Hard Lefschetz Theorem for Sasakian manifolds

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joint work with B. Cappelletti-Montano (Univ. Cagliari) and I. Yudin (CMUC)

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Reference

B. Cappelletti-Montano, A.D.N., I. Yudin,
Hard Lefschtez Theorem for Sasakian manifolds.
arXiv:1306.2896

Kähler manifolds

Let (M^{2n}, g) be a Riemannian manifold, ω a 2-form such that

$$\omega^n \quad \text{is a volume form,} \qquad d\omega = 0.$$

Define $J : TM \to TM$ by

$$\omega(X, Y) = g(X, JY), \qquad \text{for any } X, Y \in \Gamma(TM).$$

Now assume that

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Sasakian manifolds

Let (M^{2n+1}, g) be a Riemannian manifold, η a 1-form, such that

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Define $\varphi : TM \to TM$ by

$$\mathrm{d}\eta(X, Y) = 2g(X, \varphi Y), \qquad \text{for any } X, Y \in \Gamma(TM).$$

Let $\xi \in \Gamma(TM)$ be the metric dual of η and assume that $\eta(\xi) = 1$. Moreover, suppose that

$$\varphi^2 = -Id + \eta \otimes \xi$$

and the Nijenhuis torsion of φ satisfies

$$N_\varphi + 2d\eta \otimes \xi = 0.$$

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Comparison

Kähler manifolds	Sasakian manifolds
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$\omega \in \Omega^2(M), \quad d\omega = 0$ $\omega^n \neq 0$ $\omega(X,Y) = g(X,JY)$	$\eta \in \Omega^1(M)$ $\eta \wedge (\mathrm{d}\eta)^n \neq 0$ $\mathrm{d}\eta(X,Y) = 2g(X,\varphi Y)$
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Example of Sasakian manifold: $S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$

Let J be the standard complex structure on $\mathbb{C}^{n+1}$

$$J(z_0, \dots, z_n) = (iz_0, \dots, iz_n)$$

and let N be the unit outward vector field normal to S^{2n+1}. Then put

$$\xi := -JN,$$

and for any $X \in \Gamma(TS^{2n+1})$, decompose JX in its tangent and normal components

$$JX = \varphi(X) + \eta(X)N.$$

Then $(S^{2n+1}, \varphi, \xi, \eta, g)$ is a compact Sasakian manifold.

The Sasakian structure of S^{2n+1} projects under the *Hopf fibration* onto the Kähler structure of $\mathbb{C}P^n$.

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Notation for harmonic forms

Let (M^m, g) be a compact oriented Riemannian manifold.
Define $\delta : \Omega^p(M) \to \Omega^{p-1}(M)$ as

$$\delta = (-1)^{m(p+1)+1} * d * .$$

The Laplacian $\triangle : \Omega^p(M) \to \Omega^p(M)$ is then defined as

$$\triangle = d\delta + \delta d.$$

We define

$$\Omega^p_\triangle(M) := \{\alpha \in \Omega^p(M) \mid \triangle \alpha = 0\}.$$

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Hard Lefschetz Theorem for Kähler manifolds

Now, let (M^{2n}, ω, g) be a compact Kähler manifold. Then, the maps

$$\omega^p \wedge - : \Omega^{n-p}_{\triangle}(M) \to \Omega^{n+p}_{\triangle}(M)$$

$$\alpha \mapsto \omega^p \wedge \alpha$$

are isomorphisms.

RECALL: Each $\omega \wedge -$ sends harmonic forms to harmonic forms.

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In a compact Sasakian manifold (M^{2n+1}, η, g) one would like to define

$$\eta \wedge (d\eta)^p \wedge - : \Omega_{\triangle}^{n-p}(M) \to \Omega_{\triangle}^{n+p+1}(M)$$
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and to get isomorphisms.

PROBLEM: Neither $d\eta \wedge -$ nor $\eta \wedge d\eta \wedge -$ send harmonic forms into harmonic forms! So, a priori the above maps are not well defined.

However, the claim turns out to be true. So, how to prove it?

