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Groups in which every non-abelian subgroup is self-centralizing

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Abstract

We study groups having the property that every non-abelian subgroup contains its centralizer. We describe various classes of infinite groups in this class, and address a problem of Berkovich regarding the classification of finite p-groups with the above property.

Keywords: centralizer, non-abelian subgroup, self-centralizing subgroup

1. Introduction

A subgroup H of a group G is *self-centralizing* if the centralizer $C_G(H)$ is contained in H. Clearly, an abelian subgroup A of G is self-centralizing if and only if $C_G(A) = A$. In particular, the trivial subgroup of G is self-centralizing if and only if G is trivial.

The structure of groups in which many non-trivial subgroups are self-centralizing has been studied in several papers. In [5] it has been proved that a locally graded group (that is, a group in which every non-trivial finitely

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generated subgroup has a proper subgroup of finite index) in which all non-trivial subgroups are self-centralizing has to be finite; therefore it has to be cyclic of prime order or a non-abelian group whose order is a product of two different primes. The papers [5] and [6] deal with the class $\mathfrak{C}$ of groups in which every non-cyclic subgroup is self-centralizing; in particular, a complete classification of locally finite $\mathfrak{C}$-groups is given.

In this paper, we study the class $\mathfrak{A}$ of groups in which every non-abelian subgroup is self-centralizing. We note that the class $\mathfrak{A}$ is fairly wide. Clearly, it contains all $\mathfrak{C}$-groups. It also contains the class of commutative-transitive groups (that is, groups in which the centralizer of each non-trivial element is abelian), see [13]. Moreover, by definition, the class $\mathfrak{A}$ contains all minimal non-abelian groups (that is, non-abelian groups in which every proper subgroup is abelian); in particular, Tarski monsters are $\mathfrak{A}$-groups.

The structure and main results of the paper are as follows. In Section 2 we derive some basic properties of $\mathfrak{A}$-groups; these results are crucial for the further investigations in the subsequent sections. In Section 3 we consider infinite nilpotent $\mathfrak{A}$-groups; for example, we prove that such groups are abelian, which reduces the investigation of nilpotent $\mathfrak{A}$-groups to finite p-groups in $\mathfrak{A}$. Infinite supersoluble groups in $\mathfrak{A}$ are classified in Section 4; for example, we prove that if such a group has no element of order 2, then it must be abelian. In Section 5 we discuss some properties of soluble groups in $\mathfrak{A}$. Lastly, in Section 6, we consider finite $\mathfrak{A}$-groups, and we derive various characterisations of finite groups in $\mathfrak{A}$. Motivated by Section 3, we focus on finite p-groups in $\mathfrak{A}$; Problem 9 of [1] asks for a classification of such groups. This appears to be hard, as there seem to be many classes of finite p-groups that belong to $\mathfrak{A}$. We show that all finite metacyclic p-groups are in $\mathfrak{A}$, and classify the finite p-groups in $\mathfrak{A}$ which have maximal class or exponent p.

2. Basic properties of $\mathfrak{A}$-groups

We collect some basic properties of $\mathfrak{A}$-groups. Since every free group lies in $\mathfrak{A}$, the class $\mathfrak{A}$ is not quotient closed. On the other hand, $\mathfrak{A}$ obviously is subgroup closed. Similarly, the next lemma is an easy observation.

Lemma 2.1. *If G is an $\mathfrak{A}$-group, then its center $Z(G)$ is contained in every non-abelian subgroup of G.*

As usual, we denote by $\Phi(G)$ the Frattini subgroup of a group G.

Lemma 2.2. *If $G \in \mathfrak{A}$ is non-abelian group, then $Z(G) \leq \Phi(G)$.*

Proof. Let M be a maximal subgroup of G. If M is abelian and $Z(G) \nleq M$, then $MZ(G) = G$, hence G is abelian, a contradiction. On the other hand, if M is non-abelian, then $Z(G) \leq C_G(M) < M$. In conclusion, $Z(G)$ lies in every maximal subgroup of G, thus $M \leq \Phi(G)$. □

Lemma 2.3. *If $G \in \mathfrak{A}$ is infinite, then every normal subgroup of G is abelian or infinite.*

Proof. Let $H \trianglelefteq G$ be non-abelian. If H is finite, then so is $C_G(H) < H$, and $|G| = [G : C_G(H)||C_G(H)| \leq |\mathrm{Aut}(H)||C_G(H)| < \infty$, a contradiction. □

The next three results will be of particular importance in Section 4. Recall that a group element is aperiodic if its order is infinite.

Lemma 2.4. *Let $G \in \mathfrak{A}$. If N is a finite normal subgroup of G, then $C_G(N)$ contains all aperiodic elements of G.*

Proof. Let x be an aperiodic element of G. Suppose, for a contradiction, that $x \notin C_G(N)$. Since $G/C_G(N)$ embeds into the finite group $\mathrm{Aut}(N)$, it is finite. Thus, there exists a positive integer n such that $x^n \in C_G(N)$. Let p be a prime number which does not divide n, so that $x^p \notin C_G(N)$. Then $N\langle x^p\rangle$ is non-abelian, hence $C_G(N\langle x^p\rangle) < N\langle x^p\rangle$. Since $x^n \in C_G(N\langle x^p\rangle)$, there exist $s \in N$ and $\alpha \in \mathbb{Z}$ such that $x^n = s(x^p)^\alpha$. Now $s = x^{n-p\alpha} \in N \cap \langle x\rangle = \{1\}$ yields $x^{n-p\alpha} = 1$, hence $n = p\alpha$, contradicting our choice of p. □

Corollary 2.5. *Let $G \in \mathfrak{A}$. If G is generated by aperiodic elements, then every finite normal subgroup of G is central.*

Lemma 2.6. *Let $G \in \mathfrak{A}$ and let $a \in G$ be non-trivial of infinite or odd order. If $a^x = a^{-1}$ for some $x \in G$, then x is periodic and its order is a power of 2. Moreover, $C_G(\langle a, x\rangle) = \langle x^2\rangle$.*

Proof. If x is aperiodic, then $\langle a, x^3\rangle$ is non-abelian, hence $C_G(\langle a, x^3\rangle) < \langle a, x^3\rangle = \langle a\rangle\langle x^3\rangle$. Since $x^2 \in C_G(\langle a, x^3\rangle)$, we can write $x^2 = a^\alpha(x^3)^\beta$ for some $\alpha, \beta \in \mathbb{Z}$. Since x^2 commutes with x, we must have $a^\alpha = a^{-\alpha}$, and $a^\alpha = 1$ follows. Then $x^2 = x^{3\beta}$, which yields the contradiction $2 = 3\beta$. This proves that x must be periodic, say with order n. Suppose n is divisible by an odd prime p. Since $\langle a, x^p\rangle$ is not abelian, $x^2 \in C_G(\langle a, x^p\rangle) < \langle a, x^p\rangle = \langle a\rangle\langle x^p\rangle$.

