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GROUPS IN WHICH EVERY NON-ABELIAN SUBGROUP IS SELF-NORMALIZING

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Abstract. We study groups having the property that every non-abelian subgroup is equal to its normalizer. This class of groups is closely related to an open problem posed by Berkovich. We give a full classification of finite groups having the above property. We also describe all infinite soluble groups in this class.

1. Introduction

In his book [1], Berkovich posed the following problem:

Problem 1.1 ([1], Problem 9). *Study the p-groups G in which $C_G(A) = Z(A)$ for all non-abelian $A \leq G$.*

Classification of such p-groups appears to be difficult, as there seem to be many classes of finite p-groups enjoying the above property. In a recent paper [5], the finite p-groups which have maximal class or exponent p and satisfy Berkovich's condition are characterized. In addition to that, the infinite supersoluble groups in which every non-abelian subgroup is self-centralizing are completely classified. Some relaxations of Berkovich's problem are considered in [3, 4] where locally finite or infinite supersoluble groups G in which every non-cyclic subgroup A satisfies $C_G(A) = Z(A)$ are described.

Recently, Pavel Zalesskii suggested to us another related problem:

Problem 1.2. *Classify finite groups G in which $N_G(A) = A$ for all non-abelian $A \leq G$.*

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For the purposes of this paper, a subgroup A of a group G is called *self-centralizing* if $C_G(A) \leq A$. Moreover, A is called *self-normalizing* if $N_G(A) = A$. Let us denote by Σ the class of all groups G satisfying the property stated in Problem 1.2, that is, that every non-abelian subgroup of G is self-normalizing. Motivation for considering this problem is twofold. Firstly, every group in Σ clearly satisfies the property that every non-abelian subgroup is self-centralizing. Thus the class Σ fits into the framework set by the above mentioned Berkovich's problem. Secondly, the class of groups in which every non-abelian subgroup is self-normalizing can be considered as a particular case of the following general situation. Fix a property $\mathcal{P}$ related to the subgroups of a given group and consider the class of all groups G in which every subgroup H either has the property $\mathcal{P}$ or is of bounded index in its normalizer $N_G(H)$. There is an abundance of literature studying restrictions these kind of conditions impose on groups. For example, taking $\mathcal{P}$ to be the property of being normal, the authors of [7] investigate finite p-groups in this class. If we take $\mathcal{P}$ to be commutativity and set the upper bound of $|N_G(H) : H|$ to 1, then we obtain the class Σ.

One of the purposes of this paper is to completely characterize the finite groups in Σ. This is done in Section 2. We show that these groups are either soluble or simple. Finite non-abelian simple groups in Σ are precisely the groups $\mathrm{Alt}(5)$ and $\mathrm{PSL}(2, 2^{2n+1})$, where $n \geq 1$ (see Theorem 2.17). The structure of finite soluble non-nilpotent groups in Σ is described in Theorem 2.13. In contrast with Berkovich's problem, finite nilpotent non-abelian groups in Σ are precisely the minimal non-abelian p-groups, whose structure is well-known.

In Section 3 we deal with infinite groups in Σ. We completely describe the structure of infinite soluble groups in Σ in Theorems 3.2 and 3.3. On the other hand, the structure of infinite soluble groups with the property that every non-abelian subgroup is self-centralizing is still not completely known; the paper [5] only deals with the supersoluble case. Finally, we prove in Theorem 3.4 that every infinite locally finite group in Σ is metabelian.

2. Finite groups

The following three results hold in general for groups in Σ, not just for finite groups. They will be used throughout the paper without further reference.

Proposition 2.1. *The following properties hold:*

(i) *The class Σ is subgroup and quotient closed.*
(ii) *If $G \in \Sigma$, then every non-abelian subgroup of G is self-centralizing.*

(iii) *Non-abelian groups in Σ are directly indecomposable.*

Proposition 2.2. *Let $G \in \Sigma$.*

(i) *Every proper subnormal subgroup of G is abelian.*
(ii) *If G is not perfect, then it is metabelian.*

Lemma 2.3. *Let $G \in \Sigma$ be a soluble group which is either infinite or non-nilpotent finite, and F its Fitting subgroup. Then F is abelian and of prime index in G.*

Proof. If G is infinite then F is abelian by [5, Theorem 3.1]. Otherwise, F is abelian by (i) of Proposition 2.2. By (ii) of Proposition 2.2 we have $G' \leq F$, so G/F is an abelian group having no non-trivial proper subgroups, and hence it has prime order. □

The following easy observation immediately follows from the definition.