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Important subspaces

$$\omega \in \Omega^{p,\nu}_{\blacksquare}(M) \overset{def}{\Longleftrightarrow} \begin{cases} \boxed{\Delta\omega = \nu\omega} \\ \\ \\ \end{cases}$$

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Harmonic p-forms

By definition,

$$\Omega^{p,0}_{\blacksquare}(M) \subset \Omega^{p}_{\triangle}(M)$$

On the other hand, for $p \le n$, every harmonic p-form belongs to $\Omega^{p,0}_{\blacksquare}(M)$ since $d\omega = 0$, $\delta\omega = 0$, and [Tachibana]

$$i_{\xi}\omega = 0.$$

Thus,

Property

Let M be a compact Sasakian manifold of dimension $2n+1$. For $p \le n$,

$$\Omega^{p,0}_{\blacksquare}(M) = \Omega^{p}_{\triangle}(M).$$

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Some information on the spectrum of $\triangle$

Proposition

Let M^{2n+1} be a compact Sasakian manifold.

(i) *The only values of ν for which the space $\Omega_{\blacksquare}^{p,4\nu}(M)$ is not zero are of the form $\nu = k(n-p+k+1)$ for some integer $k \geq 0$ such that $(p-n)/2 \leq k \leq p/2$*

(ii) *The only values of ν for which the space $\Omega_{\bullet}^{p,4\nu}(M)$ is not zero are of the form $\nu = k(n-p+k-1)$ for some integer $k \geq 0$ such that $(p+1-n)/2 \leq k \leq (p+1)/2$.*

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Some information on the spectrum of $\triangle$

Theorem

Let M be a compact Sasakian manifold.

(i) $\omega \in \Omega_{\blacksquare}^{p,4\nu}(M) \Longrightarrow \eta \wedge \omega \in \Omega_{\bullet}^{p+1,4(\nu-p+n)}(M)$.

(ii) $\omega \in \Omega_{\bullet}^{p,4\nu}(M) \Longrightarrow i_\xi \omega \in \Omega_{\blacksquare}^{p-1,4(\nu+p-n-1)}(M)$.

We get the pair of inverse isomorphisms

$$\Omega_{\blacksquare}^{p,4\nu}(M) \underset{i_\xi}{\overset{\eta\wedge -}{\rightleftarrows}} \Omega_{\bullet}^{p+1,4(\nu-p+n)}(M) \ . \tag{1}$$

Some information on the spectrum of $\triangle$

Proposition

Let M be a compact Sasakian manifold and $\nu \neq 0$.

(i) $\omega \in \Omega_{\bullet}^{p,4\nu}(M) \Longrightarrow d\omega \in \Omega_{\blacksquare}^{p+1,4\nu}(M)$ *and* $d\omega \neq 0$.

(ii) $\omega \in \Omega_{\blacksquare}^{p,4\nu}(M) \Longrightarrow \delta\omega \in \Omega_{\bullet}^{p-1,4\nu}(M)$ *and* $\delta\omega \neq 0$.

Thus for $\nu \neq 0$, we have the pair of isomorphisms

$$\Omega_{\bullet}^{p,4\nu}(M) \underset{\delta}{\overset{d}{\rightleftarrows}} \Omega_{\blacksquare}^{p+1,4\nu}(M)\ , \tag{2}$$

for any $0 \leq p \leq 2n$.

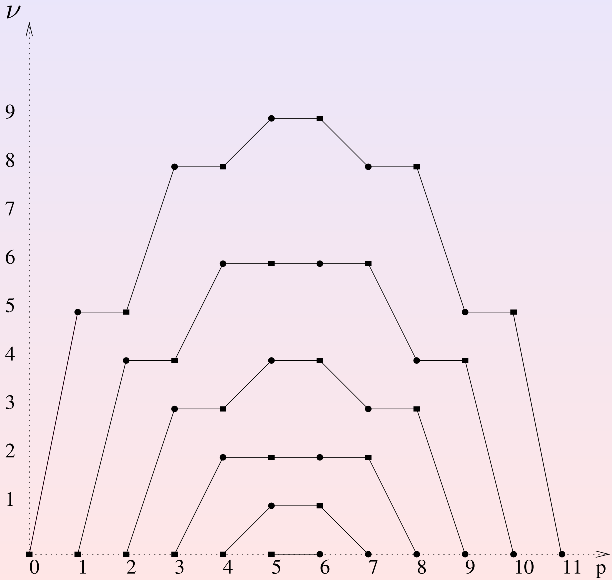
Some information on the spectrum of $\triangle$

