Topology of 3-cosymplectic manifolds

Antonio De Nicola

joint work with B. Cappelletti Montano and I. Yudin

CMUC, Department of Mathematics,
University of Coimbra

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Outline

Almost contact manifolds

- An almost contact manifold (M, ϕ, ξ, η) is an odd-dimensional manifold M which carries a $(1,1)$-tensor field ϕ, a vector field ξ, a 1-form η, satisfying

$$\phi^2 = -I + \eta \otimes \xi \quad \textit{and} \quad \eta(\xi) = 1.$$

- It follows that

$$\phi\xi = 0 \quad \text{and} \quad \eta \circ \phi = 0.$$

- An almost contact manifold manifold of dimension $2n+1$ is said to be a contact manifold if

$$\eta \wedge (d\eta)^n \neq 0.$$

Normality

- An almost contact manifold (M, ϕ, ξ, η) is said to be normal if
$$[\phi, \phi] + 2d\eta \otimes \xi = 0.$$

- M is normal iff the almost complex structure J on the product $M \times \mathbb{R}$ defined by setting
$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right),$$
for any $X \in \Gamma(TM)$ and $f \in C^{\infty}(M \times \mathbb{R})$ is integrable.

Almost contact metric manifolds

- Every almost contact manifold admits a compatible metric g, i.e. such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y),$$

for all $X, Y \in \Gamma(TM)$. It follows that $\eta(X) = g(X, \xi)$.

- Also, $\Phi(X, Y) := g(X, \phi Y)$ defines a 2-form

- By putting $\mathcal{H} = \ker(\eta)$ one obtains a $2n$-dim. distribution on M and TM splits as the orthogonal sum

$$TM = \mathcal{H} \oplus \langle \xi \rangle .$$

Cosymplectic manifolds

Definition

A cosymplectic manifold is a normal almost contact metric manifold (M, ϕ, ξ, η, g) such that $d\eta = 0$ and $d\Phi = 0$.

D. Chinea, M. de León, J.C. Marrero
Topology of cosymplectic manifolds,
J. Math. Pures Appl. **72** (1993), 567–591.

Topology of cosymplectic manifolds

Among their results for a compact cosymplectic manifold (M, ϕ, ξ, η, g) of dimension $2n+1$:

- $b_0 \leq b_1 \leq \ldots \leq b_n = b_{n+1} \geq b_{n+2} \geq \ldots \geq b_{2n+1}$
- $b_{2p+1} - b_{2p}$ is even. In particular b_1 is odd.
- The first nontrivial example
- A version of the strong Lefschetz theorem
- ...
- and many others.

3-structures

- An almost 3-contact metric manifold is a $(4n+3)$-dim. smooth manifold M endowed with three almost contact structures $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$ satisfying, for any even permutation (α, β, γ) of $(1, 2, 3)$, the relations

$$\phi_\gamma = \phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha,$$
$$\xi_\gamma = \phi_\alpha \xi_\beta, \quad \eta_\gamma = \eta_\alpha \circ \phi_\beta,$$

  and a Riemannian metric g compatible with each of them. (For odd permutations, there is a change of signs).
- M is said to be normal (sometimes hyper-normal) if each almost contact structure is normal.
- The distribution $\mathcal{H} := \bigcap_{\alpha=1}^{3} \ker(\eta_\alpha)$ has dimension $4n$, and TM splits as the orthogonal sum

$$TM = \mathcal{H} \oplus \mathcal{V},$$

  where $\mathcal{V} := \langle \xi_1, \xi_2, \xi_3 \rangle$.

3-cosymplectic manifolds

Definition

An almost 3-cosymplectic manifold is an almost 3-contact metric manifold $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ such that $d\eta_\alpha = 0$ and $d\Phi_\alpha = 0$, for each α. It is called 3-cosymplectic if it is normal.

Theorem (Ianus et al.)

Any almost 3-cosymplectic manifold is 3-cosymplectic.

3-cosymplectic manifolds and 3-Sasakian manifolds ($d\eta_\alpha = \Phi_\alpha$) are special cases of 3-quasi-Sasakian manifolds ($d\Phi_\alpha = 0$), which we studied in recent years.