Lemma 2.4. *Let G be a finite group. Then G belongs to Σ if and only if the following holds: for every non-abelian subgroup H of G, the number of conjugates of H in G is equal to $|G : H|$.*

The next lemma gives a well-known characterization of minimal non-abelian finite p-groups, see [1, Exercise 8a, p. 29].

Lemma 2.5. *Let G be a finite p-group. The following are equivalent:*

a) *G is minimal non-abelian.*
b) *$\mathrm{d}(G) = 2$ and $|G'| = p$.*
c) *$\mathrm{d}(G) = 2$ and $Z(G) = \Phi(G)$.*

Proposition 2.6. *Let G be a nilpotent group. Then $G \in \Sigma$ if and only if G is either abelian or a finite minimal non-abelian p-group for some prime p.*

Proof. Let G be a nilpotent group in Σ, and suppose that G is not abelian. By [5, Theorem 3.1], G has to be finite, and Proposition 2.1 implies that G is a p-group. Let H be a proper subgroup of G. Then $H < N_G(H)$ by [14, 5.2.4], hence H is abelian. This proves that G is minimal non-abelian. The converse is clear. □

Finite soluble groups in which all Sylow subgroups are abelian are called *A-groups*, cf. [10, Seite 751]. Next we show that finite soluble non-nilpotent groups in Σ are A-groups.

Lemma 2.7. *Let $G \in \Sigma$ be a soluble non-nilpotent group and let $p = |G : F|$. Then the Sylow p-subgroups of G are cyclic and, for primes $r \neq p$, the Sylow r-subgroups are abelian.*

Proof. If $r \neq p$, then the Sylow r-subgroups of G are contained in F and so are abelian by 2.3. Now let P be a Sylow p-subgroup of G and assume that P is not cyclic. Since $C_G(F) = F$, there exists a prime r and a Sylow r-subgroup R of G such that P and R do not commute. Since R is normal in G, $X = PR$ is a subgroup of G. Since P is not cyclic, there are maximal subgroups P_1 and P_2 of P with $P_1 \neq P_2$. It follows that X normalizes both P_1R and P_2R. As $X > P_1R$ and $X > P_2R$, we conclude that P_1R and P_2R are abelian. But then $P = P_1P_2$ commutes with R, a contradiction. Hence P is cyclic. □

Suppose that an element x acts on an abelian group A. Consider the induced homomorphism $\partial_x \colon A \to A$, $\partial_x(a) = a^{1-x}$. We will describe groups belonging to the class $\boldsymbol{\Sigma}$ based on the following property of ∂_x:

$$(\mathbb{I}) \qquad \forall B \leq A\colon\ (\partial_x(B) \leq B \implies \partial_x(B) = B).$$

Note that x acts fixed point freely on A if and only if ∂_x is injective. The property $\mathbb{I}$ implies injectivity of ∂_x, since taking $B = \ker \partial_x$ immediately gives $B = 1$. Furthermore, the property $\mathbb{I}$ implies that ∂_x is an epimorphism by taking $B = A$. Therefore having property $\mathbb{I}$ implies that ∂_x is an isomorphism.

The following proposition shows how property $\mathbb{I}$ is related to $\boldsymbol{\Sigma}$.

Proposition 2.8. *Let $G = \langle x \rangle \ltimes A$ with x^p acting trivially on A for some prime p and x acting fixed point freely on A. Then $G \in \boldsymbol{\Sigma}$ if and only if ∂_x has property $\mathbb{I}$.*

Proof. Assume first that $G \in \boldsymbol{\Sigma}$. If ∂_x does not have property $\mathbb{I}$, then there exists a subgroup $B \leq A$ such that $\partial_x(B) \subsetneq B$. Consider the subgroup $H = \langle x \rangle \ltimes \partial_x(B)$ of G. Take an element $b \in B \backslash \partial_x(B)$ and observe that $x^b = x\partial_x(b) \in H$. Therefore b normalizes H and does not belong to H. This implies that H is abelian, and so $\partial_x(\partial_x(B)) = 1$. By injectivity of ∂_x, it follows that $B = 1$, a contradiction with $\partial_x(B) \subsetneq B$. Therefore ∂_x has property $\mathbb{I}$.