Therefore, using the isomorphisms (1) and (2), we have

$$\begin{array}{c}
\xrightarrow{\qquad\qquad\qquad L \qquad\qquad\qquad} \\
\Omega^{p,4\nu}_{\blacksquare}(M) \underset{i_\xi}{\overset{\eta\wedge -}{\rightleftarrows}} \Omega^{p+1,4(\nu-p-1+n)}_{\bullet}(M) \underset{\delta}{\overset{d}{\rightleftarrows}} \Omega^{p+2,4(\nu-p-1+n)}_{\blacksquare}(M). \\
\xleftarrow[\qquad\qquad\qquad \Lambda \qquad\qquad\qquad]{}
\end{array}$$

This shows that $L = (d\eta) \wedge -$ and its adjoint Λ induce inverse isomorphisms between the spaces in the diagram.

HLT for Sasakian manifolds (n=5)



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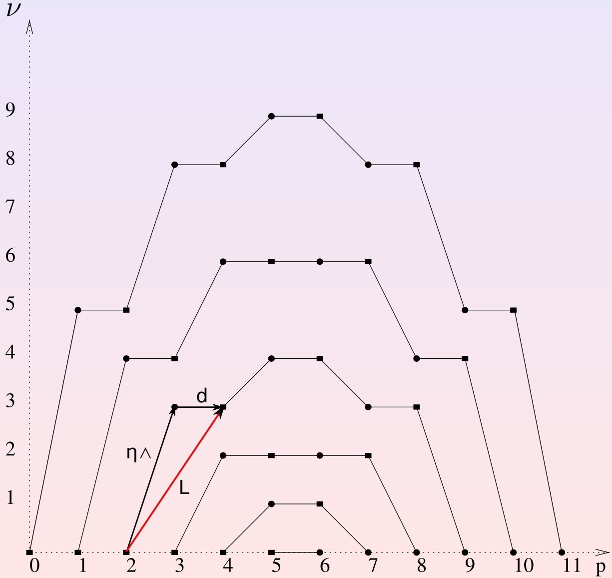

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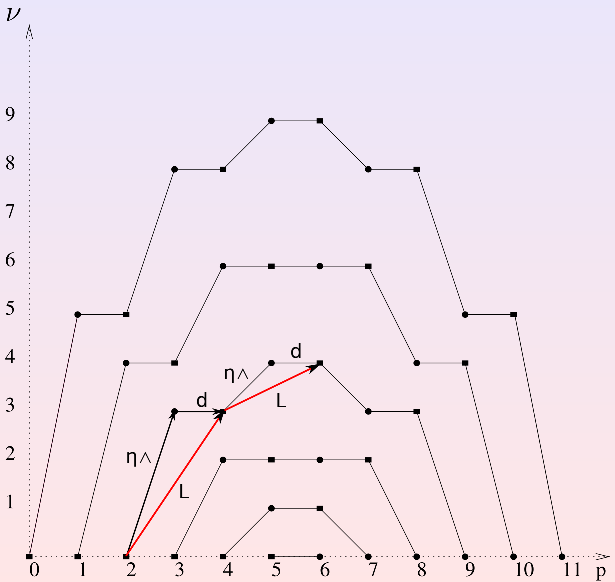

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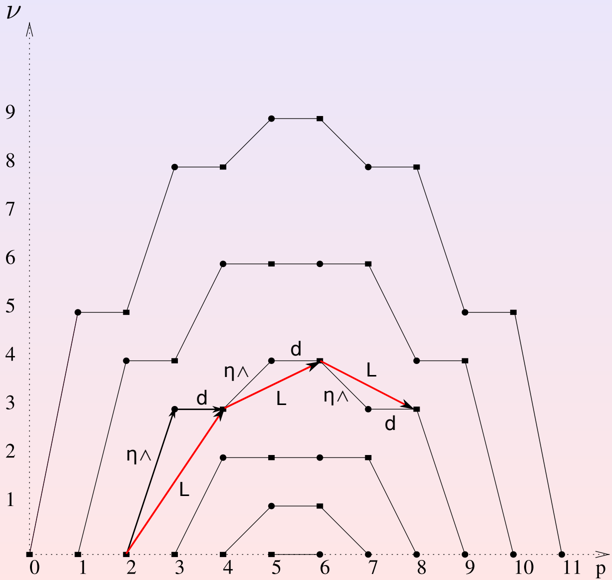

