Model Theory of Finite Groups

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First-order sentences/formulae

$(\forall x \forall y \forall z)([x, y, z] = 1)$	G nilp. of class $\leqslant 2$	Yes!
$(\forall x \in G')(\forall z)([x, z] = 1)$	G nilp. of class $\leqslant 2$	No!
$(\forall x_1 \forall x_2 \forall x_3 \forall x_4)(\exists y_1, y_2)([x_1, x_2][x_3, x_4] = [y_1, y_2])$	every element of G' is a commutator	
$(\forall x_1 \forall x_2 \exists y)(y \neq x_1 \wedge y \neq x_2)$	$\lvert G\rvert \geqslant 3$	
$(\forall x_1 \forall x_2 \forall x_3 \forall x_4)(\bigvee_{1 \leqslant i < j \leqslant 4} x_i = x_j)$	$\lvert G\rvert \leqslant 3$	
$(\forall x)(x^6 = 1 \to x = 1)$	no elements of order $2, 3$	
$g^4 = 1 \wedge g^2 \neq 1$	g has order 4	
$(\exists n)(g^n = 1)$	g has finite order	No!
$(\forall x \in G')(x^7 = 1)$	G' has exponent dividing 7	No!

Definable sets

... sets of elements $g \in G$ (or in $G^{(n)} = G \times \cdots \times G$) defined by first-order formulae, possibly with parameters from G.

Examples: • $\mathrm{Z}(G)$, defined by $(\forall y)([x, y] = 1)$

• $\mathrm{C}_G(h)$, defined by $[x, h] = 1$

• $X_h = \{[h^{-1}, h^g] \mid g \in G\}$, $\quad W_h = \bigcup\{X_{h^g} \mid g \in G, [X_h, X_{h^g}] \neq 1\}$.

• Centralizers of definable sets are definable:
Say $S = \{s \mid \varphi(s)\}$; then $\mathrm{C}_G(S) = \{t \mid \forall g(\varphi(g) \to [g, t] = 1)\}$

So $\exists$ f.o. formula ω_h with $\omega_h(g)$ iff $g \in \mathrm{C}_G\mathrm{C}_G(W_h)$

• $\delta(x, y)$: $\delta(h_1, h_2)$ iff $\mathrm{C}^2_G(W_{h_1}) = \mathrm{C}^2_G(W_{h_2})$

$\{(h_1, h_2) \mid \delta(h_1, h_2)\}$ definable in $G^{(2)}$, a definable equiv. relation

• $\exists$ $\beta(x)$: $\beta(h)$ iff $\mathrm{C}^2_G(W_h)$ commutes with its distinct conjugates.

Classes of finite groups defined by a sentence

($\exists$ only $\aleph_0$ such!)

(1) {groups of order $\leqslant n$}, {groups of order $\geqslant n$}, {groups with no elements of order n}

(2) Let $H = \{h_1, \ldots, h_n\}$ be finite, $h_i h_j = h_{\mu(i,j)}$

Mult. table gives $\theta_H(x_1, \ldots, x_n)$: $\left(\bigwedge_{i \neq j}(x_i \neq x_j) \wedge \bigwedge_{i,j}(x_i x_j = x_{\mu(i,j)})\right)$
Use it to define formulae ϕ_H, ψ_H:
$G \models \phi_H$: $\exists$ subgroup $\cong H$, $\qquad$ $G \models \psi_H$: $G \cong H$.

Sentences for non-abelian (finite) simple groups?

Hard to find a first-order sentence corresponding to

$(\forall k \neq 1)(\forall g)(\exists r \in \mathbb{N})(\exists x_1, \ldots, x_r)(g = k^{x_1} k^{x_2} \ldots k^{x_r})$.

E.g. let $k = (12)(34), g = (1\,2\, \ldots\, n)$ in A_n, n odd.
$|\operatorname{supp} k^x| = 4$, so $|\operatorname{supp} k^{x_1} k^{x_2} \ldots k^{x_r}| \leqslant 4r$, need $r \geqslant \frac{1}{4} n$.

Felgner's Theorem (1990). $\exists$ sentence σ (in the f.-o. language of group theory) such that (for G finite) $G \models \sigma \Leftrightarrow G$ is non-abelian simple.