Conversely, assume now that ∂_x has property $\mathbb{I}$. To prove that G belongs to $\boldsymbol{\Sigma}$, take a non-abelian subgroup H of G. Note that H must contain an element of the form xa for some $a \in A$. Then since ∂_x is surjective, we have $a = \partial_x(b)$ and so $xa = x^b$. After possibly replacing H by $H^{b^{-1}}$, it suffices to consider the case when $b = 1$, and therefore $x \in H$. We can thus write $H = \langle x \rangle \ltimes B$ for some $B = H \cap A \leq A$. Let us show that H is self-normalizing in G. To this end, take an element $x^jc \in N_G(H)$. Then $x^{x^jc} = x^c = xc^{-x}c$, and so we must have $\partial_x(c) \in B$. Conversely, any element $x^jc \in A$ with the property that $\partial_x(c) \in B$ normalizes H, since for any $b \in B$ we also have $b^{x^jc} = (b^{x^j})^c = b^{x^j} \in B$.

Thus $N_G(H) = \langle x\rangle \ltimes \partial_x^{-1}(B)$. By property $\mathbb{I}$, we have $\partial_x^{-1}(B) = B$, which implies $N_G(H) = \langle x\rangle \ltimes B = H$, as required. □

Example 2.9. Let ζ_p be a primitive complex p-th root of unity for some prime p. Then ζ_p acts by multiplication on the abelian group $\mathbb{C}$ and we can form $G = \langle\zeta_p\rangle \ltimes \mathbb{C}$. The group G has a subgroup $H = \langle\zeta_p\rangle \ltimes \mathbb{Z}[\zeta_p]$. Now, note that ζ_p acts fixed point freely on $\mathbb{C}$, and multiplication by $1-\zeta_p$ is invertible on $\mathbb{C}$. Therefore ∂_{ζ_p} is an isomorphism of $\mathbb{C}$. However, multiplication by $1-\zeta_p$ maps $\mathbb{Z}[\zeta_p]$ into its augmentation $\{\sum_i \lambda_i\zeta_p^i \mid \sum_i = 0\}$. In particular, the restriction of ∂_{ζ_p} on $\mathbb{Z}[\zeta_p]$ is not surjective. Therefore the isomorphism ∂_{ζ_p} does not have property $\mathbb{I}$. In particular, neither G nor H belong to the class Σ.

Example 2.10. Take $C_p = \langle x\rangle$ for some prime p and consider a $\mathbb{Z}[C_p]$-module A. Assume that ∂_x is an isomorphism of A. To verify whether or not ∂_x has property $\mathbb{I}$, it suffices to show that the restriction of ∂_x on every cyclic submodule of A is surjective. To this end, suppose that B is a cyclic $\mathbb{Z}[C_p]$-module with ∂_x having trivial kernel on B. Therefore B is isomorphic to a quotient of the ring $\mathbb{Z}[C_p]$ by some ideal J. Denote $D = x-1$ and $N = x^{p-1}+x^{p-2}+\cdots+1$ as elements in $\mathbb{Z}[C_p]$. Observe that injectivity of ∂_x is equivalent to saying that whenever $D\cdot z \in J$ for some $z \in B$, it follows that $z \in J$. Now, we have that $D\cdot N = 0$, and so it follows that $N \in J$. Therefore J is the preimage of an ideal in the ring $\mathbb{Z}[C_p]/N\mathbb{Z}[C_p] \cong \mathbb{Z}[\zeta_p]$, where ζ_p is a complex primitive p-th root of unity. Note that multiplication by D is surjective on $\mathbb{Z}[C_p]/J$ if and only if we have $\mathrm{im}D + J = \mathbb{Z}[C_p]$.

