Geometry and topology of 3-quasi-Sasakian manifolds

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joint work with B. Cappelletti Montano (Univ. Cagliari) and I. Yudin (CMUC)

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Almost contact manifolds

- An almost contact manifold (M, ϕ, ξ, η) is an odd-dimensional manifold M which carries a $(1,1)$-tensor field ϕ, a vector field ξ, a 1-form η, satisfying

$$\phi^2 = -I + \eta \otimes \xi \quad \textit{and} \quad \eta(\xi) = 1.$$

- It follows that

$$\phi\xi = 0 \quad \text{and} \quad \eta \circ \phi = 0.$$

- An almost contact manifold manifold of dimension $2n+1$ is said to be a contact manifold if

$$\eta \wedge (d\eta)^n \neq 0.$$

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Normality

- An almost contact manifold (M, ϕ, ξ, η) is said to be normal if

$$[\phi, \phi] + 2d\eta \otimes \xi = 0.$$

- M is normal iff the almost complex structure J on the product $M \times \mathbb{R}$ defined by setting, for any $X \in \Gamma(TM)$ and $f \in C^\infty(M \times \mathbb{R})$,

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right)$$

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Almost contact metric manifolds

- Every almost contact manifold admits a compatible metric g, i.e. such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all $X, Y \in \Gamma(TM)$.

- By putting $\mathcal{H} = \ker(\eta)$ one obtains a $2n$-dim. distribution on M and TM splits as the orthogonal sum

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Quasi-Sasakian manifolds

- A quasi-Sasakian structure on a $(2n+1)$-dimensional manifold M is a normal almost contact metric structure (ϕ, ξ, η, g) such that $d\Phi = 0$, where Φ is defined by

$$\Phi(X, Y) = g(X, \phi Y).$$

- They were introduced by Blair in 1967 in the attempt to unify Sasakian geometry ($d\eta = \Phi$) and cosymplectic geometry ($d\eta = 0,\ d\Phi = 0$).
- A quasi-Sasakian manifold is said to be of rank $2p+1$ if

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3-quasi-Sasakian manifolds

Definition

A 3-quasi-Sasakian manifold is given by a $(4n+3)$-dimensional manifold M endowed with three quasi-Sasakian structures $(\phi_1, \xi_1, \eta_1, g)$, $(\phi_2, \xi_2, \eta_2, g)$, $(\phi_3, \xi_3, \eta_3, g)$ satisfying the following relations, for any even permutation (α, β, γ) of $\{1, 2, 3\}$,

$$\phi_\gamma = \phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha,$$
$$\xi_\gamma = \phi_\alpha \xi_\beta, \quad \eta_\gamma = \eta_\alpha \circ \phi_\beta.$$

(For odd permutations, there is a change of signs).

The class of 3-quasi-Sasakian manifolds ($d\Phi_\alpha = 0$) includes as special cases the 3-cosymplectic manifolds ($d\eta_\alpha = 0,\ d\Phi_\alpha = 0$), and the 3-Sasakian manifolds ($d\eta_\alpha = \Phi_\alpha$).

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The canonical foliation of a 3-quasi-Sasakian manifold

Theorem

Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold. Then the 3-dimensional distribution $\mathcal{V} := \langle \xi_1, \xi_2, \xi_3 \rangle$ is integrable. Moreover, it defines a totally geodesic and Riemannian foliation.

- The distribution $\mathcal{H} := \bigcap_{\alpha=1}^{3} \ker(\eta_\alpha)$ has dimension $4n$, and TM splits as the orthogonal sum

$$TM = \mathcal{H} \oplus \mathcal{V}.$$

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Structure of the leaves of $\mathcal{V}$

Theorem

Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ *be a 3-quasi-Sasakian manifold. Then, for any even permutation* (α, β, γ) *of* $\{1, 2, 3\}$ *and for some* $c \in \mathbb{R}$

$$[\xi_\alpha, \xi_\beta] = c\xi_\gamma.$$

So we can divide 3-quasi-Sasakian manifolds in two main classes according to the behaviour of the leaves of $\mathcal{V}$: those 3-quasi-Sasakian manifolds for which each leaf of $\mathcal{V}$ is locally $SO(3)$ (or $SU(2)$) (which corresponds to take in the above theorem the constant $c \neq 0$), and those for which each leaf of $\mathcal{V}$ is locally an abelian group (the case $c = 0$).