$\sigma = \sigma_1 \wedge \sigma_2$ with

$$\sigma_1 = (\forall x \forall y)(x \neq 1 \wedge \mathrm{C}_G(x,y) \neq \{1\}$$
$$\rightarrow \bigcap_{g \in G}(\mathrm{C}_G(x,y)\mathrm{C}_G(C_G(x,y)))^g = \{1\}),$$

and

$\sigma_2 =$ 'each element is a product of κ_0 commutators' for a fixed $\kappa_0 \in \mathbb{N}$.

(Now we know that we can take $\kappa_0 = 1$:
'Yes' for the Oré conjecture (Liebeck, O'Brien, Shalev, Tiep, 2010):
all elements of non-abelian (finite) simple groups are commutators.)

σ_1 works as finite simple groups are 2-generator groups.

Ulrich Felgner

Sentences characterizing finite soluble groups:

E.g. (1) $\{\forall x \forall y (x^m = y^n = 1 = (xy)^r) \to x = 1 \mid$
$m, n, r \in \mathbb{N},$ pairwise coprime$\}$

E.g. (2) defined by 'no $g \neq 1$ is a prod. of commutators $[g^h, g^k]$'; that is, ρ_n holds $\forall n$

$$\rho_n: (\forall g \forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_n)(g = 1 \vee g \neq [g^{x_1}, g^{y_1}] \dots [g^{x_n}, g^{y_n}]).$$

Question. Is there a single sentence describing the finite soluble groups?

$$\rho_n : (\forall g \forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_n)(g = 1 \vee g \neq [g^{x_1}, g^{y_1}] \ldots [g^{x_n}, g^{y_n}]).$$

Theorem (JSW 2005) Finite G is soluble iff it satisfies ρ_{56}.

For $C \subseteq G$ write
$\delta(C) = \{[c, d] \mid c, d \in C\}$,
$\delta^n(C) = \{\text{products of } n \text{ elements of } \delta(C)\}$.

Theorem (JSW 2005). Each minimal non-soluble G has a conj. class C with $G = \delta^{56}(C)$.

Write $S = G/\Phi(G)$, $K = [\Phi(G), G]$; so $S/\mathrm{Z}(S)$ is min. simple.

$\forall$ minimal non-sol. G there's C with
(i) $G = K\delta^2(C)$; (ii) $\exists\ c_1, c_2 \in G$ with $G = \langle c_1, c_2 \rangle$.

Theorem (JSW 2005). Let $q > 8$, and V be a simple $\mathbb{F}_2\Gamma$-module where $\Gamma = \mathrm{Sz}(q)$. Then $\dim \mathrm{H}^2(\Gamma, V) \leqslant \dim V$.

The (*soluble*) *radical* $\mathrm{R}(G)$ of a finite group G is the largest soluble normal subgroup of G.

Theorem (JSW 2008). There's a f.-o. formula $r(x)$ such that if G is finite and $g \in G$ then $g \in \mathrm{R}(G)$ iff $r(g)$ holds in G.

Ultrafilters

Let I be an index set. An ultrafilter on I is a set $\mathcal{U}$ of subsets of I such that

(i) $\emptyset \notin \mathcal{U}$, (ii) if $S_1 \in \mathcal{U}$ and $S_1 \subseteq S_2$ then $S_2 \in \mathcal{U}$
(iii) if $S_1, S_2 \in \mathcal{U}$ then $S_1 \cap S_2 \in \mathcal{U}$
(iv) for each $S \subseteq I$ either $S \in \mathcal{U}$ or $I \setminus S \in \mathcal{U}$
E.g., for $x \in I$, $\{S \subseteq I | x \in S\}$ is a principal ultrafilter.
Non-principal ultrafilters $\mathcal{U}$ exist by Zorn's lemma.

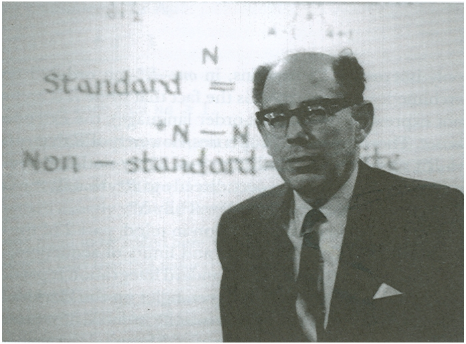
Ultraproducts of fields/nonstandard reals

Let $(F_i \mid i \in \mathbb{N})$ be a family of fields.
$C := \prod F_i$, Cartesian product containing all 'sequences' (x_i) with $x_i \in X_i$.
$I := \{ (x_i) \in C \mid \{ i \mid x_i = 0 \} \in \mathcal{U} \}$; so I is an ideal in C.
The ultraproduct $\prod F_i/\mathcal{U}$ is C/I.
(x_i), (y_i) in C have the same image in $\prod F_i/\mathcal{U}$ iff they agree on a set in $\mathcal{U}$.

