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QUASI-METRIC SPACES AND POINT-FREE GEOMETRY (An extended abstract of a paper published on Math. Struct. in Comp. Science, 16 (2006) 115-137)

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Abstract. An approach to point-free geometry based on the notion of a quasi-metric is proposed in which the primitives are the regions and a non symmetric distance between regions. The intended models are the bounded regular closed subsets of a metric space together with the Hausdorff excess measure.

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1. Introduction

Recently in Computer Science the interest in point-free geometry increased in connection with the question of a suitable formalization of the naive spatial knowledge. The motivation of this new searching field lies in a dissatisfaction, from a computational point of view, with the complexity of the Euclidean geometry based on the notion of point. The possibility of considering a geometry in which the notion of point is not assumed as a primitive was at first examined by A. N. Whitehead in *An Inquiry Concerning the Principles of Natural Knowledge*, in *The Concept of Nature* and in *Process and Reality.* In particular, in this last book the primitives are the *regions* and the *connection relation*, that is the relation between two regions that either overlap or have at least a common boundary point. Such a point-free approach to geometry was formalized and investigated by several authors (see, for example [3], [6]). Namely, one considers structures (Re, C) where the elements in Re are called *regions* and C is a binary relation in Re called *connection relation*. The *inclusion relation* $\leq$ is defined by setting $x \leq y$ if and only if $C(x) \subseteq C(y)$ where, as usual, for any region z, we set $C(z) = \{z' \in Re : zCz'\}$. Successively, in [5] it was proposed the notion of *pointless pseudo-metric space* $(Re, \leq, m, D)$ in which the inclusion, the distance $m : Re \times Re \to R^+$ and the diameter D: $Re \to R^+$

are all assumed as primitives. A "canonical" model is obtained by setting Re equal to the class of bounded regular open subsets of a metric space $(M, \delta), \leq$ equal to the set theoretical inclusion and by defining m and D by setting

$$m(x,y) = \inf\{\delta(P,Q) : P \in x, Q \in y\} \quad ; \quad D(x) = \sup\{\delta(P,P') : P, P' \in x\}$$

for any pair x, y of subsets of M. Such a class of structures was previously defined in [16] in the framework of computability theory.

In this note we sketch a new approach to point-free geometry where the unique primitive is the notion of *quasi-metric*, i.e. a distance-like measure lacking in symmetry property (see, for example [4], [11], [12] and [13]). Namely, we examine a particular class of quasi-metrics, the *quasi-metric spaces of regions.* The intended model is the *excess measure* e_δ defined by setting, for any pair x and y of nonempty closed bounded subsets of a metric space (M,δ),

$$e_\delta(x,y) = \sup\{\delta(P,y) : P \in x\},$$

where, in turn, $\delta(P,y) = \inf\{\delta(P,Q) : Q \in y\}$. Such a measure is well-known in literature since the Hausdorff distance d_H is defined by setting $d_H(x,y) = \max\{e_\delta(x,y), e_\delta(y,x)\}$. An advantage of such an approach with respect to the quoted researches is that we are not forced to assume the inclusion relation and the diameter as primitives. Indeed, these notions can be defined in a very simple way from the quasi-metric. Obviously, the main step in our theory is the definition of *point* and of *distance between points* in order to associate any quasi-metric space of regions (Re, d) with a point-based metric space.

We remark that this paper in the present form does not face the computational dimension of point-free geometry which is on the basis of the recent literature in generalized metric spaces. Anyway it looks to be possible to reformulate the notions and the results contained in it in constructive terms.

In the end, we wish to thank the referees for their fruitful suggestions and comments.

2. Preliminaries

In the following R denotes the set of real numbers and $R^+ = \{x \in R : x \geq 0\}$.

Definition 2.1. A *quasi-metric space* is a structure (Re, d) such that Re is a nonempty set and where $d : Re \times Re \to R^+$ is a mapping such that, for any $x, y, z \in Re$,
d1) $d(x,x) = 0$;
d2) $d(x,y) = 0$ and $d(y,x) = 0 \Rightarrow x = y$;
d3) $d(x,y) \leq d(x,z) + d(z,y)$.

Then, the metric spaces are the quasi-metric spaces satisfying the symmetric property

d0) $d(x,y) = d(y,x)$.

The proof of the following proposition is trivial.

Proposition 2.2. *Let (Re, d)be a quasi-metric space and define the mapping $d_H : Re \times Re \to R^+$ by setting*

$$d_H(x, y) = d(x, y) \vee d(y, x),$$

then (Re, d_H) is a metric space.

We call (Re, d_H) the *symmetrization* of (Re, d). The quasi-metric spaces are related with the partial orders in the following way:

Proposition 2.3. *Let (Re, d) be a quasi-metric space, then the relation $\leq$ defined by setting*:

$$x \leq y \quad \Leftrightarrow \quad d(x, y) = 0$$

for every x, y∈ Re is a partial order. Conversely, let $\leq$ be any partial order in a set Re and define the mapping $d : Re \times Re \to R^+$ by setting

$$d(x, y) = \begin{cases} 0 & \text{if } x \leq y \\ 1 & \text{otherwise.} \end{cases}$$

Then (Re, d) is a quasi-metric space whose associated partial order is $\leq$.

Since our goal is to give a basis for point-free geometry, we call *regions* the elements of *Re* and *inclusion relation* the relation $\leq$ defined in Proposition 2.3. Also, we define the diameter of a region as follows:

Definition 2.4. Given a quasi-metric space (Re, d), we call *diameter* of a region $x \in Re$ the number

$$d(x) = \sup\{d(x_1, x_2) : x_1 \leq x,\ x_2 \leq x\}. \tag{2.1}$$

We say that x is *bounded*, if $d(x) \neq \infty$.

Observe that the notion of diameter is assumed as a primitive by several authors (see for example [9], [10] and [1]). Obviously, $d(x) = 0$ if and only if x is an atom. In the case (Re, d) is a metric space, then the associated partial order $\leq$ coincides with the identity relation and therefore all diameters are equal to zero and all regions are atoms. When the quasi-metric space is defined by a partial order as in Proposition 2.3, we have that $d(x) = 0$ if x is an atom and $d(x) = 1$ otherwise.

Proposition 2.5. *Any quasi-metric $d : Re \times Re \to R^+$ is order-preserving with respect to the first variable and order-reversing with respect to the second variable. Also the diameter $d : Re \to R^+$ is order-preserving and, for any region x,*

$$d(x) = \sup\{d(x, x') : x' \leq x\}. \tag{2.2}$$

Definition 2.6. Given two quasi-metric spaces (Re, d) and (Re', d') and a mapping $h : Re \to Re'$, we say that h is *non expansive* provided that $d'(h(x), h(y)) \leq d(x, y)$. We say that h is an *isometry* provided that $d(x, y) = d'(h(x), h(y))$.

We conclude this section by noticing that the class of quasi-metric spaces defines a category in a natural way.