The rank of a 3-quasi-Sasakian manifold

In a 3-quasi-Sasakian manifold one has, in principle, the three odd ranks r_1, r_2, r_3 of the 1-forms η_1, η_2, η_3, since we have three distinct, although related, quasi-Sasakian structures. We prove that these ranks coincide and their value has great influence on the geometry of the manifold.

The rank of a 3-quasi-Sasakian manifold

Theorem

Let $(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold. Then the 1-forms η_1, η_2 and η_3 have all the same rank $4l + 3$, for some $l \leq n$, or rank 1, according to $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$ with $c \neq 0$, or $[\xi_\alpha, \xi_\beta] = 0$, respectively.

- The above theorem allows to define the rank of a 3-quasi-Sasakian manifold $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ as the rank shared by the 1-forms η_1, η_2 and η_3.

Theorem

Every 3-quasi-Sasakian manifold of rank 1 is 3-cosymplectic.

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Every 3-quasi-Sasakian manifold of maximal rank is 3-α-Sasakian.

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Toward a decomposition theorem

Besides the vertical distribution $\mathcal{V}$ we proved that the following two fundamental distributions are Riemannian and totally geodesic.

- $\mathcal{E}^{4m} := \{X \in \mathcal{H} \mid i_X d\eta_\alpha = 0, \quad \text{for } \alpha = 1, 2, 3\}$,
- $\mathcal{E}^{4l+3} := \mathcal{E}^{4l} \oplus \mathcal{V}$,

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3-quasi-Sasakian manifolds of rank $4l+3$

The following decomposition theorem holds.

Theorem

Let $(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ *be a 3-quasi-Sasakian manifold of rank* $4l+3$ *with* $[\xi_\alpha, \xi_\beta] = 2\xi_\gamma$*. Then* M^{4n+3} *is locally the Riemannian product of a 3-Sasakian manifold* M^{4l+3} *and a hyper-Kähler manifold* M^{4m}*, with* $m = n - l$*.*

Nontrivial examples of 3–quasi-Sasakian manifolds

Example

- Let M be a compact Riemannian manifold and G a finite group freely acting on M. Then from the Hodge theory we can obtain
$$H^*(M/G) \cong H^*(M)^G .$$

- Now, let M and N are two compact manifolds with G-action. Then G acts on the product $M \times N$ and we get
$$H^k(M \times N)^G = \bigoplus_{q+p=k} (H^q(M) \otimes H^p(N))^G ,$$
since $H^q(M) \otimes H^p(N)$ are G-invariant subspaces.

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Example (continued)

Now, take $M = S^{4n-1} \subset \mathbb{H}^n$ and $N = \mathbb{T}^4 = \mathbb{H}/\mathbb{Z}^4$.
Let $\mathbb{Z}_4$ (the cyclic group of order 4) act on $S^{4n-1}$by

- $\sigma \cdot (q_1, \dots, q_n) = (iq_1, \dots, iq_n)$,
  and on $\mathbb{T}^4$ by
- $\sigma \cdot [q] = [iq]$.

We get

$$H^k \left(S^{4n-1} \otimes \mathbb{T}^4\right)^{\mathbb{Z}_4} = H^k \left(\mathbb{T}^4\right)^{\mathbb{Z}_4} \oplus H^{k-4n+1} \left(\mathbb{T}^4\right)^{\mathbb{Z}_4}.$$

It follows that the Poincaré polynomial of $\left(S^{4n-1} \times \mathbb{T}^4\right)/\mathbb{Z}_4$ is

$$\left(1 + t^{4n-1}\right)\left(1 + 4t^2 + t^4\right).$$

Thus, $\left(S^{4n-1} \times \mathbb{T}^4\right)/\mathbb{Z}_4$ cannot be a product of 3-Sasakian and hyper-Kähler manifolds.