As before, we can write $x^2 = x^{p\beta}$ for some $\beta \in \mathbb{Z}$, that is, $p\beta = 2 + cn$ for some $c \in \mathbb{Z}$. But this is not possible since p is odd and dividing n.

Let now $g \in C_G(\langle a, x\rangle)$. Since $\langle a, x\rangle$ is non-abelian, $C_G(\langle a, x\rangle) < \langle a, x\rangle = \langle a\rangle\langle x\rangle$. Then we can write $g = a^\alpha x^\beta$ for suitable $\alpha, \beta \in \mathbb{Z}$. Thus $g = g^x = a^{-\alpha}x^\beta$, hence $a^{2\alpha} = 1$ and so $a^\alpha = 1$, that is, $g = x^\beta$. Since $a = a^g = a^{(x^\beta)}$, it follows that β is even, and therefore $g \in \langle x^2\rangle$. This completes the proof. $\square$

The following results are easy, but useful, observations.

Lemma 2.7. *A group G is in the class $\mathfrak{A}$ if and only if $C_G(\langle x, y\rangle) < \langle x, y\rangle$ for every pair of non-commuting elements $x, y \in G$.*

Corollary 2.8. *A finite group G is in the class $\mathfrak{A}$ if and only if $C_G(K) < K$ for every minimal non-abelian subgroup K of G.*

3. Infinite nilpotent $\mathfrak{A}$-groups

Every abelian group lies in $\mathfrak{A}$, thus, compared with $\mathfrak{C}$-groups, the structure of $\mathfrak{A}$-groups which are soluble or nilpotent is less restricted. Our first result reduces the study of nilpotent $\mathfrak{A}$-groups to finite nilpotent $\mathfrak{A}$-groups.

Theorem 3.1. *Every nilpotent $\mathfrak{A}$-group is abelian or finite.*

Proof. First let G be an $\mathfrak{A}$-group of class $c = 2$; we will show that G is finite. If H is any non-abelian subgroup of G, then $G' \leq Z(G) < H$ and $H \trianglelefteq G$. Therefore G is metahamiltonian, and hence G' is a finite p-group (see, for instance, [8, Theorem 2.1]). Let $a, b \in G$ with $[a, b] \neq 1$, and put $H = \langle a, b\rangle$. Since $G \in \mathfrak{A}$, we have $C_G(H) = C_G(a) \cap C_G(b) = Z(H)$. Thus

$$|G : Z(H)| = |G : (C_G(a) \cap C_G(b))| \leq |G : C_G(a)||G : C_G(b)| \leq |G'|^2,$$

so $|G : Z(H)|$ is finite. Since $H' \leq G'$ is finite, in order to prove that G is finite it suffices to show that $|Z(H) : H'|$ is finite. For every positive integer n coprime to p we get $[a^n, b^n] = [a, b]^{n^2} \neq 1$, and so $Z(H) \leq \langle a^n, b^n\rangle$. It follows that

$$Z(H)/H' \subseteq \bigcap_{\gcd(n,p)=1} \langle a^n, b^n\rangle H'/H' = \bigcap_{\gcd(n,p)=1} (H/H')^n,$$

which is obviously finite.

Now consider an $\mathfrak{A}$-group G of nilpotency class $c > 2$ and use induction on c. Suppose, for a contradiction, that G is infinite. For all $a \in G \setminus G'$, the nilpotency class of $\langle a \rangle G'$ is less than c. If $\langle a \rangle G'$ is abelian for all $a \in G \setminus G'$, then $G' \leq Z(G)$ and $c \leq 2$, which is a contradiction to our assumption that $c > 2$. Thus, $H = \langle a \rangle G'$ is non-abelian for some $a \in G \setminus G'$; by the induction hypothesis, H is finite. This contradicts Lemma 2.3, thus G is finite. □

Let G be a Chernikov p-group, that is, G is an extension of a direct product of a finite number k of Prüfer p-groups by a finite group; the number $\delta(G) = k$ is an invariant of G.

Proposition 3.2. *If G is a non-abelian infinite locally nilpotent $\mathfrak{A}$-group, then G is a Chernikov p-group with $\delta(G) \geq p - 1$.*

Proof. Every finitely generated subgroup of G is nilpotent, so by Theorem 3.1 it is either abelian or finite. This implies that all torsion-free elements of G are central. It follows from [4, Proposition 1]) that G is periodic, hence G is direct product of groups of prime-power order, see for instance [12, Proposition 12.1.1]. In fact, only one prime can occur since G is an $\mathfrak{A}$-group, that is, G is a locally finite p-group for some prime p. Since G is non-abelian, there exist non-commuting $a, b \in G$. The subgroup $H = \langle a, b \rangle$ is finite and non-abelian, thus $C_G(H) < H$ is finite and G has finite Prüfer rank by [7, Theorem 5]. By a result of Blackburn [3], the group G is a Chernikov p-group. Finally, $\delta(G) \geq p - 1$ by a result of Chernikov, see for instance [3]. □

As a consequence of Proposition 3.2 and [12, 12.2.5] we get the following corollary; recall that a group is hypercentral if its upper central series terminates at the whole group.