Los' Theorem. If θ a first-order sentence then $\prod F_i/\mathcal{U} \models \theta$ iff $\{ i \mid F_i \models \theta \} \in \mathcal{U}$. Hence if each F_i satisfies θ, so does $\prod F_i/\mathcal{U}$.

First order in language of field theory–or ordered field theory if all F_i are ordered fields.
If all $F_i \cong F$ then constant map $f \mapsto (f)$ to C induces embedding $F \hookrightarrow C/I$. So if $F = \mathbb{R}$ then $\mathbb{R} \hookrightarrow C/I$. The element $h = (1, \frac{1}{2}, \frac{1}{3}, \dots) + I$ satisfies $nh < 1$ for all $n \in \mathbb{N}$, it's an infinitesimal.
Corollary (Robinson). Calculus without limits (Leibniz' idea, ca. 1670).

Abraham Robinson (1918–1974), developer of non-standard analysis (1960s)



Gottfried Wilhelm Leibniz (1646–1716), conceiver of infinitesimals, towering above us all

Some sentences valid for all finite groups

- $x \mapsto x^n$ injective iff $x \mapsto x^n$ surjective: $(\forall x_1 \forall x_2)(x_1^n = x_2^n \to x_1 = x_2) \leftrightarrow (\forall x \exists y)(x = y^n)$
- $C_G(x) \leqslant C_G(x^y) \to C_G(x) = C_G(x^y)$
- Higman:

$\langle x, y, z, w \mid x^y = x^2, y^z = y^2, z^w = z^2, w^x = w^2 \rangle$ is non-trivial but has no finite images $\neq 1$.
So finite groups satisfy
$(\forall a, b, c, d)(a^b \neq a^2 \vee b^c \neq b^2 \vee c^d \neq c^2 \vee d^a \neq d^2 \vee a = 1)$.

- Similarly finite groups (but not all groups) satisfy

$(\forall a, b, \alpha, \beta)(a^{2b} \neq a^3 \vee \alpha^{2\beta} \neq \alpha^3 \vee [a, b] \neq 1 \vee [\alpha, \beta] \neq 1 \vee a = b = \alpha = \beta = 1)$.

Pseudo-finite (psf) groups

... infinite models for the theory of finite groups; i.e., infinite groups satisfying all first-order sentences valid in all finite groups.

So they satisfy e.g.

- $(\forall x)(x^n = 1 \to x = 1) \to (\forall y)(\exists z)(y = z^n)$ and other f.o. 'injective $\Rightarrow$ surjective' sentences
- $\mathrm{C}_G(x) \leqslant \mathrm{C}_G(x^y) \to \mathrm{C}_G(x) = \mathrm{C}_G(x^y)$
- 'Higman sentence'.

Similarly psf fields.

Psf examples. (1) From Los', ultraproducts of finite groups are psf.

(2) For $n \geqslant 2$ and K psf, $\mathrm{SL}_n(K)$ and $\mathrm{PSL}_n(K)$ are psf; $\mathrm{PSL}_n(K)$ is simple.

If K is psf, L a Lie type and if $G \equiv \mathrm{L}(K)$, then G is simple psf.

Pseudo-finite (psf) groups

... infinite models for the theory of finite groups; i.e., infinite groups satisfying all first-order sentences valid in all finite groups.

Study of them begun by Felgner; further developed by me, Macpherson + Tent, and Ould-Houcine + Point.

Simple psf groups Let K be a psf field, L a Lie type, $G \equiv \mathrm{L}(K)$. Then $\mathrm{L}(K)$ is simple psf – e.g. $\mathrm{PSL}_2(K)$ with K psf.

Theorem (JSW 1995 (+Ryten 2007)). If G is simple psf then $G \cong \mathrm{L}(K)$ for some psf field F and Lie type L.

A psf group S is definably simple if $\nexists$ definable normal subgroups except 1, S.

Proposition (Felgner). If G is psf then G is definably simple iff $G \equiv$ an UP of finite simple groups.

