Hall Classes of Groups

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Camille Jordan

January 5, 1838 – January 22, 1922

The Nilpotency criterion of Philip Hall (1958)

Let G be a group containing a nilpotent normal subgroup N such that G/N' is nilpotent. Then also G is nilpotent.

Philip Hall (1904–1982)

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A few years later Boris Plotkin (1961) proved that also locally nilpotent groups form a Hall class

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In fact, any Hall class containing $\mathfrak{A}$ must contain all nilpotent groups

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This is for instance the case of the class $\mathfrak{S}$ of all soluble groups and of most of the natural classes of generalized soluble groups

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but a short and elegant proof was later given
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If G is any group and k is any positive integer,
there exists a G-epimorphism

$$(G/G') \otimes \big(\gamma_k(G)/\gamma_{k+1}(G)\big) \longrightarrow\!\!\!\!\!\rightarrow \gamma_{k+1}(G)/\gamma_{k+2}(G)$$

where $\gamma_i(G)$ is the i-th term of the lower central series of G,
for each integer $i \geqslant 1$, and each factor
is considered as a G-module by conjugation

To obtain Hall's nilpotency criterion, it is enough to notice that an abelian normal subgroup A of a group G is contained in $\zeta_k(G)$ for some positive integer k if and only if it is polytrivial as a G-module (by conjugation), and that the class of polytrivial G-modules is closed with respect to tensor products and homomorphic images

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Here a G-module A is called *polytrivial*
if it admits a chain of submodules

$$\{0\} = A_0 < A_1 < \ldots < A_k = A$$

each of whose factors is a trivial G-module

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A group G is *paranilpotent* if it has a finite normal series
$\{1\} = G_0 < G_1 < \ldots < G_t = G$
such that each G_{i+1}/G_i is abelian
and all its subgroups are G-invariant

Of course, supersoluble groups are paranilpotent
and any paranilpotent groups is hypercyclic

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• Engel groups

A group G is a *Engel group* if for all $x, y \in G$ there is a positive integer $n = n(x, y)$ such that $[x,_n y] = 1$

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be group classes. We denote by $\mathfrak{X}\mathfrak{Y}$ the class of all groups G containing a normal subgroup N such that $N \in \mathfrak{X}$ and $G/N \in \mathfrak{Y}$

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The condition $\mathfrak{X} = \mathbf{N}_0\mathfrak{X}$ means of course that a theorem of Fitting type holds for the class $\mathfrak{X}$, i.e. in any group the product of two normal $\mathfrak{X}$-subgroups belongs to $\mathfrak{X}$

If we choose as $\mathfrak{Y}$ the class $\mathfrak{F}$ of all finite groups,
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More in general which classes of the form $\mathfrak{X}\mathfrak{Y}$,
where $\mathfrak{Y}$ is a Hall class but $\mathfrak{X}$ is a class producing
a sort of obstruction at the bottom, have the Hall property?

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If q is any prime number, there exists a soluble group G,
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and is not finite-by-(locally nilpotent), but contains
a unipotent normal subgroup N which is nilpotent of class 2
and such that G/N' is finite-by-abelian

The answer for the class $\mathfrak{F}\mathfrak{N}$ is in general negative, even within the universe of linear groups

If q is any prime number, there exists a soluble group G, which is linear over a field of characteristic q and is not finite-by-(locally nilpotent), but contains a unipotent normal subgroup N which is nilpotent of class 2 and such that G/N' is finite-by-abelian

This example shows that none of the relevant Hall classes mentioned above produces a new Hall class when a finite obstruction at the bottom occurs

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of all finite-by-soluble groups is a Hall class

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But we have constructed a group G, which is linear of degree 8 over a field of characteristic 2 and is not finite-by-soluble, but contains a nilpotent normal subgroup N such that G/N' is finite-by-abelian

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Theorem

Let G be a linear group over a field $\mathfrak{K}$ and let N be a nilpotent normal subgroup of G such that G/N' is finite-by-nilpotent. Then G is finite-by-nilpotent provided one of the following conditions holds:

(a) $\mathfrak{K}$ has characteristic 0;

(b) N or G is connected;

(c) N_u is connected or abelian.

A similar result can be proved if the class $\mathfrak{N}$ is replaced by other relevant Hall classes, like for instance those of hypercentral, hypercyclic and paranilpotent groups

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Let $\mathfrak{X}$ be a group class such that $\mathfrak{N} \leqslant \mathfrak{X} = \mathbf{S}\mathfrak{X} = \mathbf{Q}\mathfrak{X} = \mathbf{R}_0\mathfrak{X}$ and suppose that $\mathfrak{X}$ contains all (finite central)-by-$\mathfrak{X}$ groups. If all linear $\mathfrak{X}$-groups are nilpotent-by-finite and $\mathfrak{X}\mathfrak{F}$ is a Hall class in a suitable universe of linear groups, then also $\mathfrak{F}\mathfrak{X}$ is a Hall class in the same universe.

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This is a consequence of a general result, for which we need the following definition.

If G is a group and $\mathcal{M}$ is a class of G-modules, we say that a normal section H/K of G is $P_{\mathfrak{F}}(\mathcal{M})$ if it has a finite G-invariant series each of whose infinite factors is abelian and belongs to $\mathcal{M}$, when regarded as G-module by conjugation

Lemma

Let G be a group and let $\mathcal{M}$ be a tensorial class of G-modules which is closed with respect to forming submodules and whose members are finitely generated as $\mathbb{Z}$-modules. If G contains a nilpotent normal subgroup N such that G/N' is $P_{\mathfrak{F}}(\mathcal{M})$, then G itself is $P_{\mathfrak{F}}(\mathcal{M})$.

Lemma

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Another consequence of this lemma is that finite-by-supersoluble groups form a Hall class

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In some sense, there is a strange dichotomy between the classes $\mathfrak{F}\mathfrak{X}$ and $(\mathbf{L}\mathfrak{F})\mathfrak{X}$ with respect to the Hall property, and very often it turns out that one is a Hall class if and only if the other is not

As a first evidence of this phenomenon, we note that the class of all (locally finite)-by-(finitely generated nilpotent) groups is not a Hall class (even in the universe of linear groups)

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There exists a torsion-free nilpotent linear group G over a field of characteristic 0 which is not finitely generated, but G/G' is periodic-by-(finitely generated abelian)

Theorem

The class $(\mathbf{L}\mathfrak{F})\mathfrak{X}$ of all groups which are (locally finite)-by-$\mathfrak{X}$ is a Hall class, for any choice of $\mathfrak{X}$ as the class of nilpotent, hypercentral, locally nilpotent, paranilpotent, hypercyclic, or locally supersoluble groups

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Each of the proofs involves considerations on tensor products and needs information on the possibility of switching factors of certain types in a finite normal series. A typical example is the following lemma for the class $(\mathbf{L}\mathfrak{F})\mathfrak{N}$

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A group is (locally finite)-by-nilpotent if and only it has a normal series of finite length whose factors are either locally finite or central

AGTA Workshop - Reinhold Baer Prize 2022

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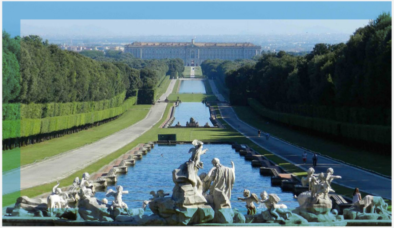

The Reinhold Baer Prize is dedicated to the great German algebraist who built an important school of group theory. The aim of the prize is to support the scientific activity of young researchers, regardless of their geographical origin. The prize was issued in 2017, and the winner of this fifth edition will be announced during this workshop.

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