Consider two special cases. First let $J = N\mathbb{Z}[C_p]$. This corresponds to the module $\mathbb{Z}[\zeta_p]$ from Example 2.9. Since $\mathrm{im}N = \ker D$ (see [15, Lemma 9.26]), we have that $\mathrm{im}D + J = \mathrm{im}D + \mathrm{im}N$. Dividing the polynomial N by D in $\mathbb{Z}[\langle x\rangle]$, we get the remainder $p \in \mathbb{Z}$. Whence $\mathrm{im}D+\mathrm{im}N$ contains $p\mathbb{Z}[C_p]$. On the other hand, $\mathrm{im}D+\mathrm{im}N$ is not the whole of $\mathbb{Z}[C_p]$, since ∂_{ζ_p} is not surjective on $\mathbb{Z}[\zeta_p]$. Consider now the case when J is the ideal generated by N and a prime q distinct from p. Thus $\mathrm{im}D + J$ contains $p\mathbb{Z}[C_p]$ and q. It follows that $\mathrm{im}D + J = \mathbb{Z}[C_p]$, and ∂_x is surjective in this case. Moreover, the map ∂_x will be surjective on all cyclic submodules of $\mathbb{Z}[C_p]/J$, as the ideals corresponding to these submodules all contain J. Therefore the group $\langle x\rangle \ltimes \mathbb{Z}_q[\zeta_p]$ belongs to Σ.

Lemma 2.11. *Let x be an automorphism of order p of an abelian group A. If ∂_x is surjective, then $A = pA$, i.e., A is p-divisible.*

Proof. Consider A as a $\mathbb{Z}[\langle x\rangle]$-module. In this sense, the operator ∂_x corresponds to the element $1 - x \in \mathbb{Z}[\langle x\rangle]$. We have $(1 - x)^p \equiv 0$

modulo $p\mathbb{Z}[\langle x\rangle]$, and so the image of $(\partial_x)^p\colon A \to A$ is a subgroup of $p\mathbb{Z}[\langle x\rangle]A = pA$. As ∂_x is assumed to be surjective, it follows that $A = pA$. □

Corollary 2.12. *Let $G = \langle x\rangle \ltimes A$ with x^p acting trivially on A for some prime p. Assume that A is free abelian of finite rank. Then G does not belong to Σ.*

Proof. By Lemma 2.11, the map ∂_x is not surjective, and so ∂_x does not have property $\mathbb{I}$. It follows from Proposition 2.8 that G does not belong to Σ. □

Theorem 2.13. *Let G be a finite soluble non-nilpotent group. Then $G \in \Sigma$ if and only if G splits as $G = \langle x\rangle \ltimes A$, where $\langle x\rangle$ is a p-group for some prime p, A is an abelian p'-group, x^p is central and x acts fixed point freely on A.*

Proof. Assume first that $G \in \Sigma$. By Lemma 2.7, all Sylow subgroups of G are abelian. It follows that $G' \cap Z(G) = 1$ by [14, 10.1.7], and G splits as $G = \langle x\rangle \ltimes G'$ with x^p in the Fitting subgroup of G for some prime p. Whence $\langle x^p\rangle \leq Z(G)$. If an element $xa \in G$ is central, then $a^x = a$ and so a must be central. As x is not central, we must have $Z(G) = \langle x^p\rangle$ and $C_G(x) = \langle x\rangle$. Observe that as G belongs to Σ, the map ∂_x is surjective on G', and so by Lemma 2.11 the group G' must be of p'-order. Now, if $\langle x\rangle$ is not of prime power order, then it splits as a product $\langle x\rangle = A_p \times A_{p'}$ with A_p a p-group and $A_{p'}$ a p'-group. Then $A_p \ltimes G'$ is a non-abelian proper normal subgroup of G, a contradiction. Whence $\langle x\rangle$ is of p-power order. Note that x acts fixed point freely on G' since $C_G(x) \cap G' = 1$. Thus $G/Z(G)$ is a Frobenius group with complement of order p.

Conversely, take $G = \langle x\rangle \ltimes A$ with the stated properties. Therefore x^p acts trivially on A and ∂_x is an injective endomorphism of A. As A is finite, it immediately follows that ∂_x is surjective and that it satisfies property $\mathbb{I}$. It now follows from Lemma 2.8 that G belongs to Σ. □

Notice that in Theorem 2.13 we have that $\langle x\rangle$ is the Sylow p-subgroup of G and $F = \langle x^p\rangle \ltimes A$, so $A = F/\langle x^p\rangle$.

