On commuting probability for subgroups of a finite group

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Part 1: Introduction

The probability that two randomly chosen elements of a finite group G commute is given by

$$Pr(G) = \frac{|\{(x, y) \in G \times G \ : \ xy = yx\}|}{|G|^2}.$$

The above number is called the *commuting probability* (or the *commutativity degree*) of G.

This is a well studied concept. In the literature one can find publications dealing with problems on the set of possible values of $Pr(G)$ and the influence of $Pr(G)$ over the structure of G.

The most famous result: If G is nonabelian, then $Pr(G) \leq 5/8$.

The bound is attained in Q_8 and D_8.

In 1989 P. M. Neumann proved the following.

Theorem

Let G be a finite group such that $Pr(G) \geq \epsilon$. Then G has a normal subgroup T such that both the index $[G : T]$ and the order $|[T, T]|$ of the commutator subgroup of T are ϵ-bounded.

That is, G is bounded-by-abelian-by-bounded.

If K is a subgroup of G, write

$$Pr(K,G) = \frac{|\{(x,y) \in K \times G \ : \ xy = yx\}|}{|K||G|}.$$

This is the probability that an element of G commutes with an element of K (the relative commutativity degree of K in G).

This is a natural concept that was studied in a number of papers over the last 20 years.

We have established the following extension of P. M. Neumann's theorem. This is a joint result with Eloisa Detomi (U. of Padova).

Proposition

Let G be a finite group having a subgroup K such that $Pr(K, G) \geq \epsilon$. Then there is a normal subgroup $T \leq G$ and a subgroup $B \leq K$ such that the indexes $[G : T]$ and $[K : B]$ and the order of the commutator subgroup $[T, B]$ are ϵ-bounded.

The Neumann theorem can be obtained from this in the particular case where $K = G$.

Some remarks related to the proposition

Remark 1: The subgroup $N = \langle [T, B]^G \rangle$ has ϵ-bounded order.

Proof.

Since $[T, B]$ is normal in T, it follows that there are only boundedly many conjugates of $[T, B]$ in G and they normalize each other. Since N is the product of those conjugates, N has ϵ-bounded order. □

Remark 2: If K is normal, then B can be chosen normal while the subgroup T can be chosen in such a way that $K \cap T \leq Z_3(T)$.

Proof.

We know that $N = \langle [T, B]^G \rangle$ has ϵ-bounded order. Let $B_0 = \langle B^G \rangle$ and note that $B_0 \leq K$ and $[T, B_0] \leq N$. So we can replace B by B_0 and assume that B is normal in G.

Consider the normal series $1 \leq N \leq B \leq K$.

The centralizers of N and K/B have bounded index in G because $|N|$ and $|K/B|$ have bounded order. The centralizer of B/N has bounded index in G because T centralizes B/N. So we replace T by the intersection of T and the centralizers. Then we will have the property that $K \cap T \leq Z_3(T)$, whence the result. □

Part 2: Applications

We deduce a number of corollaries of the proposition. In the proposition K can be any subgroup. The interesting cases arise when K is an ‘important subgroup’, for example, the generalized Fitting subgroup $F^*(G)$, or a term of the lower central series of G, or a Sylow subgroup, etc.

The generalized Fitting subgroup $F^*(G)$.

Recall that the generalized Fitting subgroup $F^*(G)$ of a finite group G is the product of the Fitting subgroup $F(G)$ and all subnormal quasisimple subgroups; here a group is quasisimple if it is perfect and its quotient by the centre is a non-abelian simple group.

This plays an important role in the "local analysis".

Theorem

Let G be a finite group such that $Pr(F^*(G), G) \geq \epsilon$. Then G has a class-2-nilpotent normal subgroup R such that both the index $[G : R]$ and the order of the commutator subgroup $[R, R]$ are ϵ-bounded.

A somewhat surprising aspect of the above theorem is that the knowledge of the commuting probability of a subgroup (in this case $F^*(G)$) enables one to draw a conclusion about G as strong as in P. M. Neumann's theorem.

The subgroup $F^*(G)$ cannot be replaced here by $F(G)$ unless the group G is soluble.

Idea of the proof.

Set $K = F^*(G)$. There are normal subgroups $T \leq G$ and $B \leq K$ such that the indexes $[G : T]$ and $[K : B]$, and the order of the commutator subgroup $[T, B]$ are ϵ-bounded. As K is normal in G, the subgroup T can be chosen in such a way that $K \cap T \leq Z_3(T)$. We have $K \cap T = F^*(T)$. Therefore $F^*(T) \leq Z_3(T)$ and we conclude that $T = F^*(T)$ and so $T \leq K$. It follows that the index of K in G (and therefore the index of B in G) is ϵ-bounded. We know that the subgroup $N = [T, B]$ has ϵ-bounded order. Conclude that $R = T \cap B \cap C_G(N)$ has ϵ-bounded index in G. Moreover R is nilpotent of class at most 2 and $[R, R]$ has ϵ-bounded order. This completes the proof.

Sylow subgroups.

We also consider finite groups with a given value of $Pr(P, G)$, where P is a Sylow p-subgroup of G.