Vertical foliation and transverse structure

In any 3-cosymplectic manifold ξ_α, η_α, ϕ_α and Φ_α are ∇-parallel. Thus,

$$[\xi_\alpha, \xi_\beta] = \nabla_{\xi_\alpha}\xi_\beta - \nabla_{\xi_\beta}\xi_\alpha = 0$$

- Thus $\mathcal{V} = \langle \xi_1, \xi_2, \xi_3 \rangle$ is involutive. It defines a Riemannian foliation with totally geodesic leaves.

Theorem (B. Cappelletti Montano and A.d.N., 2007)

Let $\left(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g\right)$ be a 3-cosymplectic manifold. If $\mathcal{V}$ is regular, then $M^{4n+3}/\mathcal{V}$ is a hyper-Kähler manifold of dimension $4n$. Consequently, any 3-cosymplectic manifold is Ricci-flat.

Horizontal foliation and local product structure

Unlike the case of 3-Sasakian geometry, in any 3-cosymplectic manifold also $\mathcal{H}$ is integrable. Indeed, for all $X, Y \in \Gamma(\mathcal{H})$,

$$\eta_\alpha([X, Y]) = -2d\eta_\alpha(X, Y) = 0$$

since $d\eta_\alpha = 0$. Thus,

- The tangent bundle splits as the orthogonal sum

$$TM = \mathcal{H} \oplus \mathcal{V}$$

  of two Riemannian foliations with totally geodesic leaves.

Theorem

Any 3-cosymplectic manifold M^{4n+3} is locally the Riemannian product of a hyper-Kähler manifold N^{4n} and a 3-dimensional flat abelian Lie group G^3.

Examples of 3-cosymplectic manifold

Example

- The standard example of a compact 3-cosymplectic manifold is given by the torus $\mathbb{T}^{4n+3}$ with the structure described in [F. Martín Cabrera, *Czechoslovak Math. J.*,1998].
- On the other hand, the standard example of a noncompact 3-cosymplectic manifold is given by $\mathbb{R}^{4n+3}$ with the structure described in [B.Cappelletti Montano-A.d.N., *J. Geom. Phys.*, 2007].
- Both the above examples are the global product of a hyper-Kähler manifold with a 3-dimensional flat abelian Lie group. In fact, as we have seen, locally this is always true.

Nontrivial examples of 3-cosymplectic manifolds

Example

Let (M^{4n}, J_α, G) be a compact hyper-Kähler manifold and $f : M^{4n} \longrightarrow M^{4n}$ be an hyper-Kählerian isometry, i.e., an isometry such that

$$f_* \circ J_\alpha = J_\alpha \circ f_*, \qquad \text{for each } \alpha \in \{1, 2, 3\}.$$

Define the action φ of $\mathbb{Z}^3$ on the product manifold $M^{4n} \times \mathbb{R}^3$ by

$$\varphi\left((k_1, k_2, k_3), (x, t_1, t_2, t_3)\right) = \left(f^{k_1+k_2+k_3}(x), t_1 + k_1, t_2 + k_2, t_3 + k_3\right).$$

Then, $M_f^{4n+3} := (M^{4n} \times \mathbb{R}^3)/\mathbb{Z}^3$ is a smooth manifold.

Nontrivial examples of 3-cosymplectic manifolds

Example (continued)

We define a 3-cosymplectic structure on $M_f^{4n+3} := (M^{4n} \times \mathbb{R}^3)/\mathbb{Z}^3$ as follows.
On $M^{4n} \times \mathbb{R}^3$, we define $\hat{\xi}_\alpha := \frac{\partial}{\partial t_\alpha}$,

$$\hat{g} = G + dt_1 \otimes dt_1 + dt_2 \otimes dt_2 + dt_3 \otimes dt_3.$$

and $\hat{\eta}_\alpha := \hat{g}(\cdot, \hat{\xi}_\alpha)$. Finally,

$$\hat{\phi}_\alpha E := J_\alpha E_{\mathcal{H}} + \sum_{\beta,\gamma=1}^{3} \epsilon_{\alpha\beta\gamma} \hat{\eta}_\beta(E) \hat{\xi}_\gamma,$$

where we have used the unique decomposition of the vector field $E = E_{\mathcal{H}} + \sum_{\beta=1}^{3} \hat{\eta}_\beta(E)\hat{\xi}_\beta$, $E_{\mathcal{H}}$ being the component tangent to M^{4n}.