Theorem 2.14. *Let G be a finite group in Σ. Then G is either soluble or simple.*

Proof. By induction on the order of G. Suppose that G is not simple, and let A be a maximal normal subgroup of G. Then A is abelian since G is in Σ, hence $C_G(A)$ contains A and is normal in G. It follows that either $C_G(A)$ is abelian or $C_G(A) = G$.

Assume first that $C_G(A)$ is abelian. Thus $A = C_G(A)$ by the maximality. Let P/A be a Sylow subgroup of G/A, and choose $x \in P \setminus A$. Then $A\langle x\rangle$ is not abelian and subnormal in P, so $A\langle x\rangle = P$ by (i) of Proposition 2.2. This shows that every Sylow subgroup of G/A is cyclic, hence G/A is soluble. Therefore G is soluble.

Hence we can assume that $C_G(A) = G$, and so $A \leq Z(G)$. Moreover, A is contained in every maximal subgroup M of G. Namely, if this is not the case, then $G = MA$ implies that $G' = M' < G$, so G' is abelian, hence G is soluble and we are done. Therefore every maximal subgroup of G is non-simple, so it is soluble by the induction hypothesis. Suppose that G has a maximal subgroup which is nilpotent. Then Proposition 2.6 implies that its Sylow 2-subgroup has nilpotency class ≤ 2, therefore it follows from [11] that G is soluble. Hence we can assume that every maximal subgroup of G is not nilpotent. In this case, every Sylow subgroup of G is abelian by Lemma 2.7. Then we have $G' \cap Z(G) = 1$, hence $G' < G$. Therefore G' is abelian, G is soluble and the proof is complete. $\square$

Theorem 2.14, Proposition 2.6 and Theorem 2.13 show that, in order to obtain a full classification of all finite groups in the class Σ, it only remains to describe the finite simple groups in Σ. At first we need a couple of auxiliary results.

Lemma 2.15. *Let $n > 2$. The dihedral group $\mathrm{Dih}(n)$ of order $2n$ belongs to Σ if and only either $n = 4$ or n is odd.*

Proof. Denote $G = \mathrm{Dih}(n) = \langle x, y \mid x^n = y^2 = 1, x^y = x^{-1}\rangle$, where $n \neq 4$. Suppose that $G \in \Sigma$. Then n is not a power of 2 by Proposition 2.6. Let p be an odd prime dividing n, and assume that n is even. Denote $H = \langle y, x^{n/p}\rangle$. Then $x^{n/2} \in Z(G) \setminus H$, hence H is not self-normalizing. Therefore n is odd.

Conversely, clearly $\mathrm{Dih}(4)$ belongs to Σ since it is minimal non-abelian. Suppose now n is odd. Let H be a non-abelian subgroup of $\mathrm{Dih}(n)$ of index m. Then H is conjugate to $K = \langle x^m, y\rangle$. Take $z = x^i y^j \in N_G(K)$, $0 \leq i < n$, $0 \leq j \leq 1$. Then $y^{x^i y^j} = x^{2(-1)^{j+1} i} y \in K$ if and only m divides i, that is, $z \in K$. This shows that $\mathrm{Dih}(n) \in \Sigma$. $\square$

Lemma 2.16. *If $q \neq 3, 5$ is an odd prime power, then $\mathrm{PSL}(2, q)$ does not belong to Σ.*

Proof. Let $G = \mathrm{PSL}(2, q)$. Since q is odd, it follows from [6] that G contains a subgroup H isomorphic to $\mathrm{Dih}((q-1)/2)$, and a subgroup K isomorphic to $\mathrm{Dih}((q+1)/2)$. If $q \equiv 1 \mod 4$, then H is not in Σ, unless $q = 5$, whereas if $q \equiv 3 \mod 4$, then $K \notin \Sigma$, unless $q = 3$ or

$q = 7$. Notice that $\mathrm{PSL}(2,7) \notin \Sigma$ since it has a subgroup isomorphic to $\mathrm{Sym}(4)$ (see [10, Theorem 8.27]), and therefore a non-abelian subgroup which is isomorphic to $\mathrm{Alt}(4)$ and is not self-normalizing. □

Theorem 2.17. *A finite non-abelian simple group G belongs to Σ if and only if it is isomorphic to* $\mathrm{Alt}(5)$ *or* $\mathrm{PSL}(2,2^n)$*, where $2^n - 1$ is a prime.*