Corollary 3.3. *Every locally nilpotent $\mathfrak{A}$-group is hypercentral.*

Theorem 3.1 and Proposition 3.2 reduce the study of nilpotent $\mathfrak{A}$-groups to finite p-groups; we provide some results on these groups in Section 6.

4. Infinite supersoluble $\mathfrak{A}$-groups

A group is supersoluble if it admits a normal series with cyclic sections. In this section we describe the structure of infinite supersoluble $\mathfrak{A}$-groups completely. If such a group has no involutions, then it must be abelian.

Theorem 4.1. *An infinite supersoluble $\mathfrak{A}$-group with no involutions is abelian.*

Proof. Let G be an infinite supersoluble $\mathfrak{A}$-group without elements of even order. Then the set T of all periodic elements in G is a finite subgroup of G by [12, 5.4.9]. As $T < G$, it easily follows that G is generated by its aperiodic elements, hence $T \leq Z(G)$ by Corollary 2.5. By a result of Zappa (see for instance [12, 5.4.8]), the quotient group G/T has a finite series

$$G_0/T = T/T \leq G_1/T \leq \cdots \leq G_s/T \leq G/T,$$

where each G_i is normal in G, each G_{j+1}/G_j is infinite cyclic, and G/G_s is a finite 2-group. We claim that each $G_i \leq Z(G)$. Since $G_0 = T$, this is true for $i = 0$. By induction on i, let us assume that $G_i \leq Z(G)$ with $0 \leq i \leq s-1$; then $G_{i+1} = \langle a \rangle G_i$ where $\langle a \rangle$ is infinite. Suppose, for a contradiction, that $G_{i+1}/G_i \not\leq Z(G/G_i)$. Then $(G/G_i)/C_{G/G_i}(G_{i+1}/G_i)$ has order 2, hence there exists $x \in G$ such that $a^x G_i = a^{-1} G_i$. It follows that $a^x = a^{-1}y$ for a suitable $y \in G_i$, hence $a^{(x^2)} = (a^{-1}y)^{-1}y = a$ and $a^{(x^2)} = a$. Now it follows from $1 = [a, x^2] = [a, x][a, x]^x$ that $[a, x]^x = [a, x]^{-1}$, and therefore $[a, x] = 1$ by Lemma 2.6, a contradiction. This shows that $G_{i+1}/G_i \leq Z(G/G_i)$, hence $G_{i+1} \leq Z_2(G)$. It follows readily that $\langle x \rangle G_{i+1}$ is nilpotent for all $x \in G$. Since G_{i+1} is infinite, $\langle x \rangle G_{i+1}$ is abelian by Theorem 3.1. This means that $G_{i+1} \leq C_G(x)$ for all $x \in G$, so $G_{i+1} \leq Z(G)$, and the claim is proved. In particular, $G_s \leq Z(G)$. Since G/G_s is a finite 2-group (hence nilpotent), it follows that G is nilpotent. Hence G is abelian by Theorem 3.1. □

Now we consider infinite supersoluble $\mathfrak{A}$-groups with involutions.

Theorem 4.2. *Let G be a non-abelian infinite supersoluble group. Then G lies in $\mathfrak{A}$ if and only if the following holds: $G = A\langle x \rangle$ with x of order 2^n and $A = \langle a_1 \rangle \times \cdots \times \langle a_t \rangle \times \langle d \rangle$ abelian with each a_i of infinite or odd order, d of order 2^h (with $d^{(2^{h-1})} = x^{(2^{n-1})}$ if $h > 0$), and $a^x = a^{-1}$ for all $a \in A$.*

Proof. Let G be an infinite supersoluble $\mathfrak{A}$-group which is not abelian. By Zappa's theorem (see [12, 5.4.8]), there exists a non-trivial normal subgroup C of G such that C has no involutions and G/C is a finite 2-group. Thus C is infinite, and therefore it is abelian by Theorem 4.1. So G/C is non-trivial.

We claim that if $x \in G \setminus C_G(C)$ and $x^2 \in C_G(C)$, then $a^x = a^{-1}$ for all $a \in C$. Indeed, from $x \notin C_G(C)$ it follows that there exists $c \in C$ such that $c^x \neq c$. Since $x^2 \in C_G(C)$ we get $1 = [c, x^2] = [c, x]^x[c, x]$, hence $[c, x]^x = [c, x]^{-1}$. By Lemma 2.6, the order of x is a power of 2 and $C_G(\langle [c, x], x \rangle) \leq \langle x^2 \rangle$. Since $(cc^x)^x = cc^x$ it follows that $cc^x \in C_G(\langle [c, x], x \rangle)$. Thus $cc^x \in \langle x^2 \rangle \cap C$, hence

$cc^x = 1$ and $c^x = c^{-1}$. Now let $a \in C$. Since $a^{(x^2)} = a$, we get $(aa^x)^x = aa^x$ and $aa^x \in C_G(\langle c, x\rangle) \leq \langle x^2\rangle$. Therefore $aa^x \in C \cap \langle x^2\rangle = \{1\}$. Thus $a^x = a^{-1}$ and the claim is proved. Notice that this implies $Z(G) \leq \langle x^2\rangle$.

Since G is supersoluble, there exists a G-invariant subgroup S of C which is infinite cyclic, say $S = \langle s\rangle$. Then $C_G(C) \leq C_G(S)$. Suppose there exists $x \in C_G(S) \setminus C_G(C)$. Note that $G/C_G(C)$ is a finite 2-group. Replacing x by a suitable power, we may assume that $x^2 \in C_G(C)$. But then the same argument as in the previous paragraph shows that $s^x = s^{-1}$, a contradiction. Thus $C_G(C) = C_G(S)$, and since S is infinite cyclic, the group $G/C_G(C)$ is easily seen to be cyclic of order two.

Since $C_G(C)/C$ is a finite 2-group, hence nilpotent, it follows that $C_G(C)$ is nilpotent. Thus $C_G(C)$ is abelian by Theorem 3.1. Write $B = C_G(C)$ and choose $x \in G$ so that $G = B\langle x\rangle$. As seen above, x has order 2^n for some $n \geq 1$, and $a^x = a^{-1}$ for all $a \in C$. Since G/B has order 2 we get $x^2 \in B$, and so $x^2 \in Z(G)$. Therefore $Z(G) = \langle x^2\rangle$.