Proposition 2.7. *The class of quasi-metric spaces defines a category QMS provided that we assume as morphisms the non expansive mappings. Let ORD be the category whose objects are the ordered sets and the morphisms are the order preserving maps. Then Proposition 2.3 defines a functor from QMS to ORD and a functor from ORD to QMS.*

3. The notion of point

In this section we will propose a suitable definition of point and distance between points in order to associate any quasi-metric space with a metric space in a natural way. To this aim recall that a *pseudo-metric space* is a structure (M, d) satisfying d0, d1 and d3 and that any pseudo-metric space (M, d) is associated with a metric space (M', d') we call *the quotient* of (M, d). Namely, we define an equivalence relation $\equiv$ in M by setting $x \equiv y$ if and only if $d(x, y) = 0$ and we set M' equal to the quotient of M modulo $\equiv$. Moreover, we define the distance between two classes $[x]$ and $[y]$ by setting $d'([x], [y]) = d(x, y)$.

Definition 3.1. A sequence $< p_n >_{n\in N}$ of regions of a quasi-metric space (Re, d) is called a *point-representing* if
a) $\lim_{n\to\infty} d(p_n) = 0$;
b) $\forall\varepsilon > 0\ \exists m : h \geq m, k \geq m \Rightarrow d(p_h, p_k) < \varepsilon$.

We denote by Pr the class of point-representing sequences. In the case (Re, d) is a metric space, the notion of point-representing sequence coincides with the usual notion of Cauchy sequence. There are quasi-metric spaces in which no point-representing sequence exists. So, we add the following axiom:

d4) A point-representing sequence exists.

Proposition 3.2. *For any* $< p_n >_{n\in N}$ *and* $< q_n >_{n\in N}$ *in Pr, the sequence* $< d(p_n, q_n) >_{n\in N}$ *is convergent.*

Proposition 3.3. *The structure* (Pr,d_c) *satifies d1 and d3.*

Proof. Axiom d1 is immediate. To prove d3, observe that, if $< p_n >_{n\in N}, < q_n >_{n\in N}$ and $< r_n >_{n\in N}$ are elements in Pr, then

$$\begin{aligned} d_c(< p_n >_{n\in N}, < q_n >_{n\in N}) &= \lim_{n\to\infty} d(p_n, q_n) \leq \lim_{n\to\infty} (d(p_n, r_n) + d(r_n, q_n)) \\ &= \lim_{n\to\infty} d(p_n, r_n) + \lim_{n\to\infty} d(r_n, q_n) \\ &= d_c(< p_n >_{n\in N}, < r_n >_{n\in N}) + d_c(< r_n >_{n\in N}, < q_n >_{n\in N}). \end{aligned}$$

It is easy to prove that d_c is not symmetric, in general (see Proposition 5.6). To obtain this property we have to add a further axiom to quasi-metric spaces. As an example, we propose the following one:

d5) $|d(x, y) - d(y, x)| \leq d(x) + d(y)$.

This axiom is in accordance with the idea that *"small"* regions are approximations of ideal points. In fact, it says that in the class of *"small"* regions the mapping d is approximately symmetric and therefore that the class of *"small"* regions can be regarded (approximately) as a metric space. Observe that all the results in this paper remain valid in the case in d5 we substitute the maximum $\mathrm{Max}\{d(x), d(y)\}$ to the sum $d(x) + d(y)$.

Definition 3.4. We call *quasi-metric space of regions* any structure (Re,d) satisfying d1-d5.

Trivially, the set of atoms of a quasi-metric space of regions is a metric space and the metric spaces coincide with the quasi-metric spaces of regions in which all the regions have diameter zero. Observe also that while any subset Re' of a quasi-metric space (Re, d) defines a quasi-metric space, in the case (Re,d) satisfies d5 it is possible that (Re', d) does not satisfy d5. This since the notion of diameter in (Re,d) is different from the notion of diameter in (Re', d).

Proposition 3.5. *The structure* (Pr, d_c) *associated with a quasi-metric space of regions is a pseudo-metric space.*

Proof. To prove the symmetric property, observe that, since $|d(p_n, q_n) - d(q_n, p_n)| \leq d(p_n) + d(q_n)$, it is $\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(q_n, p_n)$.

Such a proposition enables us to propose the following definition.

Definition 3.6. We call *metric space associated with* (Re, d) the quotient $(\bar{M}, \bar{\delta})$ of the pseudo-metric space (Pr, d_c). We call *point* any element in $\bar{M}$.

Thus, the metric space $(\bar{M}, \bar{\delta})$ associated with a metric space of regions (Re, d) is defined
- by considering the class Pr of point-representing sequences ;
- by setting $\bar{M}$ equal to the quotient of Pr modulo the equivalence $\equiv$ defined by

$$< p_n >_{n\in N} \equiv < q_n >_{n\in N} \Leftrightarrow \lim_{n\to\infty} d(p_n, q_n) = 0;$$

- by defining $\bar{\delta} : \bar{M} \times \bar{M} \to R^+$ by the equation,

$$\bar{\delta}(P, Q) = \lim_{n\to\infty} d(p_n, q_n).$$

where $P = [< p_n >_{n\in N}]$ and $Q = [< q_n >_{n\in N}]$ are points in $\bar{M}$.

Observe that if (Re, d) is a metric space, then the associated metric space $(\bar{M}, \bar{\delta})$ is the completion of (Re, d). Indeed, since all diameters are equal to zero, the proposed notion of point-representing sequence coincides with the usual notion of Cauchy sequence.

Proposition 3.7. *Let* (Re, d) *be a quasi-metric space of regions and* (Re, d_H) *be its symmetrization. Then the associated metric space* $(\bar{M}, \bar{\delta})$ *is a subspace of the metric completion of* (Re, d_H).

4. Distance between points and regions and completeness

The following proposition is useful in order to define the notion of distance between a point and a region:

Proposition 4.1. Let $< p_n >_{n\in N}$ be an element in Pr and x a region. Then, both the sequences $< d(x, p_n) >_{n\in N}$ and $< d(p_n, x) >_{n\in N}$ are convergent. Moreover

$$< p_n >_{n\in N} \equiv < p'_n >_{n\in N} \ \Rightarrow \lim_{n\to\infty} d(x, p_n) = \lim_{n\to\infty} d(x, p'_n)$$

and

$$< p_n >_{n\in N} \equiv < p'_n >_{n\in N} \ \Rightarrow \ \lim_{n\to\infty} d(p_n, x) = \lim_{n\to\infty} d(p'_n, x).$$

Assume that $< p_n >_{n\in N}$ and $< p'_n >_{n\in N}$ are two equivalent point-representing sequences. Then, from $d(x, p_n) \le d(x, p'_n) + d(p'_n, p_n)$ it follows that

$$d(x, p_n) - d(x, p'_n) \le d(p'_n, p_n) \le d(p'_n, p_n) + d(p_n) + d(p'_n).$$

Since $d(x, p'_n) \le d(x, p_n) + d(p_n, p'_n)$, we have

$$d(x, p'_n) - d(x, p_n) \le d(p_n, p'_n) \le d(p'_n, p_n) + d(p_n) + d(p'_n).$$

So:

$$|d(x, p'_n) - d(x, p_n)| \le d(p'_n, p_n) + d(p_n) + d(p'_n)$$

and this proves that $lim_{n\to\infty} d(x, p_n) = \lim_{n\to\infty} d(x, p'_n)$.

