Nilpotent aspherical Sasakian manifolds

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Aspherical manifolds

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- Two aspherical manifolds are homotopy equivalent if and only if their fundamental groups are isomorphic.
- Thus an aspherical manifold is determined, up to homotopy equivalences, by its fundamental group.

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  The quotient $\Gamma \backslash G$ is called a nilmanifold.
  It is a compact aspherical manifold M with fundamental group Γ.

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- The quotient $N_\Gamma = \Gamma \backslash G(\Gamma)$ is a nilmanifold with the same fundamental group of M.
- M and N_Γ are aspherical manifolds with the same fundamental group, hence they are homotopy equivalent.

The Borel conjecture

Conjecture (Borel)

Two homotopy equivalent compact aspherical manifolds are homeomorphic.

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- Thus, a compact aspherical manifold M with nilpotent fundamental group Γ is homeomorphic to the nilmanifold $N_\Gamma = \Gamma \backslash G(\Gamma)$.

Definition

If M is homeomorphic but not diffeomorphic to a nilmanifold, then it is called an exotic nilmanifold.

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Theorem (Farrell-Jones, 1994)

The connected sum of an exotic $n-$sphere and a nilmanifold N^n is an exotic nilmanifold, for $n > 4$.

Contact Exotic nilmanifolds

Theorem (Boyer-Galicki)

There are infinitely many Sasakian (hence contact) exotic spheres.

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The connected sum of two contact manifolds carries a contact structure.

Corollary

The connected sum of a compact contact nilmanifold and a contact exotic sphere gives an example of a contact exotic nilmanifold.

The Main Result

Theorem

Let M^{2n+1} be a compact aspherical Sasakian manifold with nilpotent fundamental group. Then M is diffeomorphic to $N_\Gamma = \Gamma \backslash H(1,n)$ where $\Gamma \cong \pi_1(M)$ is a lattice in the Heisenberg group $H(1,n)$.

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- Sasakian manifolds
- Heisenberg groups

Almost contact manifolds

- An almost contact manifold (M, φ, ξ, η) is an odd-dimensional manifold M which carries a $(1,1)$-tensor field φ, a vector field ξ, a 1-form η, satisfying

$$\varphi^2 = -I + \eta \otimes \xi \quad \textit{and} \quad \eta(\xi) = 1.$$

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- An almost contact manifold is said to be normal if

$$[\varphi, \varphi] + d\eta \otimes \xi = 0.$$

Normality

- One defines an almost complex structure J on the product $M \times \mathbb{R}$ by setting, for any $X \in \Gamma(TM)$ and $f \in C^\infty(M \times \mathbb{R})$,

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right)$$

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One has

$$[J, J] = 0 \iff [\varphi, \varphi] + d\eta \otimes \xi = 0.$$

Sasakian manifolds

- Every almost contact manifold (M, φ, ξ, η) admits a compatible metric, that is, a metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

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- A Sasakian manifold M^{2n+1} is a normal almost contact metric manifold such that

$$d\eta(X, Y) = g(X, \varphi Y), \quad \text{for all vector fields } X, Y.$$

Example of a Sasakian manifold: $H(1, n)$

An example of a manifold that admits a left-invariant Sasakian structure is the Heisenberg group $H(1, n)$. It consists of the square matrices of order $n + 2$ of the form

$$\begin{pmatrix} 1 & \mathbf{x}^T & z \\ 0 & I_n & \mathbf{y} \\ 0 & 0 & 1 \end{pmatrix},$$

where I_n is the $n \times n$ identity matrix, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $z \in \mathbb{R}$.

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where I_n is the $n \times n$ identity matrix, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $z \in \mathbb{R}$. The left invariant contact form is $\eta = dz - \mathbf{x} \cdot d\mathbf{y}$, then $\xi = \partial_z$,

$$\varphi = \sum_i [(\partial_{x_i} + x_i \partial_z) \otimes dx_i - \partial_{x_i} \otimes dy_i]$$

and

$$g = \sum_i (dx_i \otimes dx_i + dy_i \otimes dy_i) + \eta \otimes \eta.$$

Sasakian nilmanifolds

If Γ is a discrete cocompact subgroup of $H(1,n)$, then the Sasakian structure, being left-invariant, goes to the quotient. Thus, $N_\Gamma = \Gamma \backslash H(1,n)$ is a compact Sasakian nilmanifold.
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Theorem (Cappelletti Montano, –, Marrero, Yudin, 2015)

A nilmanifold $\Gamma \backslash G$ of dimension $2n + 1$ admits a Sasakian structure if and only if G is isomorphic to $H(1, n)$.

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Remark (Bazzoni, 2017)

Let M^{2n+1} be a compact aspherical Sasakian manifold with nilpotent fundamental group. Then M is homotopy equivalent to $N_\Gamma = \Gamma \backslash H(1, n)$ where $\Gamma \cong \pi_1(M)$ is a lattice in the Heisenberg group $H(1, n)$.

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Corollary (Bazzoni's remark +Borel)

Let M^{2n+1} be a compact aspherical Sasakian manifold with nilpotent fundamental group. Then M is homeomorphic to $N_\Gamma = \Gamma\backslash H(1,n)$ where $\Gamma \cong \pi_1(M)$ is a lattice in the Heisenberg group $H(1,n)$.

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Remark

The above result is equivalent to the non-existence of Sasakian exotic nilmanifolds.