Our next aim is to prove that for all $a \in B$ there exists an integer α with $a^x = a^{-1}x^{4\alpha}$. Let $a \in B$ and $c \in C \setminus \{1\}$. From $a^{(x^2)} = a$ it follows that $(aa^x)^x = aa^x$, hence $aa^x \in C_G(\langle c, x\rangle) \leq \langle x^2\rangle$. Thus $aa^x = x^\beta$ for some even integer $\beta = 2\gamma$. If $n = 1$ then $x^\beta = 1$, so $a^x = a^{-1}$ and the claim is proved. Thus it remains to consider the case $n > 1$. If γ is odd, then $(ax)^2 = axax = x^{2(1+\gamma)} \in \langle x^4\rangle$. If $c \in C \setminus \{1\}$, then $c^{ax} = c^{-1}$, hence by Lemma 2.6 we get $C_G(\langle c, ax\rangle) \leq \langle (ax)^2\rangle \leq \langle x^4\rangle$. On the other hand $x^2 \in C_G(\langle c, ax\rangle)$, so $x^2 = 1$, a contradiction since $n > 1$. This proves that γ is even, so 4 divides β, as required. Therefore $\beta = 4\alpha$ for some integer α.

Now we consider $A = \{c \in B \mid c^x = c^{-1}\}$, which is a non-trivial abelian normal subgroup of G. If $a \in B$, then $a^x = a^{-1}x^{4\alpha}$, and $(ax^{-2\alpha})^x = (ax^{-2\alpha})^{-1}$ proves $ax^{-2\alpha} \in A$, hence $a \in A\langle x\rangle$. This means that $G = B\langle x\rangle = A\langle x\rangle$. Write $A = \langle a_1\rangle \times \cdots \times \langle a_t\rangle \times D$, where each a_i has infinite or odd order and D is a finite 2-group. If D is non-trivial and $y \in D$ has order 2, then $y^x = y$, hence $y \in Z(G)$; this implies that D has a unique element of order 2. Since D is abelian, this means that D is cyclic, say $D = \langle d\rangle$. Let 2^h be the order of d. If $h > 0$, then $d^{(2^{h-1})} = x^{(2^{n-1})}$, hence the structure of G is as required.

For the converse, let $G = A\langle x\rangle$ be as in the statement. Then $B = \langle x^2\rangle A$ is abelian and $|G : B| = 2$. Thus, for every non-abelian $H \leq G$ there exists $y = x^i c \in H$ with i odd and $c \in A$. Since $G = B\langle y\rangle$, we have $C_G(y) = C_B(y)\langle y\rangle = Z(G)\langle y\rangle = \langle x^2, y\rangle$. Note that $y^2 = x^{2i}$, and so $\langle y^2\rangle = \langle x^2\rangle$. Thus $C_G(H) \leq C_G(y) = \langle y\rangle \leq H$, hence $G \in \mathfrak{A}$, as desired. $\square$

5. On soluble $\mathfrak{A}$-groups

We now briefly discuss soluble groups in $\mathfrak{A}$; we start with a result on the Fitting subgroup of an infinite soluble $\mathfrak{A}$-group.

Theorem 5.1. *If $G \in \mathfrak{A}$ is infinite, then the Fitting subgroup of G is abelian.*

Proof. Suppose the Fitting subgroup F of G is non-abelian, and let $a, b \in F$ be non-commuting. Now $H = \langle a \rangle^G \langle b \rangle^G \trianglelefteq G$ is normal, nilpotent, and non-abelian, so H is finite by Theorem 3.1; this contradicts Lemma 2.3. □

In [6] it has been proved that every infinite soluble $\mathfrak{C}$-group is metabelian. However, it is not possible to bound the derived length of soluble $\mathfrak{A}$-groups, even in the torsion-free case. For example, the standard wreath product of n copies of the infinite cyclic group is a finitely generated soluble commutative-transitive group, and therefore an $\mathfrak{A}$-group, with derived length n, see [13, Corollary 18]. Torsion-free polycyclic $\mathfrak{C}$-groups are abelian, see [6]; this result is no longer true for $\mathfrak{A}$-groups.

Example 5.2. Let $F = \langle a \rangle \times \langle b \rangle$ be free abelian of rank 2 and $G = F \rtimes \langle c \rangle$, where c is the automorphism of F defined by $a^c = ab$ and $b^c = a$. Clearly, $\langle c \rangle$ acts fixed-point-freely on F, hence G is commutative-transitive, see[13, Lemma 7]. Thus, G is a torsion-free polycyclic $\mathfrak{A}$-group of derived length 2.

Proposition 5.3. *Let $G \in \mathfrak{A}$ be soluble. If G has a non-abelian subnormal subgroup which is polycyclic, then G is polycyclic.*

Proof. If $H \leq G$ is non-abelian, polycyclic, and subnormal in G, then $C_G(H) < H$ is polycyclic, and the result follows by [7, Theorem 2.1]. □

Recall that an element g of a group G is a right Engel element if for each $x \in G$ there exists a positive integer n such that the left-normed commutator $[g, {}_n x] = 1$. Let $R(G)$ denote the set of all right Engel elements of G. It is shown in [12, 12.3.2 (ii)] that $R(G)$ contains the hypercenter of G; on the other hand, it is well known that $R(G)$ may be larger than the hypercenter of G even when G is soluble.