Likewise, we proceed to prove the second implication.

Proposition 4.1 enables us to give the following definitions:

Definition 4.2. Let x be a region and $P = [< p_n >_{n\in N}\ ]$ be a point. Then, we set

$$\underline{d}(P, x) = \lim_{n\to\infty} d(p_n, x) \text{ and } \underline{d}(x, P) = \lim_{n\to\infty} d(x, p_n).$$

Trivially, $\underline{d}(P, x)$ is order-reversing with respect to the second variable and $\underline{d}(x, P)$ is order-preserving with respect to the first variable. Also $\underline{d}(P, x) \neq \underline{d}(x, P)$, in general.

Proposition 4.3. *Let x, x' be two regions and P, P' two points. Then the following inequalities hold true,*

$$\begin{array}{llll}
1) & \bar{\delta}(P, P') \le \underline{d}(P, x) + \underline{d}(x, P'); & 2) & d(x, x') \le \underline{d}(x, P) + \underline{d}(P, x'); \\
3) & \underline{d}(P, x) \le \bar{\delta}(P, P') + \underline{d}(P', x); & 4) & \underline{d}(P, x) \le \underline{d}(P, x') + d(x', x); \\
5) & \underline{d}(x, P) \le \underline{d}(x, P') + \bar{\delta}(P', P); & 6) & \underline{d}(x, P) \le d(x, x') + \underline{d}(x', P); \\
7) & \underline{d}(P, x) \le \underline{d}(x, P) + d(x); & 8) & \underline{d}(x, P) \le \underline{d}(P, x) + d(x); \\
9) & |\underline{d}(P, x) - \underline{d}(x, P)| \le d(x). & &
\end{array}$$

Proof. To prove 1), observe that

$$\bar{\delta}(P, P') = \lim_{n\to\infty} d(p_n, p'_n) \le \lim_{n\to\infty} d(p_n, x) + \lim_{n\to\infty} d(x, p'_n) = \underline{d}(P, x) + \underline{d}(x, P').$$

To prove 2), observe that $d(x, x') \le d(x, p_n) + d(p_n, x')$ and therefore that

$$d(x, x') \le \lim_{n\to\infty} d(x, p_n) + \lim_{n\to\infty} d(p_n, x) \le \underline{d}(x, P) + \underline{d}(P, x').$$

In a similar way we can prove 3), 4), 5), 6). To prove 7), observe that, since $d(p_n, x) \leq d(x, p_n) + d(x) + d(p_n)$ and $\lim_{n\to\infty} d(p_n) = 0$,

$$\underline{d}(P, x) = \lim_{n\to\infty} d(p_n, x) \leq \lim_{n\to\infty} d(x, p_n) + d(x) = \underline{d}(x, P) + d(x).$$

In a similar way we can prove 8). Finally, 9) is an immediate consequence of 7) and 8).

Theorem 4.4. *Let (Re, d) be a quasi-metric space of regions. Then the associated metric space $(\bar{M}, \bar{\delta})$ is complete.*

Proof. To prove that $(\bar{M}, \bar{\delta})$ is complete, observe that if $P = [< p_n >_{n\in N}]$ is an element of $\bar{M}$, then for any ε >0 there is a region s such that $d(s) \leq \varepsilon, \underline{d}(P, s) < \varepsilon$ and $\underline{d}(s, P) \leq \varepsilon$. In fact, let $m \in N$ be such that $d(p_h) \leq \varepsilon$ and $d(p_h, p_k) \leq \varepsilon$ for any $h \geq m$ and $k \geq m$. Then, in particular, $d(p_m) \leq \varepsilon, d(p_m, p_n) \leq \varepsilon$ and $d(p_n, p_m) \leq \varepsilon$ for any $n \geq m$ and therefore, by setting $s = p_m$, we obtain that $d(s) \leq \varepsilon$ and that $\underline{d}(s, P) = \lim_{n\to\infty} d(p_m, p_n) \leq \varepsilon$ and $\underline{d}(P, s) = \lim_{n\to\infty} d(p_n, p_m) \leq \varepsilon$. Let $< P_n >_{n\in N}$ be a Cauchy sequence of elements of the metric space $(\bar{M}, \bar{\delta})$ and, for any $n \in N$, let s_n be a region such that $d(s_n) \leq 1/n, \underline{d}(s_n, P_n) \leq 1/n$ and $\underline{d}(P_n, s_n) \leq 1/n$. Then,

$$d(s_h, s_k) \leq \underline{d}(s_h, P_h) + \bar{\delta}(P_h, P_k) + \underline{d}(P_k, s_k) \leq 1/h + \bar{\delta}(P_h, P_k) + 1/k$$

and therefore $< s_n >_{n\in N}$ is a sequence representing a point $P \in \bar{M}$. Also, since

$$\bar{\delta}(P, P_n) \leq \underline{d}(P, s_n) + \underline{d}(s_n, P_n) \leq \underline{d}(P, s_n) + 1/n,$$

and $\lim_{n\to\infty} \underline{d}(P, s_n) = 0$, we have that $P = \lim_{n\to\infty} P_n$.

5. Canonical examples: the Hausdorff excess spaces

An interesting class of quasi-metric spaces is related to the Hausdorff distance. Indeed, assume that (M, δ) is a metric space. Then, given $P \in M$ and x a nonempty subset of M, we define $\delta(P, x)$ by setting

$$\delta(P, x) = \inf\{\delta(P, Q) : Q \in x\}. \tag{5.1}$$

If x, y are nonempty subsets of M, we set

$$m(x, y) = \inf\{\delta(P, Q) : P \in x, Q \in y\} \tag{5.2}$$

or, equivalently,

$$m(x, y) = \inf\{\delta(P, y) : P \in x\}. \tag{5.3}$$

Also, we define the *excess function* e_δ by setting, for any x and y in $P(M) - \{\emptyset\}$,

$$e_\delta(x, y) = \sup\{\delta(P, y) : P \in x\}. \tag{5.4}$$

Obviously, it is possible that $e_\delta(x, y) = \infty$. If we confine ourselves to the class $B(M)$ of all closed, bounded, nonempty subsets of M, then $e_\delta(x, y)$ is always finite. Both the maps m and e_δ extend the distance δ, indeed, for any $P, Q \in M$,

$$e_\delta(\{P\}, \{Q\}) = m(\{P\}, \{Q\}) = \delta(P, Q).$$

We define the diameter $D(x)$ of an element x in $B(M)$ by setting

$$D(x) = \sup\{\delta(P, P') : P \in x, P' \in x\}. \tag{5.5}$$

Observe that, given any $x \in P(M) - \{\emptyset\}$ and denoting by $cl(x)$ the closure of x, we have that

$$cl(x) = \{P \in M : \delta(P, x) = 0\}. \tag{5.6}$$

Then, it is immediate to prove that, for any $x, y \in P(M) - \{\emptyset\}$,

$$e_\delta(x, y) = e_\delta(cl(x), cl(y)), \tag{5.7}$$

$$m(x, y) = m(cl(x), cl(y)) \tag{5.8}$$

and

$$D(x) = D(cl(x)). \tag{5.9}$$

Proposition 5.1. *Let P and Q be elements in M and x, y elements in $B(M)$. Then*

$$\delta(P, x) \leq \delta(P, Q) + \delta(Q, x). \tag{5.10}$$

$$m(x, y) \leq e_\delta(x, y) \leq m(x, y) + D(x). \tag{5.11}$$

$$|e_\delta(x, y) - e_\delta(y, x)| \leq \max\{D(x), D(y)\}. \tag{5.12}$$

Theorem 5.2. *Let (M, δ) be a metric space and $e_\delta : B(M) \times B(M) \to R^+$ be the related excess function. Then $(B(M), e_\delta)$ is a quasi-metric space of regions whose associated partial order is the set theoretical inclusion and whose diameter is the diameter function D defined by* (5.5).