Start of proof that G simple psf $\Rightarrow G \cong \mathrm{L}(K)$ with K psf and L a Lie type:

By Felgner's result, $G \equiv \prod_{i \in I} G_i/\mathcal{U}$, an UP of finite simple groups.

Easy Fact. Let $I = I_1 \cup \cdots \cup I_r$. Then (i) $I_j \in \mathcal{U}$ for some j,
(ii) $\mathcal{V} = \{ X \cap I_j \mid X \in \mathcal{U} \}$ is an ultrafilter on I_j and
(iii) $\prod_I G_i/\mathcal{U} \cong \prod_{I_j} G_i/\mathcal{V}$.

So in the UP, can assume all or none of the groups are alternating.

UPs of finite simple groups

From CFSG (together with Fact) any infinite UP of simple groups is isom. to some $\prod G_i/\mathcal{U}$ such that:

(a) $\forall i,\ G_i \cong \mathrm{Alt}(n_i)$, where $n_i \geqslant 5$; or

(b) $\forall i,\ G_i \cong {}^{\varepsilon}\mathrm{X}_{n_i}(q_i)$, where $\varepsilon \in \{1, 2, 3\}$ is fixed, $\mathrm{X} \in \{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{G}\}$ is fixed, n_i, q_i vary.

Felgner: if $\prod G_i/\mathcal{U} \equiv$ an inf. simple group then
(a) can't arise (Felgner)
in (b) the n_i are bounded (JSW); so can assume all n_i equal.

(F. Point, 1999) For each Lie type L, any UP of groups of type L is a group of type L.

Any (infinite) UP of finite simple groups of bounded rank is isom. to some $\mathrm{L}(K)$ and is psf.

Definably simple groups need not be simple

Let $I = \{ n \in \mathbb{N} \mid n \geqslant 5 \}$, $G_n = A_n$. Define

$$x_n = (12)(34), \quad y_n = \begin{cases} (1, 2, \ldots, n) & n \text{ odd}, \\ (1, 2, \ldots, n-2)(n-1, n) & n \text{ even}. \end{cases}$$

Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$, x, y the images in $\prod G_n/\mathcal{U}$ of $(x_n), (y_n)$. Suppose y a product of d conjugates of x. Equate components in some set $S \in \mathcal{U}$; so y_n is a product of d conjugates of x_n for all $n \in S$. But a product of d conjugates of x_n moves $\leqslant 4d$ points so $\neq y_n$ if $n > 4d$. Thus S is finite, contradiction.

More general (and a bit harder):

Proposition (Felgner). If $G \equiv \prod A_{n_i}/\mathcal{U}$ where $n_i \geqslant 5$ for all $i \in I$ and if G is infinite then G is not simple.

G finite: simple component = non-abelian simple subgroup S that commutes with its distinct G-conjugates ($\Leftrightarrow S$ subnormal).
G psf: definably simple component = definably simple definable subgroup that commutes with its distinct conjugates.

If G is psf, then $\mathrm{R}(G)$ and $G/\mathrm{R}(G)$ are psf or finite.

Proposition (JSW 2017). If G is psf with $\mathrm{R}(G) = 1$ then every non-triv. def. normal subgroup of G contains minimal def. normal subgroups M; each such M is $S \times \mathrm{C}_M(S)$ for a def. simple component S of G.

Theorem (2017). Let G be psf with $\mathrm{R}(G) = 1$ and with only finitely many def. simple components. Then G has a series

$$1 \leqslant G_1 \leqslant G_2 \leqslant G$$

of characteristic def. subgroups with G_1 the direct product of the (fin. many) def. simple components, G_2/G_1 metabelian, G/G_2 finite.

key: a f.-o. description of components and perfect minimal normal subgroups of finite groups.

Theorem. $\exists$ f.o. formulae $\pi(h, y)$, $\pi'(h)$, $\pi'_{\mathrm{c}}(h)$, $\pi'_{\mathrm{m}}(h)$ such that for every finite G, the direct products of simple components of G are the sets $\{\pi(h, x) \mid x \in G\}$ for the $h \in G$ satisfying $\pi'(h)$.
The simple components: the $\{\pi(h, x) \mid x \in G\}$ for which $\pi'_{\mathrm{c}}(h)$ holds.
The non-ab. min. normal subgps.: $\{\pi(h, x) \mid x \in G\}$ with $\pi'_{\mathrm{m}}(h)$.