Proof. Let $G \in \Sigma$ be finite non-abelian simple. Let P_p be a Sylow p-subgroup of G. It follows from [9, Theorem 1.1] that if $p > 3$, then P_p is abelian. For $p = 3$, the same result implies that either P_3 is abelian or $G \cong \mathrm{PSL}(2, 3^{3^a})$, where $a \geq 1$. The latter cannot happen by Lemma 2.16. Hence we conclude that P_3 needs to be abelian. If P_2 is also abelian, then all Sylow subgroups of G are abelian, and it follows from [2] that G belongs to one of the following groups: J_1, or $\mathrm{PSL}(2,q)$, where $q > 3$ and $q \equiv 0, 3, 5 \mod 8$. Note that the latter condition can be reduced to $q \equiv 0 \mod 8$ or $q = 5$ by Lemma 2.16. If P_2 is non-abelian, then it is minimal non-abelian by Proposition 2.6, and hence P_2 is nilpotent of class two. By [8], G is isomorphic to one of the following groups: $\mathrm{PSL}(2,q)$, where $q \equiv 7, 9 \mod 16$, $\mathrm{Alt}(7)$, $\mathrm{Sz}(2^n)$, $\mathrm{PSU}(3, 2^n)$, $\mathrm{PSL}(3, 2^n)$ or $\mathrm{PSp}(4, 2^n)$, where $n \geq 2$. The first family can be ruled out by Lemma 2.16.

It suffices to see which of the above listed groups belong to Σ. It follows from ATLAS that the Janko group J_1 has a subgroup isomorphic to $\mathrm{Dih}(3) \times \mathrm{Dih}(5)$, hence it is not in Σ by Proposition 2.1. Also, note that $\mathrm{Alt}(7)$ has a subgroup isomorphic to $\mathrm{Sym}(4)$, therefore $\mathrm{Alt}(7) \notin \Sigma$. If $G = \mathrm{Sz}(2^n)$ and P_2 is its Sylow 2-subgroup, then $|P_2'| = 2^n$ by [16], hence the Suzuki groups do not belong to Σ. Similarly, if G is $\mathrm{PSU}(3, 2^n)$ or $\mathrm{PSL}(3, 2^n)$, then the derived subgroup of a Sylow 2-subgroup of G has order 2^n, whereas if $G = \mathrm{PSp}(4, 2^n)$, then $|P_2'| = 2^{2n}$. This shows that neither of these groups belongs to Σ.

We are left with the groups $G = \mathrm{PSL}(2,q)$, where $q = 5$ or $q \equiv 0 \mod 8$. The subgroup structure of G is described in [6]. It is straightforward to verify that $\mathrm{PSL}(2,5) = \mathrm{Alt}(5) \in \Sigma$. Consider now $q = 2^n$ with $n \geq 3$. Suppose first $q-1$ is not a prime. Let d be a proper divisor of $q-1$. Then it follows from [6, section 250] that G has a single conjugacy class of size $q+1$ of a subgroup H isomorphic to $C_2^n \rtimes C_d$. Therefore $|G : N_G(H)| = q+1$, and $|G : H| = (q^2-1)/d$. As $d < q-1$, it follows that $H \neq N_G(H)$, therefore $G \notin \Sigma$. On the other hand, let $q-1$ now be a prime. Going through the list of subgroups of G given in [6], along with the given data on the number of conjugates of these subgroups, we see that apart from the subgroups in section 250, one has that for every non-abelian subgroup H of G, the number of conjugates of H is equal to $|G : H|$. As for the remaining subgroups, note that they must be of order $2^m d$ for some integer m and some divisor d of $q-1$. There are only two such options, one corresponding to abelian groups of order 2^m and the other corresponding to subgroups of order $q(q-1)$. Each of the latter belongs to a system of $(q^2-1)2^{n-m}/(2^k-1)$ conjugate groups for some k dividing n. Note that since $q-1$ is a prime, n must also be a prime. When $k = 1$, the group under consideration is abelian; therefore $k = n$. We thus have that the number of conjugates of each of these subgroups is equal to their index in G. By Lemma 2.4, this shows that if $2^n - 1$ is a prime, then $\mathrm{PSL}(2, 2^n)$ belongs to Σ. $\square$

3. Infinite groups

Let $G \in \Sigma$ be an infinite finitely generated soluble group, and let F denote the Fitting subgroup of G. Then F is polycyclic and $G = \langle x\rangle F$ for every element $x \in G \setminus F$, by Lemma 2.3. We will denote by $h(F)$ the Hirsch length of F. In what follows, the set of all periodic elements of a group G will be denoted by $T(G)$. For a prime p, let $T_p(G)$ be the set of elements in G of p-power order, and let $T_{p'}(G)$ be the set of elements in G of order coprime to p.