Proof. To prove the triangle inequality, observe that,

$$\delta(P, y) \leq \delta(P, Q) + \delta(Q, y) \leq \delta(P, Q) + \sup_{Q' \in z} \delta(Q', y) = \delta(P, Q) + e_\delta(z, y),$$

whenever Q belongs to z. Therefore

$$\delta(P, y) \leq \inf_{Q \in z} \delta(P, Q) + e_\delta(z, y) = \delta(P, z) + e_\delta(z, y).$$

Consequently,

$$\begin{aligned} e_\delta(x, y) &= \sup\{\delta(P, y) : P \in x\} \leq \sup\{\delta(P, z) + e_\delta(z, y) : P \in x\} \\ &= \sup\{\delta(P, z) : P \in x\} + e_\delta(z, y) = e_\delta(x, z) + e_\delta(z, y). \end{aligned}$$

Let x, y be elements in $B(M)$, then since y is a closed set,

$$e_\delta(x, y) = 0 \Leftrightarrow \delta(P, y) = 0 \text{ for any } P \in x \Leftrightarrow \subseteq y.$$

This proves both d1, d2 and that the partial order associated with $(B(M), e_\delta)$ is the inclusion. To prove that $e_\delta(x) = D(x)$, observe that, since $e_\delta(x, x') \leq e_\delta(x, \{P\})$ for any $P \in x'$,

$$\begin{aligned} e_\delta(x) &= \sup\{e_\delta(x, x') : x' \subseteq x, x' \in B(M)\} \\ &\leq \sup\{e_\delta(x, \{P'\}) : P' \in x'\} = \sup_{P \in x} \sup_{P' \in x} e_\delta(\{P\}, \{P'\}) = D(x). \end{aligned}$$

Also,

$$\begin{aligned} e_\delta(x) &= \sup\{e_\delta(x_1,x_2) : x_1 \subseteq x, x_2 \subseteq x, x_1 \in B(M), x_2 \in B(M)\} \\ &\geq \sup\{e_\delta(\{P_1\},\{P_2\}) : P_1 \in x, P_2 \in x\} \\ &= \sup\{\delta(P_1,P_2) : P_1 \in x, P_2 \in x\} = D(x). \end{aligned}$$

By (5.12) we can conclude that $(B(M), e_\delta)$ is a quasi-metric space of regions.

Observe that the symmetrization of $(B(M), e_\delta)$ is the well-known Hausdorff distance.

Definition 5.3. Let (M, δ) be a metric space, then we call *full Hausdorff excess space* the space $(B(M), e_\delta)$ and *Hausdorff excess space* any subspace of $(B(M), e_\delta)$.

In [15] one proves that any quasi-metric space is isometric to a Hausdorff excess space (see also [7]). As an immediate consequence, we obtain the following extension theorem.

Theorem 5.4. Any quasi-metric space can be extended into a quasi-metric space of regions.

Theorem 5.5. *Let $(M, \delta\ )$ be a metric space, and $(\bar{M}, \bar{\delta})$ be the metric space associated with $(B(M), e_\delta)$. Also, define the map $h : M \to \bar{M}$ by setting, for any $P \in M, h(P) = [< p_n >_{n\in N}]$ where $p_n = \{P\}$ for any $n \in N$. Then h is an isometry such that $h(M)$ is dense in $\bar{M}$. Consequently, $(\bar{M}, \bar{\delta})$ is the metric completion of (M, δ) and, when (M, δ) is complete, $(\bar{M}, \bar{\delta})$ coincides with (M, δ).*

Proof. It is evident that h is an isometry. To prove that $h(M)$ is dense in $\bar{M}$, let $P = [< p_n >_{n\in N}]$ be any element in $\bar{M}$. Moreover, for any $n \in N$, let P_n be an element in p_n. We claim that $\lim_{n\to\infty} h(P_n) = P$, i.e. that

$$\lim_{n\to\infty} \bar{\delta}(h(P_n), P) = \lim_{n\to\infty} (\lim_{m\to\infty} e_\delta(\{P_n\}, p_m)) = 0.$$

Indeed,

$$e_\delta(\{P_n\}, p_m) \leq e_\delta(\{P_n\}, p_n) + e_\delta(p_n, p_m) = e_\delta(p_n, p_m).$$

Since $< p_n >_{n\in N}$ is a point-representing sequence, given any $\varepsilon >0$ there exists an integer h such that $e_\delta(p_n, p_m) \leq \varepsilon$ for any $n \geq h$ and $m \geq h$. Consequently,

$$\bar{\delta}(h(P_n), P) = \lim_{m\to\infty} e_\delta(\{P_n\}, p_m) \leq \lim_{m\to\infty} e_\delta(p_n, p_m) \leq \varepsilon$$

for any $n \geq h$. Thus, $\lim_{n\to\infty} \bar{\delta}(h(P_n), P) = 0$ and this proves that $h(M)$ is dense in $\bar{M}$. Since by Theorem 4.5 the space $(\bar{M}, \bar{\delta})$ is complete, we can conclude that $(\bar{M}, \bar{\delta}\ )$ is the metric completion of (M, δ).

In accordance with such a theorem, in the following we identify any point P in M with the point $h(P)$ in $\bar{M}$. Then, we consider $\bar{\delta}$ as an extension of δ, and the excess $e_{\bar{\delta}}$ in $(\bar{M}, \bar{\delta})$ as an extension of the excess e_δ in (M, δ). Finally, observe that if $x \in B(M)$, then it is possible that $h(x)$ is not closed in the space $(\bar{M}, \bar{\delta})$ and therefore it is possible that $h(x)$ is not an element of $B(\bar{M})$.

A suitable modification of the excess function shows the independence of d5.

Proposition 5.6. *Let* (M, δ) *be a metric space and set, for any* $x \in B(M)$ *and* $y \in B(M)$,

$$d_\delta(x, y) = e_\delta(x, y) + |e_\delta(x) - e_\delta(y)|. \tag{5.13}$$

Then $(B(M), d_\delta)$ *is a quasi-metric space such that the map* $d_c : Pr \times Pr \to R^+$ *defined by* (3.1) *is not symmetric. Therefore* $(B(M), d_\delta)$ *does not satisfy* d5.