Proposition 5.4. *Let $G \in \mathfrak{A}$ be locally soluble. If G has a non-abelian finite subgroup which is contained in $R(G)$, then G is locally finite.*

Proof. Let $H \leq R(G)$ be non-abelian finite. Then $C_G(H) < H$ is finite as G is an $\mathfrak{A}$-group, and hence G is locally finite by [7, Theorem 2.4]. □

Corollary 5.5. *Let $G \in \mathfrak{A}$ be locally soluble. If the hypercenter of G contains a pair of non-commuting periodic elements, then G is locally finite.*

Proof. Let a and b non-commuting periodic elements contained in the hypercenter of G. Then $H = \langle a, b \rangle$ is a finite non-abelian subgroup which is contained in $R(G)$, and the result follows from Proposition 5.4. $\square$

6. Finite $\mathfrak{A}$-groups

Here we study finite groups in $\mathfrak{A}$, in particular, p-groups. The proof of the following preliminary lemma is similar to the proof of [5, Lemma 2.1].

Lemma 6.1. *Let $\mathcal{F}$ be a subgroup closed class of finite groups. Then $\mathcal{F} \subseteq \mathfrak{A}$ if and only if for every $G \in \mathcal{F}$ the following holds: if $K \leq G$ is non-abelian, then $Z(G) \leq K$.*

Proof. If $\mathcal{F} \subseteq \mathfrak{A}$, then every non-abelian $K \leq G$ of $G \in \mathcal{F}$ satisfies $Z(G) \leq C_G(K) \leq K$. For the converse, denote by $\mathcal{F}_n$ the subset of $\mathcal{F}$ of groups of order n, and proceed by induction on n. Clearly, $\mathcal{F}_1 \subseteq \mathfrak{A}$, so we may assume that $n > 1$. Let $G \in \mathcal{F}_n$ and let $K \leq G$ be non-abelian. Let $z \in C_G(K)$ and define $H = \langle K, z \rangle$; note that $z \in C_H(K)$. If $H = G$, then $z \in Z(G)$, hence $z \in K$ by assumption. If $H < G$, then $H \in \mathcal{F}_m$ for some $m < n$, hence $H \in \mathfrak{A}$ by the induction hypothesis, and therefore $z \in C_H(K) \leq K$. In conclusion, $C_G(K) \leq K$ for every non-abelian $K \leq G$, hence $G \in \mathfrak{A}$. $\square$

Lemma 6.2. *Let $\mathfrak{X}$ be a subgroup closed class of finite groups. Suppose that $Z(G) \leq \Phi(G)$ for every non-abelian $G \in \mathfrak{X}$. Then $\mathfrak{X} \subseteq \mathfrak{A}$.*

Proof. Let $G \in \mathfrak{X} \setminus \mathfrak{A}$ be of smallest possible order. Denote the family of all subgroups of G by $\mathcal{F}$. Note that every proper subgroup of G is in $\mathfrak{A}$. Let $K \in \mathcal{F}$, and let H be a non-abelian subgroup of K. We wish to prove that $Z(K) \leq H$. If $K \neq G$, then $K \in \mathfrak{A}$, therefore $Z(K) \leq C_K(H) \leq H$. If $K = G$, we can assume without loss of generality that $H \neq G$. Let M be a maximal subgroup of G containing H. As $M \in \mathfrak{A}$, we conclude that $C_M(H) \leq H$. Since $Z(G) \leq \Phi(G)$, it follows that $Z(G) \leq C_M(H) \leq H$, as required. Now Lemma 6.1 yields $\mathcal{F} \subseteq \mathfrak{A}$, which contradicts our assumption. $\square$

Corollary 6.3. *A non-abelian finite group L lies in $\mathfrak{A}$ if and only if all its maximal subgroups lie in $\mathfrak{A}$ and $Z(L) \leq \Phi(L)$.*

Proof. Clearly, if $L \in \mathfrak{A}$ and $M < L$ is maximal and non-abelian, then $M \in Y$, hence $Z(L) \leq C_L(M) \leq M$. If M is abelian, then $Z(L) \leq M$, since otherwise $L = MZ(L)$ is abelian, a contradiction.

Now consider the converse. Suppose the claim is not true and choose a minimal counterexample, that is, a group $L \notin \mathfrak{A}$ of smallest possible order such that $Z(L) \leq \Phi(L)$ and $M \in \mathfrak{A}$ for all maximal subgroups $M < L$. Let $\mathcal{F}$ be the set of all subgroups of L. Let $G \in \mathcal{F}$ and $K \leq G$ be non-abelian. We aim to show that $Z(G) \leq K$. If this holds for all such G and K, then Lemma 6.1 proves that $L \in \mathcal{F} \subseteq \mathfrak{A}$, a contradiction. If $G < L$, then $G \leq M$ for some non-abelian maximal subgroup $M < L$. By assumption, $M \in \mathfrak{A}$, hence $G \in \mathfrak{A}$ and $Z(G) \leq C_G(K) \leq K$. Now consider $G = L$. If $K = G$, then $Z(G) \leq K$, so let $K \leq M < G$ for some non-abelian maximal subgroup $M < G$. By assumption, $Z(G) \leq M \in \mathfrak{A}$, hence $Z(G) \leq C_M(K) \leq K$. □

6.4. Finite p-groups in $\mathfrak{A}$

Motivated by Section 3, we now concentrate on finite p-groups. The next two lemmas give the well-known characterizations of minimal non-abelian finite p-groups, see [14, Lemma 2.2] and [1, Exercise 8a, p. 29].

Lemma 6.5. *Let G be a finite p-group. The following are equivalent:*

a) *G is minimal non-abelian.*
b) $\mathrm{d}(G) = 2$ *and* $|G'| = p$.
c) $\mathrm{d}(G) = 2$ *and* $Z(G) = \Phi(G)$.

Lemma 6.6. *Every minimal non-abelian p-group is isomorphic to one of:*

a) $K_1 = Q_8$,
b) $K_2 = \langle a, b \mid a^{p^m} = b^{p^n} = 1, a^b = a^{1+p^{m-1}} \rangle$ *with* $m \geq 2$ *and* $n \geq 1$*; this group is metacyclic of order* p^{n+m},
c) $K_3 = \langle a, b, c \mid a^{p^m} = b^{p^n} = c^p = 1, [a, b] = c, [c, a] = [c, b] = 1 \rangle$ *with* $m + n > 2$ *if* $p = 2$*; this group is not metacyclic of order* p^{m+n+1}*, and the derived subgroup* K_3' *is a maximal cyclic subgroup.*

Recall that a finite p-group G lies in $\mathfrak{A}$ if and only if every minimal non-abelian $K \leq G$ satisfies $C_G(K) \leq K$; in particular, $Z(G) \leq K$. Since minimal non-abelian p-groups are classified, this poses strict conditions on $Z(G)$. We denote by $\Omega_1(G)$ the subgroup of G generated by all elements of order p, and G^p is the subgroup of G generated by all p-th powers of elements of G. As usual, C_n denotes a cyclic group of order n.