A similar result in even dimensions

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If X is a compact aspherical Kähler manifold with nilpotent fundamental group, then X is biholomorphic to a complex torus. More generally, if X is a compact aspherical Kähler manifold with virtually solvable fundamental group, then X is biholomorphic to a finite quotient of a complex torus.

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Thus, if the fundamental group of a compact aspherical Sasakian manifold M is a solvable group S, then there is a nilpotent normal subgroup Γ of finite index. Hence we can find a finite smooth cover $\bar{M} \to M$ which is a compact aspherical Sasakian manifold with fundamental group Γ. Thus $\bar{M} \cong N_\Gamma$ and M is diffeomorphic to a finite quotient N_Γ/G (with $G = S/\Gamma$) of N_Γ.

Corollary

If M^{2n+1} is a compact aspherical Sasakian manifold with (virtually) solvable fundamental group, then M is diffeomorphic to a finite quotient of the Heisenberg nilmanifold.

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1. We establish a correspondence between quasi-isomorphisms of CDGAs

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and smooth homotopy equivalences

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2. We show that if M is Sasakian there exists a quasi-isomorphism $\rho\colon \bigwedge T_e^* G(\Gamma) \to \Omega^\bullet(M)$ with good properties.
3. We show that the corresponding smooth homotopy equivalence $h : M \to N_\Gamma$ is a diffeomorphism.

Commutative Differential Graded Algebras (CDGAs)

Definition

A CDGA (A, d) is a graded vector space $A = \bigoplus_{k \in \mathbb{N}} A^k$ with

- a graded commutative product

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Given a CDGA we can always form its cohomology

$$H^k(A) = \frac{\ker d : A^k \to A^{k+1}}{\mathrm{Im}\, d : A^{k-1} \to A^k}.$$

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$$(H(M), \cup, d = 0);$$

- the Chevalley-Eilenberg complex $\left(\bigwedge \mathfrak{g}^*, \wedge, d^{CE}\right)$ of a Lie algebra $\mathfrak{g}$ with the multiplication of the exterior algebra.

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Definition

A *quasi-isomorphism* is a morphism of CDGAs $f : A \to B$ such that it induces an isomorphism in cohomology.

Lie group-valued maps and CDGA morphisms

Let M be a manifold, G a Lie group and $\mathfrak{g}$ its Lie algebra. If $m : M \to G$ is a smooth map, then $m^*|_{\bigwedge \mathfrak{g}^*} \colon \bigwedge \mathfrak{g}^* \to \Omega(M)$ is a morphism of CDGAs.

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Proposition (Cartan, cf. Sharpe's book)

Let $\widetilde{M}$ be a 1-connected manifold, G a Lie group and $\mathfrak{g}$ its Lie algebra. For every morphism of CDGAs

$$\mu \colon \bigwedge \mathfrak{g}^* \to \Omega(\widetilde{M})$$

there exists a smooth map $m : \widetilde{M} \to G$ such that

$$\mu = m^*|_{\bigwedge \mathfrak{g}^*}.$$

Nomizu theorem

Theorem (Nomizu)

Let $N_\Gamma = \Gamma \backslash G$ *be a compact nilmanifold. Then there is a quasi-isomorphism*

$$\psi_\Gamma \colon \left(\bigwedge \mathfrak{g}^*, d^{CE} \right) \to (\Omega(N_\Gamma), d)$$

where $\left(\bigwedge \mathfrak{g}^*, d^{CE} \right)$ *is the Chevalley-Eilenberg CDGA.*

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where $\left(\bigwedge \mathfrak{g}^*, d^{CE}\right)$ is the Chevalley-Eilenberg CDGA.

$$\begin{array}{ccc}
\bigwedge \mathfrak{g}^* = \Omega(G)^G & \overset{i}{\hookrightarrow} & \Omega(G)^\Gamma \\
& \searrow^{\psi_\Gamma} & \downarrow \cong \\
& & \Omega(\Gamma\backslash G)
\end{array}$$

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Not all quasi-isomorphisms of CDGAs $\rho\colon \bigwedge \mathfrak{g}^* \to \Omega(M)$ are of this kind!
But...

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- As $\widetilde{M}$ is 1-connected, by the Cartan integration result there exists a smooth map $f : \widetilde{M} \to G(\Gamma)$ such that $f^*|_{\bigwedge \mathfrak{g}^*} = \pi^* \circ \rho$. One shows that f is Γ-invariant and hence

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\quad f \quad} & G(\Gamma) \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow \\ M & \overset{\bar{f}}{\dashrightarrow} & f(\Gamma)\backslash G(\Gamma) \end{array}$$

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  The induced map h is a smooth map between smooth manifolds.
- As $\widetilde{M}$ and $G(\Gamma)$ are contractible, by a standard homotopy argument we get that h is also a homotopy equivalence.
- Finally, it is a routine computation to check that

$$\rho \circ a = h^* \circ \psi_\Gamma,$$

  where $a = A^*|_{\bigwedge \mathfrak{g}^*}$

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We use these properties to construct a normal almost contact structure $(\varphi_\mathfrak{h}, \eta_\mathfrak{h}, \xi_\mathfrak{h})$ on N_Γ.

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is a surjective holomorphism which is a homotopy equivalence. By using some complex analytic geometry we get that the map h_f is biholomorphic, hence h is a diffeomorphism.

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