Proof. Trivially, $d_\delta(x, x) = 0$. To prove the triangle inequality, observe that

$$\begin{aligned} d_\delta(x,y) &= e_\delta(x,y) + |e_\delta(x) - e_\delta(y)| \\ &\leq e_\delta(x,z) + e_\delta(z,y) + |e_\delta(x) - e_\delta(z)| + |e_\delta(z) - e_\delta(y)| = d_\delta(x,z) + d_\delta(z,y). \end{aligned}$$

Also, if $\leq$ is the partial order defined by d_δ then

$$x \leq y \Leftrightarrow x \subseteq y \text{ and } e_\delta(x) = e_\delta(y). \tag{5.14}$$

This shows that both d1 and d2 hold and therefore that $(B(M), d_\delta)$ is a quasi-metric space.

Let P, Q and R be points such that $\delta(P, Q) < \delta(P, R)$, let $< p_n >_{n \in N}$ be the sequence constantly equal to P and $< q_n >_{n \in N}$ the sequence constantly equal to $\{P, Q\}$. Then, since both $\{P\}$ and $\{P, Q\}$ are atoms, these sequences belong to Pr. On the other hand,

$$d_c(< p_n >_{n \in N}, < q_n >_{n \in N}) = d(P, \{Q, R\}) = \delta(P, Q) + \delta(Q, R)$$

while

$$d_c(< q_n >_{n \in N}, < p_n >_{n \in N}) = d(\{Q, R\}, P) = \delta(P, R) + \delta(Q, R)$$

and therefore $d_c(< p_n >_{n \in N}, < q_n >_{n \in N}) \neq d_c(< q_n >_{n \in N}, < p_n >_{n \in N})$.

6. The set of points of a region

We relate points and regions by the following definition.

Definition 6.1. Let P be a point and r a region. Then we say that P is *a point of* r provided that $\underline{d}(P, r) = 0$. We denote by $Pt(r)$ the set of points of r.

Proposition 6.2. *For any region r,* $Pt(r)$ *is a closed subset of* $(\bar{M}, \bar{\delta})$.

Proposition 6.3. *Let* $P = [< p_n >_{n \in N}]$ *be a point, then*

$$\lim_{n \to \infty} \underline{d}(P, p_n) = \lim_{n \to \infty} \underline{d}(p_n, P) = 0. \tag{6.1}$$

Let $(P_n)_{n \in N}$ *be a sequence of points such that* $P_n \in Pt(p_n)$. *Then*

$$\lim_{n \to \infty} P_n = P. \tag{6.2}$$

Proposition 6.4. *Let* P *be a point and* y *a region. Then*

$$\underline{d}(P, y) \leq \bar{\delta}(P, Pt(y)). \tag{6.3}$$

Theorem 6.5. *Let*(M, δ) *be a metric space. Then* $(B(M), e_\delta)$ *satisfies d6. Moreover, if* $(\bar{M}, \bar{\delta})$ *is the metric space associated with* $(B(M), e_\delta)$, *then, for every* $y \in B(M), Pt(y)$

is the closure of y in $(\bar{M},\bar{\delta})$. In particular, if (M,δ) is complete, since $(\bar{M},\bar{\delta})$ coincides with $(M,\delta), Pt(y)=y$.

Proof. Let $P=[<p_n>_{n\in N}]\in\bar{M}$ be a point and $y\in B(M)$ a region. Then, by Proposition 6.4, it is sufficient to prove that

$$\lim_{n\to\infty} e_\delta(p_n,y)\geq \inf\{\bar{\delta}(P,P'):P'\in\bar{M} \text{ and } P'\in Pt(y)\}.$$

Let $(P_n)_{n\in N}$ be a sequence of elements in M such that $P_n\in p_n$. Then, by Proposition 6.3, $\lim_{n\to\infty} P_n=P$ and therefore, since the function $f:\bar{M}\to R$ defined by setting $f(P)=\bar{\delta}(P,y)$ is continuous, and $e_\delta(p_n,y)\geq\delta(P_n,y)$,

$$\begin{aligned}\lim_{n\to\infty} e_\delta(p_n,y) &\geq \lim_{n\to\infty}\delta(P_n,y)=\bar{\delta}(\lim_{n\to\infty}P_n,y)=\bar{\delta}(P,y)\\ &= \inf\{\bar{\delta}(P,Q):Q\in\bar{M}\text{ and } Q\in y\}\geq\inf\{\bar{\delta}(P,P'):P'\in Pt(y)\}.\end{aligned}$$

To prove the second part of the theorem denote by $cl(y)$ the closure of y in the space $(\bar{M},\bar{\delta})$. Then, since $Pt(y)$ is a closed set containing y then $Pt(y)\supseteq cl(y)$. Moreover, let $P=[<p_n>_{n\in N}]$ be a point and, for any $n\in N$, let P_n be an element of p_n. Then

$$\begin{aligned}P\in Pt(y) &\Leftrightarrow \lim_{n\to\infty} e_\delta(p_n,y)=0\Rightarrow\lim_{n\to\infty}\delta(P_n,y)=0\Leftrightarrow\bar{\delta}(\lim_{n\to\infty}P_n,y)=0\\ &\Leftrightarrow \bar{\delta}(P,y)=0\Leftrightarrow P\in cl(y).\end{aligned}$$

This proves that $Pt(y)\subseteq cl(y)$ and therefore that $Pt(y)=cl(y)$.

In order to prove the independence of d6, we propose an example inspired by the notion of fuzzy subset of a metric space. Namely, we confine ourselves to the three-valued fuzzy subsets which we represent as a pair (x,y) of subsets such that $x\subseteq y$. The set x is interpreted as the set of elements whose membership degree is 1 and y as the set of elements whose membership degree is greater or equal to 0.5. In accordance, any classical subset x is identified with a pair (x,x). This enables us to prove the following proposition where, given two real numbers x and y, $x\oplus y$ denotes the average $(x+y)/2$.

Proposition 6.6. *Let (M,δ) be a metric space, set $Re=\{(x_1,x_2)\in B(M)\times B(M):x_1\subseteq x_2\}$ and define $\underline{e}_\delta$ by setting*

$$\underline{e}_\delta((x_1,x_2),(y_1,y_2))=e_\delta(x_1,y_1)\oplus e(x_2,y_2),$$

for every (x_1,x_2) and (y_1,y_2) in Re. Then $(Re,\underline{e}_\delta)$ is a quasi-metric space of regions. Such a space is a proper extension of $(B(M),e_\delta)$ whose associated metric space coincides with the one of $(B(M),e_\delta)$ and in which d6 fails.

7. Abstract excess spaces

We are interested in the spaces of regions (Re,d) for which the mapping $Pt:Re\to B(\bar{M})$ is an isometry, i.e. for which

$$d(x,y)=e_{\bar{\delta}}(Pt(x),Pt(y)).$$

The following proposition shows what happens in the general case.