Lemma 6.7. *If K is a minimal non-abelian p-group, then $Z(K) \cong C_{p^i} \times C_{p^j}$ or $Z(K) \cong C_p \times C_{p^i} \times C_{p^j}$ for some $i, j \geq 0$, and $\Omega_1(Z(K))$ is elementary abelian of rank 1, 2, or 3.*

Proof. Let K_1, K_2, K_3 be as in Lemma 6.6. If $K \cong K_1$, then $Z(G) \cong C_2$. If $K \cong K_2$, then $Z(K) \cong \langle a^p, b^p \rangle$; if $K \cong K_3$, then $Z(K) \cong \langle a^p, b^p, c \rangle$ □

The next result deals with metacyclic p-groups. We note that not all metacyclic groups are in $\mathfrak{A}$, since one can easily show, for example, that $D_{12} \notin \mathfrak{A}$. The situation is quite different if we restrict to p-groups.

Proposition 6.8. *Every metacyclic p-group is in $\mathfrak{A}$.*

Proof. Let $\mathfrak{X}$ be the class of metacyclic p-groups, and let $G \in \mathfrak{X}$ be non-abelian. Note that all subgroups of G are also metacyclic. By Lemma 6.2 it suffices to prove that $Z(G)$ is contained in $\Phi(G)$. By [10, Theorem 3.1], the group G can be given by a reduced presentation

$$G = \langle a, b \mid a^{p^m} = 1,\ b^{p^n} = a^{p^{m-s}},\ a^b = a^{\varepsilon + p^{m-c}} \rangle,$$

where the precise numerical restrictions on m, n, s and c are given in [10, Theorem 3.1], and $\varepsilon = 1$ when $p > 2$, and $\varepsilon = \pm 1$ when $p = 2$. Furthermore, it is clear that $\Phi(G) = \langle a^p, b^p \rangle$, and [10, Proposition 4.10] gives $Z(G) = \langle a^{p^u}, b^{p^v} \rangle$, where $u = v = c$ when $\varepsilon = 1$, and $u = m - 1$, $v = \max\{1, c\}$ when $\varepsilon = -1$. From here the assertion easily follows. □

Lemma 6.9. *Let $G \in \mathfrak{A}$ be a p-group. If $g \in G \setminus \Phi(G)$, then $C_G(g)$ is abelian.*

Proof. Let M be a maximal subgroup of G such that $G = \langle g, M \rangle$. First, we show that $K = C_M(g)$ is abelian. If this is not the case, then, since $G, M \in \mathfrak{A}$, we have $C_M(K) = C_G(K) = Z(K)$. By definition, g commutes with K, so $g \in C_G(K) \leq Z(K)$, which is not possible since $K \leq M$ and $g \notin M$. Thus, $K = C_M(g)$ must be abelian. Note that every $h \in G$ can be written as $h = g^j m$ for some $m \in M$ and $j = 0, .., p-1$. Now $h \in C_G(g)$ if and only if $g^{j+1}m = gg^j m = g^j mg = g^{j+1}m^g$, if and only if $m^g = m$, if and only if $m \in C_M(g) = K$. This proves that $C_G(g) = \langle g, K \rangle$. As shown above, K is abelian, hence also $C_G(g)$ is abelian. □

Lemma 6.10. *All p-groups of order p, p^2, and p^3 lie in $\mathfrak{A}$. A p-group of order p^4 lies in $\mathfrak{A}$ if and only if it is abelian, has maximal class, or $\Phi(G) = Z(G)$.*

Proof. Groups of order p^n with $n \leq 3$ are abelian or minimal non-abelian, hence lie in $\mathfrak{A}$. Let G be a group of order p^4. If G is abelian, then $G \in \mathfrak{A}$. If G has maximal class, then every maximal subgroup $M < G$ satisfies $Z(G) < G' < M$; clearly, $M \in \mathfrak{A}$ since $|M| = p^3$, hence $G \in \mathfrak{A}$ by Corollary 6.3. Now suppose G has nilpotency class 2, hence $G > Z(G) \geq G' \geq 1$. Since every maximal subgroup $M < G$ lies in $\mathfrak{A}$, we have $G \in \mathfrak{A}$ if and only if $Z(G) \leq \Phi(G)$. Since G is not abelian, we have $|G : \Phi(G)| \geq p^2$. If $Z(G) < \Phi(G)$, then we must have $G > \Phi(G) > Z(G) = G' > 1$; but $Z(G) = G'$ for a non-abelian group G of order p^4 implies that $G/Z(G)$ has order p^2, see [1, p. 11, Exercise 40], a contradiction. Thus, $\Phi(G) = Z(G)$. $\square$

We now consider p-groups of maximal class in more detail; the following remark recalls some important properties.

Remark 6.11. Let G be a group of order p^n of maximal class, and suppose $n \geq 4$. By [11, p. 56], there exists a chief series $G > P_1 > \ldots > P_n = 1$ with $P_i = P_i(G) = \gamma_i(G)$ for $i \geq 2$, and $P_1 = P_1(G) = C_G(P_2/P_4)$; we define $P_m = P_m(G) = 1$ if $m > n$. Note that $[P_i, P_j] \leq P_{i+j}$ for all $i, j \geq 1$. We note that $P_2, \ldots, P_n$ are the unique normal subgroups of G of index greater than p, see [11, Proposition 3.1.2]. Following [11, Definition 3.2.1], we say the *degree of commutativity* of a p-group of maximal class is the largest integer ℓ with the property that $[P_i, P_j] \leq P_{i+j+\ell}$ for all $i, j \geq 1$ if P_1 is not abelian, and $\ell = n - 3$ if P_1 is abelian. If $n > p + 1$, then G has positive degree of commutativity, see [11, Theorem 3.3.5]. If $s \in G \setminus P_1$ and $s_1 \in P_1 \setminus P_2$, then s and s_1 generate G; for $i \leq n$ define $s_i = [s_{i-1}, s]$. If G has positive degree of commutativity, then $s_i \in P_i \setminus P_{i+1}$ for all $i < n$, see [11, Lemma 3.2.4].