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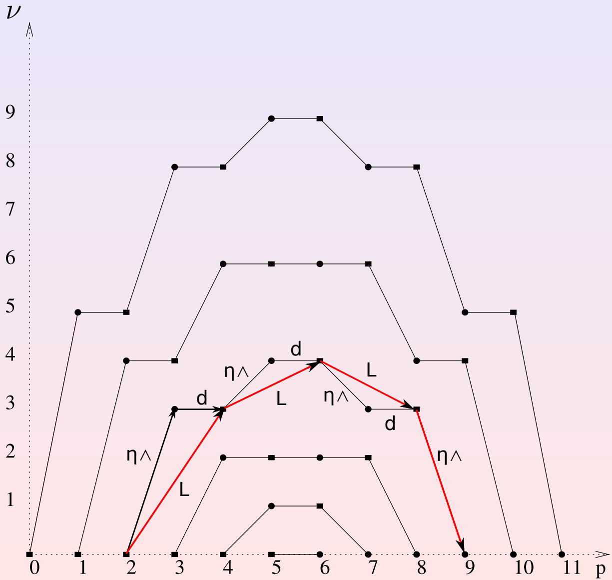

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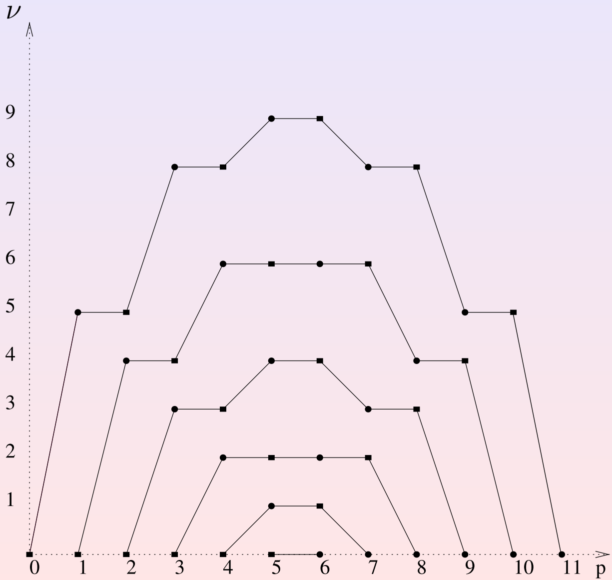

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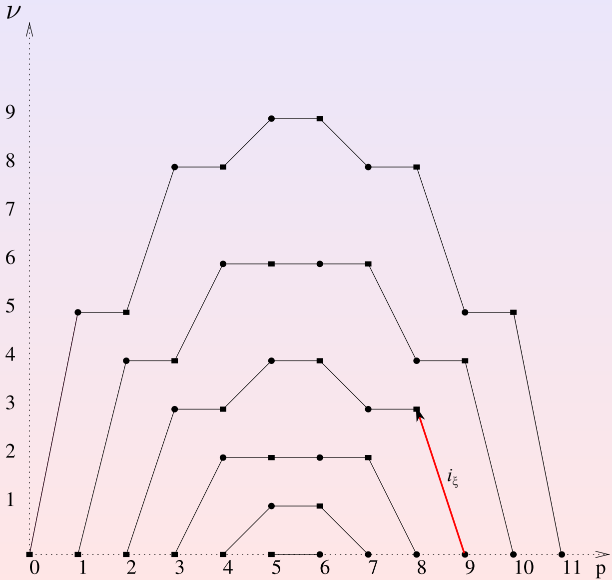

