(1) = 0$.

If $f(1) = \pm 1$, then $(A_f)_T = 0$ and $A_f = [A_f, T]$, which means that G_f is not residually nilpotent.

The definitive result is as follows.

Theorem 6. *Let $f \in \mathbb{Z}[x]$ be non-constant with $f(0) \neq 0$. Write $f = ch$ where $c = c(f)$ and $h \in \mathbb{Z}[x]$ is primitive. Then G_f is residually nilpotent if and only if $h(1) \neq \pm 1$.*

Examples

(i) If $f = 6x^2 - 2x + 4$, then $f = 2h$ where $h = 3x^2 - x + 2$. Since $h(1) = 4$, the group G_f is residually nilpotent.

(ii) If $f = 3x^2 - x - 3$, then G_f is not residually nilpotent since $f(1) = -1$.

Finite presentability

It is natural to ask which groups G_f have a finite presentation.

Theorem 7. *Let* $f = a_0 + a_1 x + \cdots + a_n x^n \in \mathbb{Z}[x]$ *where* $n > 0$ *and* $a_0, a_n \neq 0$.

(i) *If* G_f *is finitely presented, then* $a_0 = \pm 1$ *or* $a_n = \pm 1$.

(ii) *Conversely, if* $a_0 = \pm 1$ *or* $a_n = \pm 1$, *then* G_f *has a finite presentation with two generators and* $1 + \binom{n}{2}$ *relations.*

Finite presentability

It is straightforward to prove (ii) by exploiting the special form of the polynomial f.

To establish (i) we use the Bieri-Strebel invariant

$$\Sigma_A$$

of a finitely generated $\mathbb{Z}T$-module, which was discovered by R. Bieri and R. Strebel in 1978.

Finite presentability

For finite presentability Σ_{A_f} has to be a "large" subset of the set of equivalence classes of non-zero valuations on T.

Using the results of Bieri and Strebel, we show that if $a_0 \neq \pm 1$ and $a_n \neq \pm 1$, then Σ_{A_f} is empty, contradicting the finite presentability of G_f.

What can be said about the Schur multiplier

$$M(G_f) = H_2(G_f, \mathbb{Z})$$

of the group G_f? Apply the LHS-spectral sequence for homology to the exact sequence $1 \to A_f \to G_f \to T \to 1$:

Proposition 1. *Let $f \in \mathbb{Z}[x]$ be non-constant primitive with $f(0) \neq 0$.*

(i) *If $f(1) \neq 0$, then $M(G_f) \simeq (A_f \wedge A_f)_T$.*

(ii) *If $f(1) = 0$, then $M(G_f) \simeq (A_f \wedge A_f)_T \oplus \mathbb{Z}$.*

This reduces the problem to linear algebra.

The Schur multiplier

In practice $(A_f \wedge A_f)_T$ is hard to compute for high degrees. But it is 0 for $n = 1$ and has rank 1 if $n = 2$.

Some information is available in rank $n > 2$.

Theorem 8. *Let $f \in \mathbb{Z}[x]$ be primitive with degree $n > 2$, then $M(G_f)$ is torsion-free minimax of rank $n - 2$ or $n - 1$ according as $f(1) \neq 0$ or $f(1) = 0$.*

Finitely generated multipliers

It is well known that a finitely presented group has finitely generated multiplier. However, the converse is false.

Note that $M(G_f)$ is finitely generated if and only if $(A_f \wedge A_f)_T$ is finitely generated.

In the quadratic case complete information is available.

Quadratic polynomials

Theorem 9. *Let* $f = a_0 + a_1x + a_2x^2 \in \mathbb{Z}[x]$ *be primitive with* $a_0, a_2 \neq 0$.

(i) *If* $a_0 \neq a_2$, *then* $(A_f \wedge A_f)_T \simeq \mathbb{Z}_{|a_0-a_2|}$, *which is finite.*

(ii) *If* $a_0 = a_2$, *then* $A_f \wedge A_f$ *is a trivial* T*-module and* $(A_f \wedge A_f)_T \simeq \mathbb{Q}_\sigma$ *where* σ *is the set of prime divisors of* a_0. *Thus* $(A_f \wedge A_f)_T$ *is infinitely generated unless* $a_0, a_2 = \pm 1$

Corollary 3. $M(G_f)$ *is finitely generated if and only if* $a_0 \neq a_2$ *or* $a_0 = a_2$ *and* $a_0 = \pm 1$.

Quadratic polynomials

Examples

(i) Let $f = 2 + x + 2x^2$; then $M(G_f)$ is not finitely generated.

(ii) Let $h = 2 + x + 3x^2$; then $M(G_h) = 0$, but G_h is not finitely presented.

Question

Given two non-constant polynomials $f, h \in \mathbb{Z}[x]$, with $f(0), h(0) \neq 0$, when are the groups G_f and G_h isomorphic?

(i) If f and h are associate, i.e. $h = \pm f$, then $G_f \simeq G_h$.

There is another such situation when $G_f \simeq G_h$.

(ii) Let $f = a_0 + a_1 x + \cdots + a_n x^n$. Put $s = t^{-1}$, so that $G_f = \langle t \rangle \ltimes A_f = \langle s \rangle \ltimes A_{\bar{f}}$ where $\bar{f} \in \mathbb{Z}[x]$ is the *reverse* of f, defined by

$$\bar{f} = x^n f(x^{-1}) = a_n + a_{n-1}x + \cdots + a_1 x^{n-1} + a_0 x^n.$$

Hence $G_f \simeq G_{\bar{f}}$.

The Isomorphism Problem

The solution to the Isomorphism Problem is given in the final result.

Theorem 10. *Let f and h be non-constant polynomials in $\mathbb{Z}[x]$ such that $f(0), h(0) \neq 0$. Then G_f and G_h are isomorphic if and only if $h = \pm f$ or $\pm \bar{f}$.*