Lemma 6.12. *If a p-group G of maximal class has an abelian maximal subgroup, then $G \in \mathfrak{A}$.*

Proof. Let $G \notin \mathfrak{A}$ be a p-group of maximal class that has an abelian maximal subgroup A, and suppose that it is of smallest possible order. Let H be a proper non-abelian subgroup of G. It follows from [1, p. 27, Exercise 4] that H is also of maximal class; now note that $H \cap A$ is an abelian maximal subgroup of H, since $p = |G : A| = |HA : A| = |H : H \cap A|$. Thus, all proper subgroups of G belong to $\mathfrak{A}$. Taking $\mathcal{F}$ to be the family of all subgroups of G, we get a contradiction similarly as in the proof of Lemma 6.2. $\square$

There exist 3-groups of maximal class all of whose maximal subgroups are non-abelian, see [2]. Nevertheless, we can prove the following.

Lemma 6.13. *The 2- and 3-groups of maximal class lie in $\mathfrak{A}$.*

Proof. The 2-groups of maximal class are classified and all metacyclic, see [2], hence they lie in $\mathfrak{A}$ by Proposition 6.8. Now let G be a 3-group of maximal class of order 3^n with chief series $G > P_1 > \ldots > P_n = 1$ as defined above. We proceed by induction on n. It follows from Lemma 6.10 that $G \in \mathfrak{A}$ if $n \leq 4$, so let $n \geq 5$ in the following. By Lemma 6.12, we can assume that all maximal subgroups are non-abelian. It follows from [11, Corollary 3.3.6] that every maximal subgroup $M < G$ either has maximal class, or $M = P_1$. In the first case, $M \in \mathfrak{A}$ by the induction hypothesis; also, $Z(G) \leq M$ since otherwise $G = MZ(G)$, and $Z(G)Z(M) \leq Z(G)$ yields a contradiction to $|Z(G)| = 3$. If $M = P_1$, then $|M'| = 3$ by [11, Theorem 3.4.3]. It follows from [11, Corollary 3.3.6] that $M^3 = P_3$. Since $M' \leq P_3$, this yields $\Phi(M) = M'M^3 = P_3$, so $M/\Phi(M) = P_1/P_3$. This proves that M is a 2-generator group, hence minimal non-abelian by Lemma 6.5, and $M \in \mathfrak{A}$. Now Corollary 6.3 proves that $G \in \mathfrak{A}$. $\square$

Theorem 6.14. *Let G be a p-group of maximal class of order p^n. If $p \in \{2, 3\}$ or $n \leq 3$, then $G \in \mathfrak{A}$. If $p \geq 5$ and $n \geq 4$, then $G \in \mathfrak{A}$ if and only if its 2-step centralizer $P_1(G)$ is abelian.*

Proof. By Lemmas 6.13 and 6.10, it suffices to consider $p \geq 5$ and $n \geq 4$. For $n = 4$ the claim follows from the classification in [9, Satz 12.6], so let $n \geq 5$. Let G be of maximal class and define $G > P_1 > \ldots > P_n = 1$ and $s, s_1, \ldots, s_{n-1}$ as in Remark 6.11. Clearly, if P_1 is abelian, then G has an abelian maximal subgroup, hence $G \in \mathfrak{A}$ by Lemma 6.12.

For the converse, suppose that G is a counterexample of smallest order. Choose $s \in G \setminus P_1(G)$ with $|C_G(s)| = p^2$ and $s^p \in Z(G)$, which is possible by [9, Hilfssatz III.14.13], and consider $M = \langle s, P_2 \rangle$. It is easy to see that $M \in \mathfrak{A}$ has maximal class. Since $[P_2, P_3] \leq P_5$, it follows that $P_1(M) = P_2$. If P_2 is nonabelian, then this yields a contradiction to our choice of G; thus $P_2 = G'$ is abelian and G is metabelian. It follows from [2, Theorem 2.10] that G has positive degree of commutativity $\ell > 0$, cf. [2, p. 74]; in particular, if $n > p+1$, then $\ell \geq n-p-1$, see [2, Theorem 3.10]. Recall that we assume P_1 is nonabelian, thus $Z(P_1) = P_m$ for some $m \in \{2, \ldots, n-2\}$. Since $[P_1, G, P_{m-1}] = 1 = [G, P_{m-1}, P_1]$, the three-subgroup lemma [11, Proposition 1.1.8] shows that $[P_{m-1}, P_1, G] = 1$, hence $[P_1, P_{m-1}] = P_{n-1} = Z(G)$. The same argument and an induction can be used to show that $[P_1, P_{m-i}] \leq P_{n-i}$ for all $i = 1, \ldots, m-1$, which implies that $\ell = n-m-1$. Define $H = \langle x, y \rangle$

for some $x \in P_1 \setminus P_2$ and $y \in P_{m-1} \setminus P_m$. Note that $[x, y] \in P_{1+m-1+\ell} = P_{n-1}$, and $[x, y] \neq 1$ since P_2 is abelian and $P_m = Z(P_1)$; thus, $H' = Z(G) = P_{n-1}$. Since $G \in \mathfrak{A}$, it follows that $Z(P_1) \leq H$, thus $P_{m-1} = \langle y, P_m \rangle \leq H$. Since $\ell = n - m - 1 > 0$, and so $n \geq m + 2$, this implies that $P_{n-2} \leq P_{m-1} \leq H$.