Lemma 3.1. *Let $G \in \Sigma$ be an infinite finitely generated soluble group, and suppose $h(F) = 1$. Then G is abelian.*

Proof. Assume not. Since $h(F) = 1$ the group G is infinite cyclic-by-finite. It easily follows that there exists a finite normal (and hence abelian) subgroup N such that G/N is either infinite cyclic or infinite dihedral. As the latter group is not in Σ, we can write $G = \langle x\rangle \ltimes N$ where x aperiodic. Since $N \leq F$ and G is not abelian, we conclude that $x \notin F$. By Lemma 2.3 there exists a prime number p such that $x^p \in F$. Then $x^p \in Z(G)$. Then $G/\langle x^p\rangle$ is finite. Moreover, it is not nilpotent by [5, Theorem 3.1]. The Fitting subgroup of $\frac{G}{\langle x^p\rangle}$ is $\frac{F}{\langle x^p\rangle} = \frac{\langle x^p\rangle \times T(F)}{\langle x^p\rangle}$, so it equals $T(F)$ since $h(F) = 1$. Now by Theorem 2.13 it follows that $\frac{G}{\langle x^p\rangle} = \frac{\langle x\rangle}{\langle x^p\rangle} \ltimes T(F)$. Hence $G = \langle x\rangle \ltimes T(F)$. Let $q \neq p$ be any prime number which does not divide the order of $T(F)$. Thus $\frac{\langle x^q\rangle}{\langle x^{pq}\rangle} \ltimes T(F)$ is a normal proper subgroup of the factor group $G/\langle x^{pq}\rangle$. Note that $\frac{\langle x^q\rangle}{\langle x^{pq}\rangle} \ltimes T(F)$ is not abelian since $x \notin Z(G)$. Therefore $G/\langle x^{pq}\rangle \notin \Sigma$, a contradiction. $\square$

Theorem 3.2. *Let $G \in \Sigma$ be an infinite finitely generated soluble group. Then G is abelian.*

Proof. We have $G = \langle x \rangle F$ with $x^p \in F \cap Z(G)$. Consider first the case when $x^p = 1$. Thus $G = \langle x \rangle \ltimes F$.

Let us show that x acts fixed point freely on F. To this end, let $f \in F$ be an element with $f^x = f$. For any positive integer k, the quotient groups G/F^k are finite and belong to Σ. This shows that $f^x F^k = f F^k$, and thus Theorem 2.13 gives that $f F^k$ is trivial in G/F^k. This means that $f \in \bigcap_k F^k = 1$. Therefore x acts fixed point freely on F and ∂_x is injective.

Since $G \in \Sigma$, the group F can not be free abelian by Corollary 2.12. Thus the torsion subgroup $T(F)$ is not trivial. The factor group $G/T(F)$ belongs to Σ, and so the action of x on this group is trivial by Corollary 2.12. Therefore $\partial_x(F)$ is trivial in $G/T(F)$. So the image of the map ∂_x is contained in the finite group $T(F)$. This is impossible since ∂_x is injective and its domain F is infinite.

Lastly, consider the case when x^p is not trivial in G. Let us look at the group $G/\langle x^p \rangle$. If $h(G) = 1$, then G is abelian by Lemma 3.1. Thus $h(G) \geq 2$ and so $G/\langle x^p \rangle$ is an infinite finitely generated soluble group with an element of order p outside its Fitting subgroup. Therefore $G/\langle x^p \rangle$ must be abelian by the above argument. As $\langle x^p \rangle$ is contained in $Z(G)$, it follows that G is nilpotent, and so G must be abelian. The proof is now complete. $\square$

Let $G \in \Sigma$ be a soluble group. It follows easily from Theorem 3.2 that every aperiodic element of G is central. As a consequence, every non-periodic soluble group in Σ is abelian. Therefore the following result completes the description of all infinite soluble groups in Σ.

