# An eigenvalue 1 problem in representations of finite groups of Lie type

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## Introduction and background

In the theory of transformation groups a significant role belongs to the study of invariants, that is, the objects remaining unchanged under the group action. In group representations the invariants are the underlying space vectors fixed by every group element. A more specific problem is the study of invariants for individual group elements in group representations, that is, invariants in this case are the elements of the 1-eigenspace of an element in question. Many concrete questions came from applications, whereas a systematic study is still at the infant level I think.

Project A. Given a finite group $G$ and a field $P$ determine the irreducible $P$-representations of $G$ in which every group element has eigenvalue 1.

A related project is suggested by Cullinan (J. Group Theory 2019):

Project B. Determine (maximal) finite irreducible subgroups of $GL_n(P)$ whose all elements have eigenvalue 1.

Both the projects have the same nature: given an irreducible linear group, one has to decide whether some its element does not have eigenvalue 1. However, Project A has some technical advantage as some classes of groups can be treated uniformly. Note that Project B is motivated by applications to counting points of algebraic varieties over finite fields.

For brevity we use the following notion:

Definition. A representation $\phi$ of a group $G$ is called *unisingular* if 1 is an eigenvalue of $\phi(g)$ for every $g \in G$.

Using this term the above Project A can be restated as follows:

Given a finite group $G$ and a field $P$, determine the unisingular irreducible $P$-representations of $G$.

There is a lot of results in the literature for the existence and multiplicity of eigenvalue 1 for individual group elements or certain classes of them, for instance, for $p$-elements, $p$ a prime.

In this talk I concentrate on the following part of the above project:

Problem 1. Given a finite simple group $G$ of Lie type in defining characteristic $p > 0$, determine the unisingular $p$-modular irreducible representations $\phi$ of $G$.

Guralnick and Tiep (2003) determined all finite simple group $G$ of Lie type in defining characteristic $p > 0$, whose *all* $p$-modular irreducible representations are unisingular.

**Some general facts**

Finite groups of Lie type are obtained from reductive, in particular, simple, algebraic groups as the fixed point subgroups of Frobenius endomorphisms.

Definition. Let $\mathbf{G}$ be a simple (or reductive) algebraic group defined over a field of characteristic $p > 0$. A mapping $Fr : \mathbf{G} \to \mathbf{G}$ is called a *Frobenius endomorphism* if this is surjective algebraic group homomorphism and the fixed point subgroup $G = \mathbf{G}^{Fr}$ is finite. Groups $G$ obtained in this way are called *finite groups of Lie type*.

Frobenius endomorphisms of simple algebraic groups are classified in terms of a parameter $q$, so one also writes $G = G(q)$. In most cases $q$ is a $p$-power (the exceptions are the series of groups $G$ known as ${}^2B_2(q), {}^2F_4(q)$, $p = 2$, and ${}^2G_2(q), p = 3$ where $q^2$ is an odd $p$-power).

A fundamental result of the representation theory of groups of Lie type tells us that every irreducible $P$-representation of $G$ is the restriction to $G$ of a representation of $\mathbf{G}$. This allows one to use the algebraic group representation theory for an analysis of Problem 1.

The most important subgroup of $\mathbf{G}$ is a maximal torus $T$, say, which contains a conjugate of every $p'$-element of $\mathbf{G}$ (and hence of $G$). The maximal tori are conjugate so the choice of $T$ is immaterial for our purpose, we fix one of them. A significant part of the representation theory of $\mathbf{G}$, and even of $G$, is described in terms of $T$.

Let $\phi$ be a $P$-representation of $\mathbf{G}$. Then the group $\phi(T)$ is completely reducible, and the irreducible constituents of $\phi(T)$ are called *weights* of $\phi$ (and of the respective $P\mathbf{G}$-module). The weights are the rational homomorphisms $T \to P^{\times}$; they are elements of Hom$\,(T, P^{\times})$, the set of all algebraic group homomorphisms $T \to P^{\times}$. Set $\Omega =$Hom$\,(T, P^{\times})$ and let $\Omega(\phi)$ be the set of weights of $\phi$. Then $\Omega \cong \mathbb{Z}^n$, the integral lattice of finite rank $n$, which is also called *the rank of* $\mathbf{G}$.

One singles out a special basis $\omega_1, \ldots, \omega_n$ of $\Omega$ whose elements are called *fundamental weights* of $\mathbf{G}$. So every $\omega \in \Omega$ is of the form $\sum z_i \omega_i$, $z_i \in \mathbb{Z}$. The zero weight is just the zero element of $\Omega$. The weights $\sum z_i \omega_i$ with $z_i \geq 0$ are called *dominant*; they form a subset $\Omega^+$ of $\Omega$. The non-zero weights of the adjoint representation are called *the roots*; they span the root lattice $R$.