Proposition 7.1. *Let P be a point and x and y regions. Then*

$$d(x,y) \geq \sup\{\underline{d}(P,y) : P \in Pt(x)\}. \tag{7.1}$$

Consequently, if d6 is satisfied, then

$$d(x,y) \geq e_{\bar{\delta}}(Pt(x), Pt(y)) \tag{7.2}$$

and

$$d(x) \geq e_{\bar{\delta}}(Pt(x)). \tag{7.3}$$

Proof. To prove (7.1) observe that $\underline{d}(P,y) \leq \underline{d}(P,x) + d(x,y) = d(x,y)$ for any $P \in Pt(x)$. (7.2) is trivial. Also, for every $P, Q \in Pt(x)$,

$$\begin{aligned}
\bar{\delta}(P,Q) &= \lim_{n\to\infty} d(p_n,q_n) \leq \lim_{n\to\infty} d(p_n,x) + d(x,q_n) \\
&\leq \lim_{n\to\infty} (d(p_n,x) + d(q_n,x) + d(q_n) + d(x)) \\
&= \lim_{n\to\infty} d(p_n,x) + \lim_{n\to\infty g} (q_n,x) + \lim_{n\to\infty} d(q_n) + d(x) = d(x)
\end{aligned}$$

and therefore $e_{\bar{\delta}}(Pt(x)) = \sup\{\bar{\delta}(P,Q) : P \in Pt(x), Q \in Pt(x)\} \leq d(x)$.

Such a proposition suggests the following definition.

Definition 7.2. An *abstract excess space* is a quasi-metric space of bounded regions (Re, d) satisfying d6 and such that, for any point P and x, $y \in Re$,

d7) $$d(x,y) = \sup\{\underline{d}(P,y) : P \in Pt(x)\}.$$

Proposition 7.3. *Every full Hausdorff excess space is an abstract excess space.*

Proof. Let (M,δ) be a metric space and let x, y be regions in $B(M)$. Then, by denoting by cl the closure operator in $(\bar{M}, \bar{\delta})$,

$$\begin{aligned}
e_{\delta}(x,y) &= e_{\bar{\delta}}(cl(x), cl(y)) = e_{\bar{\delta}}(Pt(x), Pt(y)) \\
&= \sup_{P\in Pt(x)} \bar{\delta}(P, Pt(y)) = \sup_{P\in Pt(x)} \underline{d}(P,y).
\end{aligned}$$

In a simple way, we can prove the following representation theorem for abstract excess spaces.

Theorem 7.4. *Let (Re, d) be an abstract excess space and $(\bar{M}, \bar{\delta})$ be the associated metric space. Then*

(i) $d(x,y) = e_{\bar{\delta}}(Pt(x), Pt(y))$,
(ii) $d(x) = e_{\bar{\delta}}(Pt(x))$

and therefore $Pt : Re \to B(\bar{M})$ is an isometry from (Re, d) into $(B(\bar{M}), e_{\bar{\delta}})$ preserving diameters. Consequently, every abstract excess space is isometric to a Hausdorff excess space.

Proof. Since $d(x,y) = \sup_{P\in Pt(x)} \underline{d}(P,y)$ and $\underline{d}(P,y) = \bar{\delta}(P, Pt(y)) = \inf\{\bar{\delta}(P,Q) : Q \in Pt(y)\}$, we have that

$$d(x,y) = \sup_{P\in Pt(x)} \inf_{Q\in Pt(y)} \bar{\delta}(P,Q) = e_{\bar{\delta}}(Pt(x), Pt(y)).$$

To prove ii), observe that

$$d(x) = \sup\{d(x,x') : x' \leq x\} = \sup\{e_{\bar{\delta}}(Pt(x), Pt(x')) : x' \leq x\} \leq e_{\bar{\delta}}(Pt(x)).$$

So, by (7.3), $d(x) = e_{\bar{\delta}}(Pt(x))$.

The following proposition shows that d7 is independent of the remaining axioms. We denote by $x \div y$ the value $x - y$ if $x \geq y$ and 0 otherwise.

Proposition 7.5. *Let (M, δ) be a metric space and set, for any $x,y \in B(M)$,*

$$d(x,y) = e_\delta(x,y) + e_\delta(x) \div e_\delta(y). \tag{7.4}$$

Then $(B(M), d)$ is a quasi-metric space of regions such that the partial order is the inclusion relation, $d(x) = 2 \cdot e_\delta(x)$, and the associated metric space coincides with the one $(\bar{M}, \bar{\delta})$ of (M, δ). Moreover, while d6 is satisfied, d7 does not hold.

Proof. Trivially,

$$d(x,y) = 0 \Leftrightarrow e_\delta(x,y) = 0 \text{ and } e_\delta(x) \leq e_\delta(y) \Leftrightarrow x \subseteq y.$$

This proves both d1 and d2. To prove d3, observe that

$$\begin{aligned} d(x,y) &= e_\delta(x,y) + e_\delta(x) \div e_\delta(y) \\ &\leq e_\delta(x,z) + e_\delta(z,y) + e_\delta(x) \div e_\delta(z) + e_\delta(z) \div e_\delta(y) = d(x,z) + d(z,y). \end{aligned}$$

Also,

$$\begin{aligned} d(x) &= \sup\{d(x,x') : x' \subseteq x\} \\ &= \sup\{e_\delta(x,x') + e_\delta(x) \div e_\delta(x') : x' \subseteq x\} \\ &= \sup\{e_\delta(x,x') + e_\delta(x) - e_\delta(x') : x' \subseteq x\} \\ &= e_\delta(x) + \sup\{e_\delta(x,x') - e_\delta(x') : x' \subseteq x\} \\ &= e_\delta(x) + \sup\{e_\delta(x,\{p\}) - e_\delta(\{p\}) : p \in x\} \\ &= e_\delta(x) + \sup\{e_\delta(x,\{p\}) : p \in x\} = 2 \cdot e_\delta(x). \end{aligned}$$

Axiom d4 is trivial. To prove d5, observe that e_δ satisfies d5 and therefore

$$\begin{aligned} |d(x,y) - d(y,x)| &= |e_\delta(x,y) - e_\delta(y,x) + e_\delta(x) \div e_\delta(y) - e_\delta(y) \div e_\delta(x)| \\ &\leq |e_\delta(x,y) - e_\delta(y,x)| + |e_\delta(x) - e_\delta(y)| \\ &\leq |e_\delta(x,y) - e_\delta(y,x)| + \max\{e_\delta(x), e_\delta(y)\} \\ &\leq e_\delta(x) + e_\delta(y) + e_\delta(x) + e_\delta(y) = d(x) + d(y). \end{aligned}$$

Let $< p_n >_{n \in N}$ be a point-representing sequence in the space $(B(M), e_\delta)$. Then $\lim_{n\to\infty} d(p_n) = \lim_{n\to\infty} 2 \cdot e_\delta(p_n) = 0$. Moreover, given $\varepsilon > 0$, let m such that $e_\delta(p_h, p_k) < \varepsilon/3, e_\delta(p_h) < \varepsilon/3$ and $e_\delta(p_k) < \varepsilon/3$ for any $h \geq m$ and $k \geq m$. Then

$$d(p_h, p_k) = e_\delta(p_h, p_k) + e_\delta(p_h) \div e_\delta(p_k) \leq e_\delta(p_h, p_k) + e_\delta(p_h) + e_\delta(p_k) \leq \varepsilon$$

for any $h \geq m$ and $k \geq m$. This proves that $< p_n >_{n \in N}$ is a point-representing sequence in the space $(B(M), d)$. Conversely, since $e_\delta \leq d$, any point-representing sequence in the space $(B(M), d)$ is a point-representing sequence in the space $(B(M), d)$.