Lemma. M a direct product of non-abelian simple groups, $X \subseteq M$, X has all projections $\neq 1$. Then $M = \langle X^g \mid g \in M, [X, X^g] \neq 1 \rangle$.

Chris Parker's nicer proof. $H := \langle X \rangle$. So $[X, X^g] \neq 1 \Leftrightarrow [H, H^g] \neq 1$.
$\langle H^g \mid g \in M \rangle \lhd M$, all projections $\neq 1$, so $\langle H^g \mid g \in M \rangle = M$. Let $K = \langle H^g \mid [H, H^g] \neq 1 \rangle$.
$\mathrm{N}_M(H)$: contains the H^g that commute with H;
permutes the H^g that don't.
So $\mathrm{N}_M(H)$ normalizes K. Thus $\langle H^g \mid g \in M \rangle \leqslant \langle K, \mathrm{N}_M(H) \rangle = \mathrm{N}_M(H)K$ and $M = \mathrm{N}_M(H)K$.
$\exists\ g_0 \in M$ with $H^{g_0} \leqslant K$.
Let $g \in M$, let $g_0 = n_0 k_0$, $g = nk$ with $n_0, n \in \mathrm{N}_M(H)$, $k_0, k \in K$.
Then $H^g = H^{n n_0^{-1} g_0 k_0^{-1} k} = H^{g_0 k_0^{-1} k} \leqslant K^{k_0^{-1} k} = K$.

Lemma. M a direct product of non-abelian simple groups, $X \subseteq M$, X has all projections $\neq 1$. Then $M = \langle X^g \mid g \in M, [X, X^g] \neq 1 \rangle$.

For $h \in G$ define

$$X_h = \{[h^{-1}, h^g] \mid g \in G\} \quad \text{and} \quad W_h = \bigcup (X_h^f \mid f \in G, [X_h, X_h^f] \neq 1).$$

Lemma. Suppose $M \leqslant G$, M a direct product of non-abelian simple groups, and M commutes with its distinct G-conjugates. Suppose $h \in M$ projects non-trivially to each simple direct factor of M. Then $M = \langle W_h \rangle$.

Proof. For $g \in G$ either $g \in \mathrm{N}_G(M)$, or M, M^g commute; so $[h^{-1}, h^g] \in M$. So $X_h \subset M$, and for $f \in G$ we have $X_{h^f} = X_h^f \subseteq M^f$; thus if $[X_{h^f}, X_h] \neq 1$ then $X_{h^f} \subseteq M$. Hence $W_h \subseteq M$.
For S a simple direct factor of M, $\exists\ s \in S$ with $[h^{-1}, h^s] \neq 1$ and clearly $[h^{-1}, h^s] \in S$. So $\{[h^{-1}, h^f] \mid f \in M\}$ of M satisfies the hypothesis on X above, and $M \subseteq \langle W_h \rangle$.

Write $\mathrm{C}_G^2(X)$ for $\mathrm{C}_G(\mathrm{C}_G(X))$ for $X \subseteq G$. So $\langle X \rangle \leqslant \mathrm{C}_G^2(X)$.

Lemma. If S is a simple component of a finite group G then $S \triangleleft \mathrm{C}_G^2(S)$.

Proof. Let N be the product of all simple components; then $N = S \times T$ with $T = \mathrm{C}_N(S) \leqslant \mathrm{C}_G(S)$ so $\mathrm{C}_G^2(S) \leqslant \mathrm{C}_G(T)$. If $c \in \mathrm{C}_G(T)$ and $S^c \neq S$ then $S^c \leqslant T$ and $S \leqslant T^{c^{-1}} = T$, contradiction.

Define δ_r for $r \geqslant 1$ recursively by $\delta_1(x_1, x_2) = [x_1, x_2]$ and $\delta_r(x_1, \ldots, x_{2^r}) = [\delta_{r-1}(x_1, \ldots, x_{2^{r-1}}), \delta_{r-1}(x_{2^{r-1}+1}, \ldots, x_{2^r})]$ for $r > 1$.

Lemma. Let S be a simple component of G. So $S \lhd \mathrm{C}^2_G(S)$. Let $S \leqslant K \leqslant \mathrm{C}^2_G(S)$. Then S is the set of δ_4-values in K.
Proof. All elements of S are commutators in S, so are δ_n-values in K for all n.
Conj. in K gives homom. $K \to \mathrm{Aut}(S)$, kernel $\mathrm{C}_K(S)$; S maps to $\mathrm{Inn}(S)$.