First, let $n \leq p + 1$, so that $G/Z(G)$ has exponent p by [11, Proposition 3.3.2]. Since H is a 2-generator group and $Z(G) = H'$, it follows that $Z(G) = \Phi(H)$, hence $|H/Z(G)| \leq p^2$. Now $Z(G) < P_{n-2} \leq Z(H) \leq H$ implies that $|H : Z(H)| \leq p$, so H is abelian. This is a contradiction.

Second, consider $n > p+1$ and $\ell \geq 2$. Now [11, Corollary 3.3.6] shows that P_{m-1}/P_{m+p-2} is a subgroup of $\Omega_1(H/P_{m+p-2})$ of order $\min\{p^{p-1}, p^{n-m+1}\} = \min\{p^{p-1}, p^{\ell+2}\} \geq p^4$. But H/P_{m+p-2} is a 2-generator p-group whose derived subgroup has order at most p, and so $|\Omega_1(H/P_{m+p-2})| \leq p^3$, a contradiction.

Lastly, consider $n > p+1$ and $\ell = 1$. As mentioned above, $\ell \geq n - p - 1$, which implies that $n = p + 2$ and $m = p$. Thus, $Z(P_1) = P_p$, and [11, Corollary 3.3.6] shows that $x^p \in P_p \setminus P_{p+1}$ and $W = P_{p-2}$ has exponent p; recall $p \geq 5$. Since $P_2 = G'$ is abelian and $x \in P_1 \setminus P_2$, it follows that $C_W(x) = Z(P_1) \cap W = P_p$, hence $|\{[x, w] \mid w \in W\}| = |\{x^w \mid w \in W\}| = |W : C_W(x)| = p^2$. Together with $\{[x, w] \mid w \in W\} \subseteq P_p$, this implies that $P_p = \{[x, w] \mid w \in W\}$. In particular, there is $w \in W$ with $[x, w] = x^p$, which implies that $J = \langle x, w \rangle$ is non-abelian with order p^3 and exponent p^2, and so $|\Omega_1(J)| = p^2$. It follows from $G \in \mathfrak{A}$ that $P_p \leq J$, thus $\langle w, P_p \rangle \leq \Omega_1(J)$ and $|\Omega_1(J)| \geq p^3$. This final contradiction completes the proof. $\square$

We end this section with a classification of the p-groups in $\mathfrak{A}$ of exponent p.

Theorem 6.15. *Let $G \in \mathfrak{A}$ be a finite p-group of exponent p. If $|G| > p^p$, then G is elementary abelian. Otherwise, either G is elementary abelian, or G has maximal class and an elementary abelian subgroup of index p.*

Proof. Clearly, if G is abelian, then G is elementary abelian. Thus, in the following, suppose that G is non-abelian. By Lemma 6.6, if $p > 2$, then every minimal non-abelian $K \leq G$ must be extra-special of order p^3 and $Z(G) = Z(K) \cong C_p$; if $p = 2$, then there is no minimal non-abelian subgroup of exponent 2, hence G is elementary abelian. Thus, in the following let $p > 2$ and $|Z(G)| = p$; we prove the assertion by induction on the order of G.

By Lemma 6.10, our claim is true if $|G|$ divides p^3; if $|G| = p^4$, then the claim follows from the known classification of groups of order p^4, see [9, Satz 12.6]. So in the following we discuss the case $n \geq 5$. By the induction hypothesis, each maximal subgroup $M < G$ is either elementary abelian, or has maximal class and $M \cong \langle h \rangle \ltimes C_p^{n-2}$. Note that the latter can only happen if $n \leq p+1$, since otherwise $|M| = p^{n-1} > p^p$ and then the induction hypothesis forces M to be elementary abelian.

Suppose G has an abelian maximal subgroup M. Since $|Z(G)| = p$, it follows from [1, Exercise 4, p. 27] that G has maximal class. Define $G > P_1 > \ldots > P_n = 1$ and $s, s_1, \ldots, s_{n-1}$ as in Remark 6.11. Note that $P_1 = M$ since M is abelian and $P_1 = C_G(P_2/P_4)$, hence G has positive degree of commutativity. If $n > p$, then $s_p \in P_p \setminus P_{p+1}$ by [11, Lemma 3.2.4]. Now [11, Corollary 1.1.7(i)] yields $(ss_1)^p = s_p \neq 1$, which is a contradiction to $(ss_1)^p = 1$. This proves that if $|G| > p^p$ and G has a maximal subgroup which is elementary abelian, then G is elementary abelian.

Now suppose G has no elementary abelian maximal subgroup, and so $n \leq p+1$ as shown above. If $M < G$ is maximal, then M has maximal class and an elementary abelian subgroup $N < M$ of order p^{n-2}. Observe that $N = P_1(M)$ is characteristic in M, hence $N \trianglelefteq G$. Let $M^* \neq M$ be maximal subgroup of G, and define $N^* = P_1(M^*)$. Note that $N^* \trianglelefteq G$ is abelian, and

$$(*) \quad |NN^* : N \cap N^*| = |NN^* : N||N : N \cap N^*| = |NN^* : N||NN^* : N^*|.$$

Suppose $N \neq N^*$, so NN^* has index 1 or p in G. If $G = NN^*$, then $G' = [N, N^*] \leq N \cap N^*$ since N and N^* are abelian and normal in G; now $|G : G'| \geq p^4$ by $(*)$, contradicting $|G : M'| = p^3$. Thus, if $N \neq N^*$, then $L = NN^* < G$ is maximal. Note that $N \cap N^* \leq Z(L)$, and $|L : Z(L)| \leq |L : N \cap N^*| = p^2$ by $(*)$. Since L has maximal class, $|L| \leq p^3$, which contradicts $|G| \geq p^5$. In conclusion, we have proved $N = N^*$; in particular, N is contained in every maximal subgroup of G, and so $N = P_1(M) = \Phi(G) = \gamma_2(G)$. An induction on i now proves that $\gamma_i(G) = \gamma_{i-1}(M)$ for all $i \geq 3$, hence G has maximal class. By Theorem 6.14, the group G has an abelian maximal subgroup – a contradiction to our assumption. In conclusion, G has an elementary abelian maximal subgroup, and the claim follows. □

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