Let $< p_n >_{n\in N}$ and $< q_n >_{n\in N}$ be two point-representing sequences. Then, since $\lim_{n\to\infty}(e_\delta(p_n) \div e_\delta(q_n)) = 0$,

$$\begin{aligned}\lim_{n\to\infty} d(p_n, q_n) &= \lim_{n\to\infty}(e_\delta(p_n, q_n) + e_\delta(p_n) \div e_\delta(q_n)) \\ &= \lim_{n\to\infty} e_\delta(p_n, q_n) + \lim_{n\to\infty}(e_\delta(p_n) \div e_\delta(q_n)) = \lim_{n\to\infty} e_\delta(p_n, q_n).\end{aligned}$$

This means that the metric space associated with $(B(M), d)$ coincides with the metric space associated with $(B(M), e_\delta)$ and therefore with $(\bar{M}, \bar{\delta})$ the metric completion of (M, δ).

To prove d6, let $P = [< p_n >_{n\in N}]$ be an element in $\bar{M}$ and $y \in B(M)$. Then, by denoting by $e_{\bar{\delta}}$ the Hausdorff excess induced by $\bar{\delta}$,

$$\bar{\delta}(P, Pt(y)) = e_{\bar{\delta}}(\{P\}, Pt(y)) = \lim_{n\to\infty} e_\delta(p_n, y) = \lim_{n\to\infty} d(p_n, y) = \underline{d}(P, y).$$

Finally, to prove that d7 does not hold, let x and y be two regions such that $e_\delta(x) > e_\delta(y)$. Then

$$\begin{aligned}d(x, y) &= e_\delta(x, y) + e_\delta(x) \div e_\delta(y) > e_\delta(x, y) = e_{\bar{\delta}}(Pt(x), Pt(y)) \\ &= \sup\{\bar{\delta}(P, Pt(y)) : P \in Pt(x)\} \\ &\geq \sup\{\underline{d}(P, y) : P \in Pt(x)\}.\end{aligned}$$

8. Atom-free spaces

So far no axiom excludes the existence of regions which are atoms. This enables us to obtain a theory extending the theory of the metric spaces. Alternatively, in accordance with Whitehead's program, we can decide to confine ourselves to atom-free spaces.

Definition 8.1. We call *atom-free quasi-metric space of regions* any quasi-metric space of regions satisfying:

***d*8)** no atom exists in *Re*.

The space $(B(M), e_\delta)$ is not atom-free, obviously. Then, in order to define a notion of canonical models for the atom-free space theory, we have to look for a more reasonable definition of region in a metric space. As an example, we can refer only to the regular subsets in $B(M)$.

Definition 8.2. Let (M, δ) be a metric space and denote by $cl : P(M) \to P(M)$ and by $\text{int} : P(M) \to P(M)$ the closure operator and the interior operator, respectively. Also, define $reg : P(M) \to P(M)$ by setting $reg(x) = cl(\text{int}(x))$. Then we call *regularly closed set*, in brief *regular set*, any fixed point of *reg*.

It is easy to prove that in the class of the closed subsets of (M, δ), the operator *reg* satisfies the following properties

(i) $reg(\emptyset) = \emptyset$;

(ii) $x \subseteq y \Rightarrow reg(x) \subseteq reg(y)$;

(iii) $reg(x) \subseteq x$;

(iv) $reg(reg(x)) = reg(x)$.

Also, the class of regular sets is a Boolean algebra. We denote by $Re(M)$ the class of regular elements in $B(M)$. Equation iv) entails that $Re(M) = \{reg(x) : x \in B(M)\} - \{\emptyset\}$. An interesting class of elements in $Re(M)$ is defined by setting, for any $P \in M$ and $n \in N$,

$$B_n(P) = cl(\{P' \in M : \delta(P', P) < 1/n\}). \tag{8.1}$$

Theorem 8.3. *Let (M, δ) be a metric space, then $(Re(M), e_\delta)$ is a quasi-metric space of regions whose diameter coincides with the diameter D defined in* (5.5) *and whose order is the set-theoretical inclusion.*

Proof. Denote by $e_\delta(x)$ the diameter of an element $x \in Re(M)$ in the space $(Re(M), e_\delta)$. Then, by Theorem 5.2,

$$\begin{aligned} e_\delta(x) &= \sup\{e_\delta(x, x") : x" \subseteq x, x" \in Re(M)\} \\ &\leq \sup\{e_\delta(x, x") : x" \subseteq x, x" \in B(M)\} = D(x). \end{aligned}$$

Also, observe that for any $x \in Re(M)$ and $x' \in B(M), x' \subseteq x$ if and only if $reg(x') \subseteq x$. Then,

$$\begin{aligned} D(x) &= \sup\{e_\delta(x, x') : x' \subseteq x, x' \in B(M)\} \\ &\leq \sup\{e_\delta(x, reg(x')) : x' \subseteq x, x' \in B(M)\} \\ &= \sup\{e_\delta(x, reg(x')) : reg(x') \subseteq x, x' \in B(M)\} \\ &= \sup\{e_\delta(x, x") : x" \subseteq x, x" \in Re(M)\} \\ &= e_\delta(x). \end{aligned}$$

Since the diameter in $(Re(M), e_\delta)$ coincides with the diameter in $(B(M), e_\delta)$, we also have that d4 is satisfied.

In the following we call *small Hausdorff excess space* the space $(Re(M), e_\delta)$.

Theorem 8.4. *Let (M, δ) be a metric space and denote by $(\bar{M}, \bar{\delta})$ the metric space associated with $(Re(M), e_\delta)$. Also, denote by $k : M \to \bar{M}$ the map defined by setting* for any $P \in M$,

$$k(P) = [< B_n(P) >_{n \in N}]. \tag{8.2}$$

Then k is an isometry such that $k(M)$ is dense in $\bar{M}$. Consequently, $(\bar{M}, \bar{\delta})$ is the completion of (M, δ) and therefore $(\bar{M}, \bar{\delta})$ is isometric with the metric space associated with $(B(M), e_\delta)$.