Theorem 3.3. *Let G be an infinite non-abelian soluble periodic group. Then G belongs to Σ if and only if G splits as $\langle x \rangle \ltimes A$ with $\langle x \rangle$ a p-group for some prime p, A is a p'-group, x^p acts trivially on A and ∂_x has property $\mathbb{I}$.*

Proof. It follows from Proposition 2.8 that a group G with the above decomposition belongs to Σ. Therefore we are only concerned with proving the converse.

Assume that $G \in \Sigma$ is an infinite non-abelian soluble periodic group. We have that $G = \langle x \rangle F$ with $x^p \in F \cap Z(G)$ for some prime p. Note that since x is of finite order, say, $p^k \beta$ for some β coprime to p, we may replace x by x^β and assume from now on that $\langle x \rangle \subseteq T_p(F)$.

Let us first prove that $T_p(F) = \langle x^p \rangle$. To this end, it suffices to consider the factor group $G/\langle x^p \rangle$, and therefore we can assume that $x^p = 1$. Thus $G = \langle x \rangle \ltimes F$, and so $G = (\langle x \rangle \ltimes T_p(F)) \ltimes T_{p'}(F)$. If the group $\langle x \rangle \ltimes T_p(F)$ is not cyclic, then there is an element $z \in T_p(F)$ that commutes with x. It follows that the group $\langle x\rangle \ltimes T_p(F)$ contains the subgroup $\langle x\rangle \ltimes \langle z\rangle \cong C_p \times C_p$. Now G contains the subgroup $\langle x\rangle \ltimes T_p(F)$ that is normalized by the element z. This is a contradiction with $G \in \Sigma$. Hence the group $\langle x\rangle \ltimes T_p(F)$ is cyclic. This is possible only if $T_p(F)$ is trivial, as claimed.

We now have a splitting $G = \langle x\rangle \ltimes A$ with $A = T_{p'}(F)$, x acts non-trivially on A and x^p is central. Let us now show that x acts fixed point freely on A. It will then follow from Proposition 2.8 that ∂_x has property $\mathbb{I}$. To see this, assume that $z \in A$ is a fixed point of x. Thus $z \in Z(G)$. In particular, as $G \in \Sigma$, we have that z must be contained in every non-abelian subgroup of G. Now, as x acts non-trivially on A, there is an element $b \in A$ with $b^x \neq b$. Set $B = \langle b, b^x, \ldots, b^{x^{p-1}}\rangle$, this is an x-invariant finite subgroup of G. Therefore G possesses the finite non-abelian subgroup $\langle x\rangle \ltimes B$. By Theorem 2.13, we have that x acts fixed point freely on B. On the other hand, we must have that $z \in \langle x\rangle \ltimes B$, and so $z \in B$. This implies that z is trivial, as required. The proof is complete. □

Let G be an infinite group in Σ, and suppose that G is not soluble. Then G is perfect by (ii) of Proposition 2.2. Our last result gives information on the structure of such a group G provided that it is locally finite.

Theorem 3.4. *Let G be an infinite locally finite group in Σ. Then G is metabelian.*

Proof. Let $G \in \Sigma$ be locally finite, and suppose that G is not metabelian. Then G contains a finite insoluble subgroup, say H_0. It follows from Theorem 2.14 and Theorem 2.17 that H_0 is isomorphic to either $\mathrm{Alt}(5)$ or some $\mathrm{PSL}(2, 2^n)$ with n a prime. Pick an element $x_1 \in G$ that does not belong to H_0, and set $H_1 = \langle x_1, H_0\rangle$. We now have that the group H_1 is isomorphic to some $\mathrm{PSL}(2, 2^m)$ with m a prime. Finally let $x_2 \in G$ be an element not in H_1, and set $H_2 = \langle x_2, H_1\rangle$. The group H_2 is isomorphic to some $\mathrm{PSL}(2, 2^k)$ with k a prime. Now, as $H_2 = \mathrm{PSL}(2, 2^k)$ properly contains $H_1 = \mathrm{PSL}(2, 2^m)$, we must have $m \mid k$, which is impossible since m and k are distinct primes. □

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