Theorem

Let P be a Sylow p-subgroup of a finite group G such that $Pr(P, G) \geq \epsilon > 0$. Then G has a class-2-nilpotent normal p-subgroup L such that both the index $[P : L]$ and the order of $[L, L]$ are ϵ-bounded.

Once we have information on the commuting probability of all Sylow subgroups of G, the result is as strong as in P. M. Neumann's theorem.

Theorem

Let G be a finite group such that $Pr(P, G) \geq \epsilon$ whenever P is a Sylow subgroup. Then G has a class-2 nilpotent normal subgroup R such that both the index of R and the order of $[R, R]$ are ϵ-bounded.

Idea of the proof: For each prime $p \in \pi(G)$ choose a Sylow p-subgroup S_p in G.

The previous theorem shows that G has a normal p-subgroup L_p of class at most 2 such that both $[S_p : L_p]$ and $|[L_p, L_p]|$ are ϵ-bounded. Since the bounds on $[S_p : L_p]$ and $|[L_p, L_p]|$ do not depend on p, it follows that there is an ϵ-bounded constant C such that $S_p = L_p$ and $[L_p, L_p] = 1$ whenever $p \geq C$. Set $R = \prod_{p\in\pi(G)} L_p$. Then all Sylow subgroups of G/R have ϵ-bounded order and therefore the index of R in G is ϵ-bounded. Moreover, R is of class at most 2 and $|[R, R]|$ is ϵ-bounded, as required.

Automorphisms.

If ϕ is an automorphism of a group G, the centralizer $C_G(\phi)$ is the subgroup formed by the elements $x \in G$ such that $x^{\phi} = x$. In the case where $C_G(\phi) = 1$ the automorphism ϕ is called fixed-point-free. A famous result of Thompson says that a finite group admitting a fixed-point-free automorphism of prime order is nilpotent. Higman proved that for each prime p there exists a number $h = h(p)$ depending only on p such that whenever a nilpotent group G admits a fixed-point-free automorphism of order p, it follows that G is nilpotent of class at most h. Therefore the nilpotency class of a finite group admitting a fixed-point-free automorphism of order p is at most h.

Khukhro obtained the following "almost fixed-point-free" generalisation of this fact:

If a finite group G admits an automorphism ϕ of prime order p such that $C_G(\phi)$ has order m, then G has a nilpotent subgroup of p-bounded nilpotency class and (m, p)-bounded index.

We establish a probabilistic variation of the above results. Recall that an automorphism ϕ of a finite group G is called coprime if $(|G|, |\phi|) = 1$.

Theorem

Let G be a finite group admitting a coprime automorphism ϕ of prime order p such that $Pr(C_G(\phi), G) \geq \epsilon$. Then G has a nilpotent subgroup of p-bounded nilpotency class and (ϵ, p)-bounded index.

Idea of the proof: Recall the well-known general fact:

$$C_{G/N}(\phi) = NC_G(\phi)/N$$

for any ϕ-invariant normal subgroup N of G.

Let $K = C_G(\phi)$. We need to show that G has a nilpotent subgroup of p-bounded nilpotency class and (ϵ, p)-bounded index.

The proposition tells us that there is a normal subgroup $T \leq G$ and a subgroup $B \leq K$ such that the indexes $[G : T]$ and $[K : B]$ and the order of the commutator subgroup $[T, B]$ are ϵ-bounded. We can assume that T is ϕ-invariant. Further, we know that $N = \langle [T, B]^G \rangle$ has ϵ-bounded order. Note that N is ϕ-invariant.

Set $D = C_G(N) \cap T$ and note that D is ϕ-invariant and has bounded index in G.

Also remark that $B \cap D \leq Z_2(D)$.

Let $\bar{D} = D/Z_2(D)$. In a natural way ϕ induces an automorphism of $\bar{D}$ and we note that $C_{\bar{D}}(\phi)$ has ϵ-bounded order because $B \cap D \leq Z_2(D)$.

The Khukhro theorem now implies that $\bar{D}$ has a nilpotent subgroup of p-bounded class and (ϵ, p)-bounded index. Since $\bar{D} = D/Z_2(D)$ and since the index of D in G is (ϵ, p)-bounded, we deduce that G has a nilpotent subgroup of p-bounded class and (ϵ, p)-bounded index.

An even stronger conclusion can be derived about groups admitting an elementary abelian group of automorphisms of rank at least 2.

Theorem

Let G be a finite group admitting an elementary abelian coprime group of automorphisms A of order p^2 such that $Pr(C_G(\phi), G) \geq \epsilon > 0$ for each nontrivial $\phi \in A$. Then G has a class-2-nilpotent normal subgroup R such that both the index R and the order of $[R, R]$ are (ϵ, p)-bounded.

One informal reason why this holds is that if Q is an A-invariant Sylow subgroup of G, we have

$$Q = \prod_{1 \neq \phi \in A} C_Q(\phi).$$

Since a random element of $C_Q(\phi)$ commutes with a random element of G with high probability, we deduce that also a random element of Q commutes with a random element of G with high probability, that is, $Pr(Q, G) \geq \epsilon_0$ for some e_0 depending only on p and ϵ.

We can use the well-known fact that A normalizes a Sylow q-subgroup of G for every $q \in \pi(G)$. Now the theorem follows from our results on Sylow subgroups.

THANK YOU!