Nontrivial examples of 3-cosymplectic manifolds

Example (continued)

- We defined above $(\hat{\phi}_\alpha, \hat{\xi}_\alpha, \hat{\eta}_\alpha, \hat{g})$ on $M^{4n} \times \mathbb{R}^3$ $(\alpha = 1, 2, 3)$.
- But all these structures descend to the quotient, so that

$$M_f^{4n+3} := (M^{4n} \times \mathbb{R}^3)/\mathbb{Z}^3$$

  with the induced structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is a 3-cosymplectic manifold.
- M_f^{4n+3} is not in general a global product of a hyper-Kähler manifold M^4 by the torus $\mathbb{T}^3$.

Nontrivial examples of 3-cosymplectic manifolds

Example (continued)

Theorem

Let $M^4 = \mathbb{T}^4 = \mathbb{H}/\mathbb{Z}^4$ and $f : \mathbb{H} \to \mathbb{H} : \mathbf{q} \mapsto \mathbf{q} \cdot \mathbf{i}$. Then M_f^7 is not a global product of a compact hyper-Kähler 4-manifold and the torus $\mathbb{T}^3$.

Idea of the proof.

- A compact hyper-Kähler 4-manifold is either the Torus $\mathbb{T}^4$ or a complex $K3$ surface.
- In the first case $b_2\left(M_f^7\right)$ would be 21, in the second case $b_2\left(M_f^7\right)$ would be 25. But it can be shown by cellular homology techniques that $b_2\left(M_f^7\right) < 21$.

Basic cohomology with respect to the Reeb foliation

The spaces of basic forms with respect to $\mathcal{V} := \langle \xi_1, \xi_2, \xi_3 \rangle$ are

$$\Omega_B^k(M) := \left\{ \omega \in \Omega^k(M) \,\middle|\, i_{\xi_\alpha}\omega = 0,\ i_{\xi_\alpha} d\omega = 0 \right\}.$$

The restriction d_B of the exterior derivative d to $\Omega_B^k(M)$ sends basic forms into basic forms, defining the basic cohomology $H_B(M)$ with respect to $\mathcal{V}$ as the cohomology of the complex $(\Omega_B^*(M), d_B)$.

Betti numbers of 3-cosymplectic manifolds

Definition

$b_p^h := \dim \{\omega \in \Omega^p(M) \mid \omega \text{ is harmonic}, i_{\xi_\alpha}\omega = 0, \alpha = 1, 2, 3\}$

Theorem

Let M^{4n+3} be a compact 3-cosymplectic manifold. Then, for each integer p such that $0 \leq p \leq 2n - 1$,

(i) *b_{2p+1}^h is divisible by four.*

(ii) $b_p = b_p^h + 3b_{p-1}^h + 3b_{p-2}^h + b_{p-3}^h$.

Corollary

For each integer p such that $0 \leq p \leq 2n - 1$,

$$b_{2p} + b_{2p+1} = 4k, \qquad \text{for some } k \in \mathbb{N}.$$

Betti numbers of 3-cosymplectic manifolds

Proposition

Let M^{4n+3} be a compact 3-cosymplectic manifold. Then,

$$b^h_{2p} \geq \binom{p+2}{2} \qquad \text{for } 0 \leq p \leq n$$

From this proposition and the previous theorem we get easily

Corollary

Let M^{4n+3} be a compact 3-cosymplectic manifold. Then,

$$b_p \geq \binom{p+2}{2} \qquad \text{for } 0 \leq p \leq 2n+1$$

Action of *so* $(4,1)$ on the basic cohomology

For (α, β, γ) cyclic permutation let

$$\Xi_\alpha := \frac{1}{2}\left(\Phi_\alpha + 2\eta_\beta \wedge \eta_\gamma\right).$$

Define the operators

$$L_\alpha \colon \Omega^k(M) \to \Omega^{k+2}(M) : \omega \mapsto \Xi_\alpha \wedge \omega$$

and

$$\Lambda_\alpha := *L_\alpha * \colon \Omega^{k+2}(M) \to \Omega^k(M).$$

Theorem

The operators L_α, Λ_α, $\alpha \in \{1,2,3\}$, give a structure of so $(4,1)$-module on the basic cohomology $H^*_B(M)$.

This result is the odd-dimensional analogous of the one obtained by Verbitsky about hyper-Kähler manifolds.

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Thank you!