Automorphism groups and Lie algebras of vector fields of affine varieties.

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<u>Motivation</u>. Let Φ be an associative commutative ring with 1,

V = affine algebraic variety over Φ = = zeros of an ideal $I \triangleleft \Phi[x_1, \dots x_n]$,

$A = \Phi[x_1, \dots, x_n]/\Phi$ is the Φ-algebra of polynomial (regular) functions on V.

$Aut(A)$ = the group of polynomial automorphisms of V,

$\mathrm{Der}(A)$ = Lie algebra of vector fields on V.

Let V be irreducible, i.e. A is a domain.

K = field of fractions of A.

$\mathrm{Aut}(K)$ = the group of birational automorphisms of V,

$\mathrm{Der}(K)$ = the Lie algebra of rational vector vector fields.

If $A = \Phi[x_1, \ldots, x_n]$ then

$\mathrm{Der}(K) = \{ \sum_{i=1}^{n} f_i(x_1, \ldots, x_n) \partial/\partial x_i$; $f_i(x_1, \ldots, x_n)$ are rational functions$\}$

$A = \mathbb{C}[x_1, \dots, x_n]$, $\mathbb{C}$ complex numbers, a particularly classical case.

Cremona group, polynomial Cremona group, Cremona Lie algebra.

Aut(A) is, generally speaking, not linear, i.e. not embeddable in the group of matrices over a commutative ring.

Aut(A)

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Linear Groups

Der(A) is, generally speaking, infinite dimensional.

Old Program (explicitely formulated by J.-P. Serre):

which properties of linear groups extend to $\mathrm{Aut}(A)$?

Some Highlights.

Jordan Theorem: G finite subgroup of $GL(n, \mathbb{C})$. Then $G \triangleright H$, $|G : H| \leq J(n)$, H is abelian.

Minkowski Theorem: G finite subgroup of $GL(n, \mathbb{Z})$. Then $|G| \leq M(n)$.

V. Popov: G finite subgroup of $\mathrm{Aut}(A) \overset{?}{\Longrightarrow} G \rhd H$, H is abelian, $|G:H| \leq f(A)$.

$A = F[x_1, \ldots, x_n]$, true for the Cremona group $Cr(n) = \mathrm{Aut}\, F(x_1, \ldots, x_n)$.

Prokhorov – Shramov, C. Birkar.

For particular varieties: Popov, Zarkhin, Bandman, et al, et al.

In full generality: OPEN.

J.-P. Serre, 2009: finite subgroups of $Cr(2)$.

S. Cantat, 2011: Tits Alternative for $Cr(2)$.

Tits Alternative: every finitely generated subgroup is either virtually solvable or contains F_2.

For $n \geq 3$: OPEN.

$Cr(2)$ is very different from $Cr(n)$, $n \geq 3$.

I. Shestakov, U. Umirbaev, 2004:

$\mathrm{Aut}\, F[x_1, \dots, x_n]$, $n \geq 3$, is much more complicated than $\mathrm{Aut}\, F[x_1, x_2]$. There exist wild automorphisms.

We extend

1) Selberg's and Burnside's theorems to $Aut(A)$;

2) Engel Theorem to $Der(A)$.

<u>Selberg Theorem</u>: G finitely generated subgroup of $GL(n, F)$, char $F = 0 \Rightarrow$ G is virtually torsion free.

virtually = there is a subgroup of finite index that is torsion free.

Alperin's version of Selberg's Theorem:

A a finitely generated domain without

additive torsion $\Rightarrow GL(n, A), n \geq 2$, is virtually torsion free.

Burnside - Schur: a torsion subgroup of $GL(n, F)$ is locally finite.

Theorem 1. Let A be a finitely generated commutative Φ-algebra. Suppose that A does not have additive torsion.

Then

1) every finitely generated subgroup of $Aut(A)$ is virtually torsion free,

2) if A is finitely generated as a ring (that is, $\Phi = \mathbb{Z}$), then $Aut(A)$ is virtually torsion free.

Theorem 2. Let A be a finitely generated commutative Φ-algebra without additive torsion. Then every torsion subgroup of $\mathrm{Aut}(A)$ is locally finite.

