Left 3-Engel elements in groups

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1. Introduction

Definition. An element $a \in G$ is a left Engel element, if for each $x \in G$ there exists a positive integer $n = n(x)$ such that

$$[x,_n a] = [[\ldots[x, \underbrace{a], \ldots], a]}_{n} = 1.$$

If $n(x)$ is bounded above by n then we say that a is a left n-Engel element.

Definition. An element $a \in G$ is a right Engel element, if for each $x \in G$ there exists a positive integer $n = n(x)$ such that $[a,_n x] = 1$. If $n(x)$ is bounded above by n then we say that a is a right n-Engel element.

Remark. The inverse of any right n-Engel element is a left $(n+1)$-Engel element. (Heineken)

Remark. If $\langle a \rangle^G$ is locally nilpotent then a is left Engel in G.

The converse does not hold in general (Golod's examples). But it does for certain classes like groups satisfying max (Baer) and solvable groups (Gruenberg).

Left n-Engel Elements and Groups of Prime Power Exponent

Groups of exponent 3. If G is of exponent 3 then any $a \in G$ is a left 2-Engel element. (Burnside)

Remark. For any group G, a is left 2-Engel if and only if $\langle a \rangle^G$ is abelian.

Groups of exponent 2^n. Let G be a finitely generated group of exponent 2^n. If $a \in G$ and $a^2 = 1$ then a is left $(n+1)$-Engel

$$[x,_{n+1} a] = [x, a]^{(-2)^n} = 1.$$

If every left $(n+1)$-Engel element in G is in HP(G) and $G/G^{2^{n-1}}$ is finite, then G is finite.

Remark 1. Through the work of Lysenok/Ivanov we thus see that, for n large enough, there is a left $(n+1)$-Engel element a such that $\langle a \rangle^G$ is not locally nilpotent.

Remark 2. A group G of exponent 8 is locally finite if and only if $\langle a \rangle^G$ is locally finite for all every left 4-Engel element $a \in G$.

2. Left 3-Engel elements and sandwich groups

Theorem (Abdollahi). If $a \in G$ is left 3-Engel of p-power order then $\langle a^p \rangle^G$ is locally nilpotent.

Remark. $a \in G$ is left 3-Engel if and only if $\langle a, a^g \rangle$ is nilpotent of class at most 2 for all $g \in G$. In other words if and only if $[a, a^g, a] = 1$ for all $g \in G$.

Definition (Kostrikin). An element a of a Lie algebra L is a sandwich if $axa = 0$ and $axya = 0$ for all $x, y \in L$. (The latter condition is superfluous when the characteristic is odd).

Theorem (Kostrikin, Zel'manov). Any Lie algebra L generated by finitely many sandwiches is nilpotent.

Example. Let $L = \langle a, b, c \rangle$ be the largest Lie algebra over GF(2) subject to Id(c) being abelian, $bc = 0$ and $bxb = axa = cxc = 0$ for all $x \in L$. Then

$$c\,\underbrace{ab \cdots ab}_{n} \neq 0$$

for all positive integers n. So L is not nilpotent.

Sandwich groups

Definition. A sandwich group is a group G generated by a set X of elements such that $[x, y^g, x] = 1$ for all $x, y \in X$ and all $g \in G$.

The following are equivalent:

(1) $\langle a \rangle^G$ is locally nilpotent for every (a, G) where a is Left 3-Engel in G.
(2) Every finitely generated sandwich group is nilpotent.

Theorem 1(T, 2014). Every 3-generator sandwich group is nilpotent of class at most 5. (If the generators are of odd order then the class is at most 3).

3. Left 3-Engel elements of odd order

Theorem 2A (Jabara and T, 2019). Every sandwich group, generated by finitely many sandwiches of odd order, is nilpotent.

Theorem 2B (Jabara and T, 2019). If a is a left 3-Engel element in G, where a is of odd order, then $\langle a\rangle^G$ is locally nilpotent.

Remarks on the proof of Theorem A.

Step 1. If X is a sandwich set and $x, y \in X$ then $X \cup \{[x, y]\}$ is also a sandwich set. One then obtains a sandwich set $\bar{X}$ that is closed under taking commutators.

Step 2. If $u, v, w \in \bar{X}$ then $[u, [v, w]] = [u, v, w][u, w, v]^{-1}$. Working with commutators therefore resembles working with Lie products.

Application 1 (T, 2014). A group of exponent 5 is locally finite if and only if it satisfies the law $[z, [y, x, x, x], [y, x, x, x], [y, x, x, x]] = 1$. (It follows that a group of exponent 5 is locally finite iff all its 3-generator subgroups are finite).

Application 2 (Jabara and T, 2019). If the variety of groups of exponent 9 satisfying a law $w^3 = 1$ is locally finite the same is true for the variety of groups of exponent 9 satisfying the law $(w^3x^3)^3 = 1$.

4. The question of global nilpotency of $\langle a\rangle^G$

Theorem (Newell). If a is right 3-Engel in G, then $\langle a\rangle^G$ is nilpotent of class at most 3.

Theorem 3 (Noce, Tracey and T, 2020). There exists a locally finite 2-group G with a left 3-Engel element a where $\langle a\rangle^G$ is not nilpotent.

Theorem 4 (Hadjievangelou, Noce and T, 2021). For each odd prime p, there exists a locally finite p-group G with a left 3-Engel element a where $\langle a\rangle^G$ is not nilpotent.

Remark on the proof of Theorem 4. Let $G = \langle x, a_1, a_2, \ldots\rangle$ be largest such that: (1) $\langle a_i\rangle^G$ is abelian; (2) $\langle x\rangle^G$ is metabelian; (3) $x^p = a_1^p = a_2^p = \cdots = 1$; (4) if $s \neq x$ is a comm. in $x, a_1, a_2, \ldots$, and $t(s) \geq 2$ or $t(s) \leq -2$, then $s = 1$.

(If s is a commutator of multiweight $(m, e_1, e_2, \ldots)$ in $x, a_1, a_2, \ldots$, then $t(s) = e_1 + e_2 + \cdots - 2m$).

One sees that x is left 3-Engel and that $[x, a_1, a_2, a_3, x, a_4, a_5, x, a_6, a_7, \ldots] \neq 1$.

5. Left 3-Engel involutions

Sandwich groups of rank 4. Let p be a prime and $G = \langle a, b, c, d \rangle$ be the largest sandwich group where the generators have order p.

We have $|G| = p^{28}$ and the class of G is 5 when p is odd. For $p = 2$ we have that $|G/\gamma_\infty(G)| = 2^{794}$ and class 14? (or G is infinite).

Definition. A strong sandwich group is a group G generated by a set X of elements such that $[x, y^g, x] = 1$ and $[x, y^g, z^h, x]$ for all $x, y, z \in X$ and all $g, h \in G$.

Definition. An element a is a strong left 3-Engel element in G, if $\langle a, a^g \rangle$ is nilpotent of class at most 2 and $\langle a, a^g, a^h \rangle$ is nilpotent of class at most 3 for all $g, h \in G$

Theorem 5 The largest strong sandwich group of rank 4 generated by involution is finite of order 2^{28} and class 5.

Conjecture. If a is a strong left 3-Engel element in a group G, the $\langle a \rangle^G$ is locally nilpotent.