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Contact circles and contact spheres

(Geiges-Gonzalo,1995)

- A *contact circle* on M^3 is a pair of contact forms (η_1, η_2) such that for any $(\lambda_1, \lambda_2) \in S^1$ the 1-form $\lambda_1\eta_1 + \lambda_2\eta_2$ is also a contact form.
- A *contact p-sphere* on M^{2n+1} is given by $(\eta_1, \ldots, \eta_{p+1})$ such that for any $(\lambda_1, \ldots, \lambda_{p+1}) \in S^p$, the 1-form $\lambda_1\eta_1 + \ldots + \lambda_{p+1}\eta_{p+1}$ is also a contact form.

Theorem (Zessin, 2005)

Any 3-Sasakian manifold M^{4n+3} admits a 2-sphere of contact structures (which is both round and taut).

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Contact circles and contact spheres

- A contact sphere is said to be *taut* if all contact forms belonging to the sphere define the same volume form.
- A contact sphere is said to be *round* if for any $(\lambda_1, \ldots, \lambda_{p+1}) \in S^p$, the Reeb vector field of

$$\eta = \sum_{h=1}^{p+1} \lambda_h \eta_h \quad \text{is} \quad \xi = \sum_{h=1}^{p+1} \lambda_h \xi_h .$$

- Zessin showed that: taut $\iff$ round in dimension 3.

Almost contact spheres

Definition

Let $(\phi_1, \xi_1, \eta_1), \dots, (\phi_{p+1}, \xi_{p+1}, \eta_{p+1})$ be almost contact structures on M. We say that they define an *almost contact sphere* if for any $(\lambda_1, \dots, \lambda_{p+1}) \in S^p$ the tensors

$$\phi := \sum_{h=1}^{p+1} \lambda_h \phi_h,$$

$$\xi := \sum_{h=1}^{p+1} \lambda_h \xi_h,$$

$$\eta := \sum_{h=1}^{p+1} \lambda_h \eta_h,$$

define an almost contact structure on M.

Almost contact spheres

Theorem

Let $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$ *be an almost contact metric 3-structure on* M*. Then* M *carries an almost contact 2-sphere* (ϕ, ξ, η) *given by*

$$\begin{aligned} \phi &:= \lambda_1\phi_1 + \lambda_2\phi_2 + \lambda_3\phi_3, \\ \xi &:= \lambda_1\xi_1 + \lambda_2\xi_2 + \lambda_3\xi_3, \\ \eta &:= \lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_3\eta_3, \end{aligned}$$

where $(\lambda_1, \lambda_2, \lambda_3) \in S^2$*. Furthermore, the Riemannian metric* g *is compatible with* (ϕ, ξ, η)*, and if* $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$ *is hyper-normal, then* (ϕ, ξ, η, g) *is a normal almost contact metric structure on* M*.*

Sasakian spheres

Corollary

A 3-quasi-Sasakian manifold of rank $4l+3$ $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ defines a 2-sphere of quasi-Sasakian structures (ϕ, ξ, η, g) of the same rank (which is both round and taut).

In particular:

Corollary

Any 3-Sasakian manifold admits a contact 2-sphere of Sasakian structures (which is both round and taut).

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Topology of 3-quasi-Sasakian manifolds

3-quasi-Sasakian manifolds

$$\begin{cases} \text{3-Sasakian manifolds: top rank } 4n+3 \\ \text{3-quasi-Sasakian manifolds of intermediate ranks } 4l+3,\ 1 \leq l < n \\ \text{3-cosymplectic manifolds: minimum rank } 1 \end{cases}$$

I - Topology of 3-Sasakian manifolds

Main Results on the Betti numbers:

Theorem (Fujitani,1966)

In any compact Sasakian manifold M^{2n+1}, the odd Betti numbers b_{2k+1} are even, for $2k+1 < n$.

Theorem (Galicki-Salamon, 1996)

In any compact 3-Sasakian manifold M^{4n+3}, the odd Betti numbers b_{2k+1} are zero, for each $k < n$.