Corollary 1. $A = F[x_1, \ldots, x_n]$, char $F = 0$. Then every torsion subgroup of $\mathrm{Aut}(A)$ is virtually abelian.

Corollary 2. If A is a finitely generated ring then every torsion subgroup is finite.

Extension to PI-algebras.

A, an algebra over a commutative ring Φ, $\Phi\langle x_1,\dots,x_n\rangle$ free associative algebra.

<u>Def.</u> A is PI if there exists

$$0\neq f(x_1,\dots,x_n)=x_1\cdots x_n+\sum_{\substack{\sigma\neq 1\\ \alpha_\sigma\in\Phi}}\alpha_\sigma x_{\sigma(1)}\cdots x_{\sigma(n)}:$$

$$f(a_1,\dots,a_n)=0\quad\forall a_1,\dots,a_n\in A.$$

<u>Amitsur-Levitzki</u>: Φ associative commutative $\Rightarrow M_n(\Phi)$ is PI.

A is <u>representable</u> if $A\hookrightarrow M_n(\Phi)$, Φ is associative & commutative.

Semiprime PI $\Rightarrow$ representable.

L. Small, 1972: there exists a finitely generated PI-algebra that is not representable.

Theorem 1'. Theorem 1 for representable PI-algebras.

Theorem 2'. Theorem 2 for all PI-algebras.

These results hold for some algebras that are far from being PI.

Proposition. Let char $F = 0$. Then a finitely generated subgroup of $\mathrm{Aut}\, F\langle x_1, \dots, x_n\rangle$ is virtually torsion free.

What about positive characteristics ?

<u>O. Mathieu</u> (very recently) : F a finite field, then $\mathrm{Aut}\, F[x_1, x_2]$ is linear.

~~Ex~~ Expected : $\mathrm{Aut}\, F[x_1, x_2, x_3]$ is not linear.

Let $\mathrm{char}\, F = p > 0$.

$\mathrm{Aut}^1 F[x_1, \dots, x_n] = \{$automorphisms $x_i \to x_i + O(2)$, $1 \le i \le n\}$,

the congruence subgroup of $\mathrm{Aut}\, F[x_1, \dots, x_n]$.

It is a residually - p group.

<u>Question</u>: does $\mathrm{Aut}^1 F[x_1, \dots, x_n]$ satisfy a pro - p identity ?

My feeling: for $n \geq 3$ NO.

Lie Algebras.

A derivation $d: A \to A$ is locally nilpotent if $\forall a \in A \ \exists\, n(a) \geq 1$:

$$d^{n(a)}(a) = 0.$$

Motivation: $\exp(d)$ is an automorphism.

$L \subset \mathrm{Der}(A)$ Lie algebra that consists of locally nilpotent derivations.

Question: is L locally nilpotent?

A. Petravchuk – K. Sysak, 2017: A domain/ algebraically closed field of zero characteristic, $\dim_F L < \infty \Rightarrow L$ is nilpotent.

A. Skutin, 2021 : $A = F[x_1, \dots, x_n]$,

char $F = 0 \Rightarrow$ YES.

Theorem 3. Let A be a finitely generated commutative algebra over an associative commutative ring. Let $L \subseteq Der(A)$ be a Lie subalgebra that consists of locally nilpotent derivations. Then the Lie algebra is locally nilpotent.

The assumption of finite generation in Theorems 1,2,3 is essential.

Rational vector fields.

A a commutative domain, $K =$ field of fractions of A,

$$\mathrm{Der}(A) \subseteq \mathrm{Der}(K), \quad K\,\mathrm{Der}(K) \subseteq \mathrm{Der}(K)$$

Theorem 4. Let A be a commutative domain, $K =$ field of fractions of A. Let $L \subseteq \mathrm{Der}(A)$ be a Lie ring that consists of locally nilpotent derivations. Suppose that $\dim_K KL < \infty$. Then the Lie ring L is locally nilpotent.

COROLLARY. (rational vector fields). Let A be a finitely generated domain, let $L \subseteq \mathrm{Der}(K)$ consist of locally nilpotent derivations. Then L is locally nilpotent.

PI-algebras.

Theorem 3 holds for finitely generated PI-algebras.