$\mathrm{Aut}(S)/\mathrm{Inn}(S)$ has derived length $\leqslant 3$, all δ_3-values in $\mathrm{Aut}(S)$ lie in $\mathrm{Inn}(S)$, so all δ_3-values in K lie in $S\mathrm{C}_K(S) = S \times \mathrm{C}_K(S)$.

But $\mathrm{C}_K(S) \leqslant \mathrm{C}_G(\mathrm{C}_G(S)) \cap \mathrm{C}_G(S)$, so $\mathrm{C}_K(S)$ is abelian. So every δ_4-value in K lies in S.

Begin with:

$$\begin{array}{rl} \varphi(h,x): & (\exists y)(x=[h^{-1},h^y]); \\ \psi(h,x): & (\exists t\exists y_1\exists y_2)(\varphi(h,y_1)\wedge\varphi(h^t,y_2)\wedge\varphi(h^t,x)\wedge\ [y_1,y_2]\neq 1); \\ \gamma^1(h,x): & (\forall y)(\psi(h,y)\to[x,y]=1); \\ \gamma(h,x): & (\forall y)(\gamma^1(h,y)\to[x,y]=1); \\ \alpha(h,x): & (\exists x_1\ldots\exists x_{16})((\bigwedge_{n=1}^{16}\gamma(h,x_n))\wedge(x=\delta_4(x_1,\ldots,x_{16})). \end{array}$$

$$\begin{array}{ccccc} \varphi(h,x) & \psi(h,x) & \gamma^1(h,x) & \gamma(h,x) & \alpha(h,x) \\ x\in X_h & x\in W_h, & x\in \mathrm{C}_G(W_h), & x\in \mathrm{C}^2_G(W_h), & x \text{ a } \delta_4\text{-val. in } \mathrm{C}^2_G(W_h). \end{array}$$

Now let G be finite, S a simple component. For $h\in S\setminus\{1\}$ we have $S=\langle W_h\rangle$, so $S\leqslant \mathrm{C}^2_G(W_h)$.
From Lemma $S=$ set of δ_4-values in $\mathrm{C}^2_G(W_h)$, and $S=\{x\mid \alpha(h,x)\}$.

Next steps in proofs routine.

Similar ideas (X_h, W_h, double centralizers) used for

branch groups (JSW 2015): ambient tree is often interpretable in the branch group

right-ordered permutation groups (Andrew Glass, JSW 2016):

$\mathrm{Aut}_{\leqslant}(\Lambda) :=$ group of order-preserving permutations of ordered set Λ.

If $\mathrm{Aut}_{\leqslant}(\Lambda)$ is f.-o.-equivalent (for group language) to $\mathrm{Aut}_{\leqslant}\mathbb{R}$ then Λ is isomorphic (as ordered set) to $(\mathbb{R}, \leqslant)$ or $(\mathbb{R}, \geqslant)$.

What next for psf groups?

Abelian normal subgroups in definable images, Clifford theory?

Big problem: no Sylow theory. Maybe exists for $p = 2$ using structure of dihedral groups? (Altinel, Borovik, Cherlin?)

psf G is pseudo-(finite soluble) iff satisfies ρ_{56}, same for def. subgroups.

How to recognise (pseudo-)nilpotent def. subgroups H?
E.g. $L < H$, L definable $\Rightarrow L < \mathrm{N}_H(L)$, def. normalizer condition for H???

(Carter subgroups?)

Is the Frattini subgroup pseudo-nilpotent?

The Lincoln Impossible Problem

G an abstract group. G satisfies the normalizer condition
if $H < G$ implies $H < \mathrm{N}_G(H)$.

Then so does every subgroup of G, and every f.g. subgroup of G is nilpotent. Proofs easy.

G pro-p: G satisfies the normalizer condition for closed subgroups (NCCS)
if H closed, $H < G$ implies $H < \mathrm{N}_G(H)$.

Do (closed) subgroups inherit the property?
Are f.g. pro-p groups with the property nilpotent?
• Does a free abstract group embed in a pro-p group with NCCS?

Not hard: just infinite pro-p groups with NCCS are $\cong \mathbb{Z}_p$.
Probable Theorem. If the answer to Question • is 'no' then f.g. pro-p groups with NCCS are nilpotent.

The Lincoln Imp