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Topology of cosymplectic manifolds

Theorem (Chinea, de León, Marrero, 1993)

Let M^{2n+1} be a compact cosymplectic manifold. Then,

(i) $b_0 \leq b_1 \leq \ldots \leq b_n$.

(ii) $b_{2p+1} - b_{2p}$ *is even, for each* $p \leq n$. *In particular* b_1 *is odd.*

They also proved a version of the strong Lefschetz property.

II - Topology of 3-cosymplectic manifolds

Definition

$b_p^h := \dim \{\omega \in \Omega^p(M) \mid \omega \text{ is harmonic}, i_{\xi_\alpha}\omega = 0, \alpha = 1, 2, 3\}$

Theorem

Let M^{4n+3} be a compact 3-cosymplectic manifold. Then, for each integer p such that $0 \leq p \leq 2n-1$,

(i) b^h_{2p+1} *is divisible by four.*

(ii) $b_p = b_p^h + 3b_{p-1}^h + 3b_{p-2}^h + b_{p-3}^h$.

Corollary

For each integer p such that $0 \leq p \leq 2n-1$,

$$b_{2p} + b_{2p+1} = 4k, \qquad \text{for some } k \in \mathbb{N}.$$

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Let M^{4n+3} be a compact 3-cosymplectic manifold. Then, for each integer p such that $0 \leq p \leq 2n-1$,

(i) b_{2p+1}^h *is divisible by four.*

(ii) $b_p = b_p^h + 3b_{p-1}^h + 3b_{p-2}^h + b_{p-3}^h$.

Corollary

For each integer p such that $0 \leq p \leq 2n-1$,

$$b_{2p} + b_{2p+1} = 4k, \qquad \text{for some } k \in \mathbb{N}.$$

II - Topology of 3-cosymplectic manifolds

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III - Topology of 3-quasi-Sasakian manifolds

We introduce the operators

$$\theta_\alpha X := \begin{cases} 0, & \text{if } X \in \Gamma(\mathcal{E}^{4l+3}) \\ \phi_\alpha X, & \text{if } X \in \Gamma(\mathcal{E}^{4m}) \end{cases}$$

and the associated 2-forms $\Theta_\alpha := g(\cdot, \theta_\alpha \cdot)$.
In any 3-quasi-Sasakian manifold each Θ_α is closed. The fact that the 2-forms Θ_α are also coclosed follows from the following lemma.

Lemma

In any 3-quasi-Sasakian manifold M^{4n+3} of non-maximal rank $4l+3$ one has

$$\nabla \Theta_\alpha = 0.$$

III -Topology of 3-quasi-Sasakian manifolds

Then, the following lower bound on the Betti numbers follows.

Theorem

In any compact 3-quasi-Sasakian manifold M^{4n+3} of non-maximal rank $4l+3$, one has the inequality

$$b_{2k} \geq \binom{k+2}{2} \qquad \text{for } 0 \leq k \leq n-l$$

Corollary

The sphere S^{4n+3} does not admit any 3-quasi-Sasakian structure of non-maximal rank.

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III -Topology of 3–quasi-Sasakian manifolds

Stronger bounds on the Betti numbers of compact 3-quasi-Sasakian manifolds are obtained after recognising that there is a decomposition of the space of harmonic forms

$$\Omega^k_\triangle(M) = \bigoplus_{s+t=k} \Omega^{s,t}_\triangle(M),$$

where

$$\Omega^{s,t}_\triangle(M) := \{\omega \in \Omega^{s+t}_\triangle(M) \,|\, i_P\omega = s\omega\},$$

and P is the projection on the 3-α-Sasakian part. Then, an action of $so(4,1)$ on $\bigoplus_{t=0}^{4m} \Omega^{s,t}_\triangle(M)$ is found and one can prove the following result.

Theorem

In any compact 3-quasi-Sasakian manifold M^{4n+3} of rank $4l+3$, the odd Betti numbers b_{2k+1} are divisible by 4, for each $k < l$.

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Obrigado!