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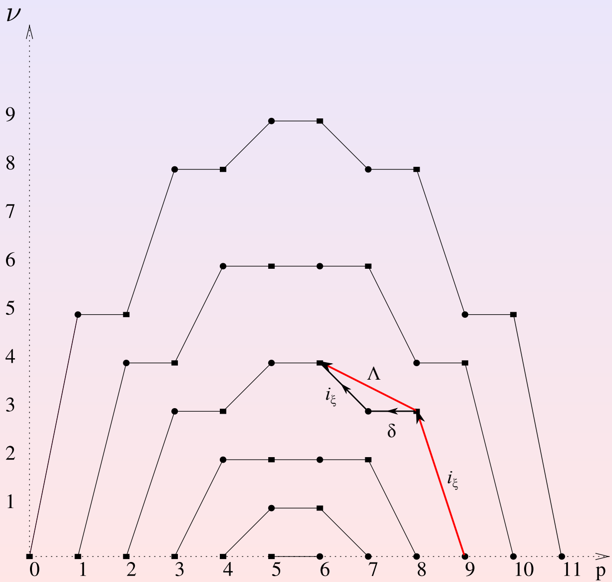

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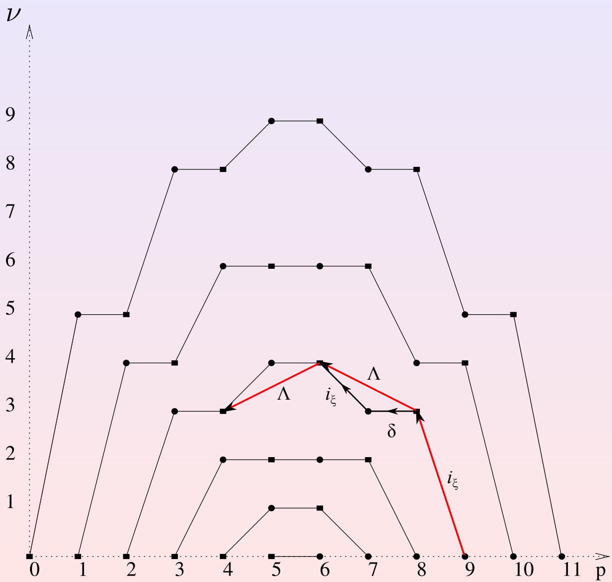
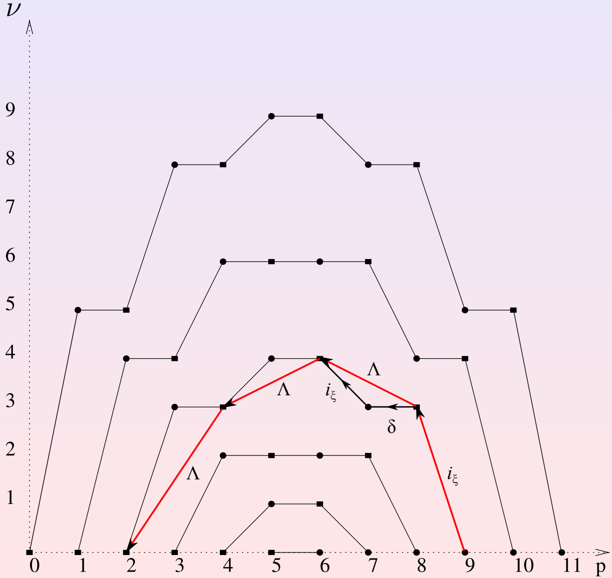


HLT for Sasakian manifolds (n=5)



Hard Lefschetz Theorem for Sasakian manifolds

Theorem

Let M a compact Sasakian manifold of dimension $2n+1$ and $p \leq n$. Then the map

$$\Omega^p_\triangle(M) \longrightarrow \Omega^{2n+1-p}_\triangle(M)$$
$$\alpha \longmapsto \eta \wedge (d\eta)^{n-p} \wedge \alpha$$

is an isomorphism.

Hard Lefschetz Theorem in cohomology

In a compact Kähler manifold (M^{2n}, ω, g) the maps

$$\begin{aligned} H^p(M) &\to H^{2n-p}(M) \\ [\,\alpha\,] &\mapsto [\,\omega^{n-p} \wedge \alpha\,], \end{aligned}$$

are isomorphisms.

Hard Lefschetz Theorem in cohomology

For a compact Sasakian manifold (M^{2n+1}, η, g) a naive guess would be to consider:

$$\begin{aligned} H^p(M) &\longrightarrow H^{2n+1-p}(M) \\ [\,\alpha\,] &\longmapsto [\,\eta \wedge (d\eta)^{n-p} \wedge \alpha\,], \end{aligned}$$

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PROBLEM:
α closed does NOT imply that $\eta \wedge (d\eta)^{n-p} \wedge \alpha$ is closed!