Another key fact of general theory is that there is a bijection $\beta : \mathrm{Irr}_P(\mathbf{G}) \to \Omega^+$. The element $\beta(\rho)$ for $\rho \in \mathrm{Irr}_P(\mathbf{G})$ is called the *highest weight of* $\rho$. Moreover, if $q$ is a prime power then the irreducible representations of $G(q)$ over $P$ are in bijection with

$$\Omega_q^+ := \{\sum z_i \omega_i,\ \ 0 \leq z_i < q.\}$$

This provides a parametrization of $\mathrm{Irr}_P(G(q))$.

## The zero weight existence

The existence of weight 0 in a representation $\phi$ of $\mathbf{G}$ is equivalent to saying that the trivial representation $1_T$ is a constituent of $\phi|_T$. So if $0 \in \Omega(\phi)$ then every element of $G$ has eigenvalue 1. This naturally leads to the following problem:

Problem 2. Determine (in terms of highest weights) the irreducible representations $\phi$ of $\mathbf{G}$ such that $0 \in \Omega(\phi)$.

The situation differs dramatically on whether $\phi$ is tensor-decomposable or not. If not, then (due to Premet's theorem (1988)) $0 \in \Omega(\phi)$ if and only if the highest weight of $\phi$ lies in $R$, the root lattice (with some exceptions; the exceptional cases were settled later (2009)).

There is a simple algorithm for verifying this condition, so in this case the solution to Problem 2 is known.

If $\phi$ is tensor-decomposable then the problem is in general open, and I cannot expect any simple solution. However, for the case where $\phi$ is a tensor product of two multiples (each is tensor-indecomposable) an efficient solution has been obtained in a recent paper by Baranov and me (J. Algebra and Appl. 2021).

If $\phi$ is a tensor product of three or more non-trivial factors then no precise result is known. It will be useful to have a non-trivial sufficient condition for $\phi$ to have weight zero, however, I have no reasonable suggestion at the moment.

## Sufficient conditions for unisingularity

As explained above, an irreducible representation of $G(q)$ extends to a representation $\phi$ of $\mathbf{G}$, and the existence of weight 0 in $\phi$ is a useful sufficient condition for unisingularity of $\phi|_{G(q)}$.

Some other sufficient conditions for unisingularity of irreducible representations of classical groups (other than unitary) was obtained in my joint paper with Cullinan (European J. Math. 2021). These have been obtained in an attempt to decide whether the Steinberg representation of a finite simple group $G$ of Lie type is unisingular.

Recall that the Steinberg representation $St$ of $G$ is an irreducible representation over the complex numbers (in fact over $\mathbb{Z}$) whose degree is $|G|_p$, the $p$-part of $|G|$. It is of $p$-defect 0 and hence $St_p = St \pmod p$, the reduction of $St$ modulo $p$, is an irreducible $P$-representation of $G$.

Question. Is $St_p$ unisingular?

If $p > 2$ then the answer is always positive; this has been observed in a paper of mine of 1990. For $p = 2$ the question remained open; the following result (obtained in the above mentioned joint paper with Cullinan) describes the current state of it:

Theorem. Let $G$ be a finite simple group of Lie type in characteristic 2 and $St_2$ the 2-modular Steinberg representation of $G$.
Suppose that $G$ is not of type $A_1(q) \cong PSL_2(q)$, ${}^2A_n(q) \cong PSU_{n+1}(q)$ ($n$ odd), or $E_7(q)$.
Then $St_2$ is unisingular.

For $G = A_1(q) = PSL_2(q)$ the answer to the above question is negative. For groups ${}^2A_n(q)$ ($n$ odd) and $E_7(q)$ with $q$ even it remains open, even for $q = 2$. This is somehow surprising in view of the fact there is a nice formula for the character of $St$, and hence for the Brauer character of $St_2$.

For $G = SL_n(2)$ and $Sp_{2n}(2)$ there are precise criterions for a 2-modular irreducible representation of $G$ to be unisingular obtained in my papers in Archiv der Math. (2018) and Comm. in Algebra (2022). So for these groups there is a final solution to Problem 1.

Theorem. Let $G = SL_{n+1}(2)$ and let $\phi$ be a non-trivial irreducible 2-modular representation with highest weight $z_1\omega_1 + \cdots + z_n\omega_n$. Then $\phi$ is unisingular if and only if $\sum_i a_{n+1-i}i \geq n$ and $\sum_i a_i i \geq n$.

The unisingularity condition for irreducible representations of $Sp_{2n}(2)$ is more complex and I do not state it here.

I have some progress for orthogonal groups over $\mathbb{F}_2$, the field of two elements. However, the case of unitary groups $SU_{n+1}(2)$, $n$ odd, seems to be much more difficult to handle.

Finally I wish to claim that Projects A,B open a wide area of research which naturally splits to many special cases of particular interest. For instance, very little is known for the case $P = \mathbf{C}$, the complex number field. Cullinan considered Project B for $n = 3$ and my joint paper with him contains some results for $n = 8$.

## Some references


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