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SOLUTION?

Take

$$\begin{aligned} H^p(M) &\longrightarrow H^{2n+1-p}(M) \\ [\,\alpha\,] &\longmapsto [\,\eta \wedge (d\eta)^{n-p} \wedge \Pi_{\triangle}\,\alpha\,] \end{aligned}$$

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NEW PROBLEM: $\Pi_{\triangle}\,\alpha$ could in general depend on the metric!

Hard Lefschetz Theorem for Sasakian manifolds

Theorem

Let (M^{2n+1}, η, g) be a compact Sasakian manifold and $p \leq n$. Let $\Pi_{\triangle} \colon \Omega^p(M) \to \Omega^p_{\triangle}(M)$ be the projection on the harmonic part. Then the map

$$\begin{aligned} Lef_p \colon H^p(M) &\longrightarrow H^{2n+1-p}(M) \\ [\,\alpha\,] &\longmapsto [\,\eta \wedge (d\eta)^{n-p} \wedge \Pi_{\triangle}\,\alpha\,], \end{aligned}$$

is an isomorphism. Furthermore, it does not depend on the choice of the Sasakian metric g on (M^{2n+1}, η).

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Independence of the metric

One has to show that $\eta \wedge (d\eta)^{n-p} \wedge (\Pi_{\triangle}\,\alpha - \Pi_{\triangle'}\,\alpha)$ is exact.
From Hodge theory one gets

$$\exists \gamma \in \Omega^{p-1} \text{ s.t. } \delta\gamma = 0 \text{ and } d\gamma = \Pi_{\triangle}\,\alpha - \Pi_{\triangle'}\,\alpha.$$

Then

$$\eta \wedge (d\eta)^{n-p} \wedge d\gamma = d(\eta \wedge (d\eta)^{n-p} \wedge \gamma) - (d\eta)^{n-p+1} \wedge d\gamma.$$

It remains to show that the last term is exact (difficult part).
We found an explicit expression:

$$(d\eta)^{n-p+1} \wedge d\gamma = -2(n-p+1)d((d\eta)^{n-p} \wedge i_{\varphi} dG\gamma),$$

where G is the Green operator of $\triangle$, i.e. $Id - \triangle G = Id - G\triangle = \Pi_{\triangle}$.

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A topological obstruction

Let (M^{2n+1}, η) be a compact contact manifold. We can define a relation between $H^p(M)$ and $H^{2n+1-p}(M)$:

$$\mathcal{R}_{Lef_p} = \left\{ \left([\beta] , [\epsilon_\eta L^{n-p}\beta] \right) \middle| \beta \in \Omega^p(M),\ d\beta = 0,\ i_\xi\beta = 0,\ L^{n-p+1}\beta = 0 \right\}.$$

Now, if (M, η) admits a compatible Sasakian metric, then $\mathcal{R}_{Lef_p}$ is the graph of the isomorphism $Lef_p : H^p(M) \longrightarrow H^{2n+1-p}(M)$.

Definition

We say that (M, η) is a *Lefschetz contact manifold* if for every $p \leq n$ the relation $\mathcal{R}_{Lef_p}$ is the graph of an isomorphism between $H^p(M)$ and $H^{2n+1-p}(M)$.

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First applications

- Let (M, η) be a Lefschetz contact manifold of dimension $2n+1$. Then the odd Betti numbers b_{2k+1} are even for $0 \leq 2k+1 \leq n$.
- Recently, jointly with J.C. Marrero we found examples of non Lefschetz K-contact manifolds in dim. 5 and 7.
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References

S. Tachibana,
On harmonic tensors in compact Sasakian spaces. *Tôhoku Math. J.* **17** *(1965), 271-284.*

B. Cappelletti-Montano, A.D.N., I. Yudin,
Hard Lefschtez Theorem for Sasakian manifolds.
arXiv:1306.2896

B. Cappelletti-Montano, A.D.N., J.C. Marrero, I. Yudin,
Examples of compact K-contact manifolds with no Sasakian metric. *arXiv:1311.3270*.

Y. Lin,
Lefschetz contact manifolds and odd dimensional symplectic geometry. *arXiv:1311.1431*.

Gracias!