Proof. Observe that, given $P \in M, < B_n(P) >_{n \in N}$ is a point-representing sequence of elements in $Re(M)$. To prove that k is an isometry, let P and Q be two elements in M, and observe that, for any $P' \in B_n(P)$ and $Q' \in B_n(Q)$,

$$\delta(P, Q) \leq \delta(P, P') + \delta(P', Q') + \delta(Q', Q) \leq 2/n + \delta(P', Q')$$

and therefore

$$\delta(P, Q) \leq 2/n + \delta(P', B_n(Q)) \leq 2/n + e_\delta(B_n(P), B_n(Q)).$$

As a consequence,

$$\delta(P,Q) \leq \lim_{n\to\infty} e_\delta(B_n(P), B_n(Q)) = \underline{\delta}(k(P), k(Q)).$$

Likewise, since

$$\delta(P',Q') \leq \delta(P',P) + \delta(P,Q) + \delta(Q,Q') \leq 2/n + \delta(P,Q),$$

we have that $e_\delta(B_n(P), B_n(Q)) \leq 2/n + \delta(P,Q)$ and therefore that

$$\delta(k(P), k(Q)) = \lim_{n\to\infty} e_\delta(B_n(P), B_n(Q)) \leq \delta(P,Q).$$

Then, $\delta(P,Q) = \underline{\delta}(k(P), k(Q))$ and this proves that $h : M \to \bar{M}$ is an isometry. To prove that $k(M)$ is dense in $(\bar{M}, \bar{\delta})$, let $P = [< p_n >_{n\in N}]$ be any element in $\bar{M}$. Moreover, for any $n \in N$, let $Q_n \in M$ be an element of the set p_n. We claim that $\lim_{n\to\infty} k(Q_n) = P$, i.e. that

$$\lim_{n\to\infty} \underline{\delta}(k(Q_n), P) = \lim_{n\to\infty} (\lim_{m\to\infty} e_\delta(B_m(Q_n), p_m)) = 0.$$

Indeed, if we denote by m the minimum distance defined by (5.2),

$$\begin{aligned} e_\delta(B_m(Q_n), p_m) &\leq e_\delta(B_m(Q_n), p_n) + e_\delta(p_n, p_m) \\ &\leq m(B_m(Q_n), p_n) + D(B_m(Q_n)) + e_\delta(p_n, p_m) \\ &= D(B_m(Q_n)) + e_\delta(p_n, p_m) \leq 2/m + e_\delta(p_n, p_m). \end{aligned}$$

On the other hand, since $< p_n >_{n\in N}$ is a point-representing sequence, given any $\varepsilon > 0$ an integer h exists such that $e_\delta(p_n, p_m) \leq \varepsilon$ for any $n \geq h$ and $m \geq h$. Consequently,

$$\underline{\delta}(k(Q_n), P) = \lim_{m\to\infty} e_\delta(B_m(Q_n), p_m) \leq \lim_{m\to\infty} e_\delta(p_n, p_m) \leq \varepsilon$$

for any $n \geq h$. Thus, $\lim_{n\to\infty} \underline{\delta}(k(Q_n), P) = 0$ and this proves that $k(M)$ is dense in $(\bar{M}, \bar{\delta})$. Since by Theorem 4.5 the space $(\bar{M}, \bar{\delta})$ is complete, we can conclude that $(\bar{M}, \bar{\delta})$ is the completion of (M, δ).

Theorem 8.5. *Let (M, δ) be a metric space, then $(Re(M), e_\delta)$ is an abstract excess space. If (M, δ) has no isolated point, then $(Re(M), e_\delta)$ is atom-free.*

Proof. Since the points in $(Re(M), e_\delta)$ coincide with the points in $(B(M), e_\delta)$, it is evident that $(Re(M), e_\delta)$ satisfies d6 and d7. To prove the second part of the theorem, we prove that an element x in $Re(M)$ is an atom iff there is an isolated point $P \in M$ such that $x = \{P\}$. Indeed, if P is an isolated point, then it is evident that $\{P\}$ is a bounded regular subset and therefore an atom in $Re(M)$. Conversely, let x be an atom in $Re(M)$ and let P be an element of $\text{int}(x)$. We claim that P is an isolated point such that $x = \{P\}$. Indeed, assume that $x \neq \{P\}$, then a point $Q \in x$ exists such that $Q \neq P$. In accordance, there exists $n \in N$ such that $B_n(P) \subseteq x$ and $Q \notin B_n(P)$. Then $B_n(P)$ is a proper sub-region of x by contradicting the hypothesis of x being atom. Since $x = \{P\}$, and x is regular, we have also that P is an isolated point.

In the case in (M, δ) there are isolated points we can again define an atom-free space by the notion of formal ball. Indeed, consider any quasi-metric space (Re, δ) and call *closed* *formal ball with center p and radious r*, every pair (p, r), where $p \in Re$ and r is a positive real number. We define in the class $Ball(Re)$ of closed formal balls in Re the function

$$d((p, \lambda), (q, \mu)) = \max\{\delta(p, q) + \lambda - \mu, 0\}.$$

It is matter of routine to prove that $(Ball(Re), d)$ is a quasi-metric space. Also, if $\leq$ is the order associated with d, then

$$(p, \lambda) \leq (q, \mu) \Leftrightarrow d((p, \lambda), (q, \mu)) = 0 \Leftrightarrow \delta(p, q) + \lambda \leq \mu.$$

Moreover, $d((p, \lambda)) = 2 \cdot \lambda$. Also, while d4 is satisfied, since it is

$$\begin{aligned} |d((p, \lambda), (q, \mu)) - d((q, \lambda), (p, \mu))| &\leq |\delta(p, q) - \delta(q, p)| + 2 \cdot \lambda + 2 \cdot \mu \\ &= |\delta(p, q) - \delta(q, p)| + d((p, \lambda)) + d((q, \mu)), \end{aligned}$$

in the case (Re, δ) is a metric space d5 is satisfied. It is also evident that such a space has no atom. In our opinion should be interesting a comparison among the just introduced notions and the completion of a generalized metric space via formal open balls proposed in [14].

9. Defining the points by nested sequences of regions

Usually in the literature in point-free geometry the notion of point is defined by referring to the class of nested sequences of regions (see, for example, [5], [19]). We can proceed in the same way in our theory of quasi-metric spaces of regions.

Definition 9.1. Given a quasi-metric space (Re, d), we call *nested-representing sequences* any order-reversing sequence $< p_n >_{n \in N}$ of regions with vanishing diameters, i.e. such that

$$\lim_{n \to \infty} d(p_n) = 0.$$

We denote by Nr the class of the nested-representing sequences. Obviously, any nested-representing sequence is a point-representing sequence in accordance with Definition 3.1. To prove that Nr is nonempty, we have to consider an axiom analogous to Axiom d4:

d4') Any region x contains a region x' such that $d(x') \leq d(x)/2$.

Trivially, d4' entails that any region contains a nested-representing sequence.

Definition 9.2. Let (Re, d) be a quasi-metric space of regions satisfying d4'. Then the *nested metric space associated with* (Re, d) is the metric space (M', δ') where

$$M' = \{[< p_n >_{n \in N}] \in \bar{M} :< p_n >_{n \in N} \in Nr\}$$

and δ' is the restriction of $\bar{\delta}$ to M'.

The space (M', δ') is different from $(\bar{M}, \bar{\delta})$, in general. As an example, if (Re, d) is a metric space, then while $(\bar{M}, \bar{\delta})$ is the completion of $(Re, d), (M', \delta')$ coincides with (Re, d). Indeed, in such a case the only point-representing sequences are the sequences constantly equal to an element of Re. This observation is in accordance with the following theorem.

Theorem 9.3. *Let (Re, d) be a quasi-metric space of regions satisfying d4', $(\bar{M}, \bar{\delta})$ and (M', δ') the associated metric space and the nested metric space, respectively. Then $(\bar{M}, \bar{\delta})$ is the metric completion of (M', δ').*

Proof. To prove that (M', δ') is dense in $(\bar{M}, \bar{\delta})$, let $P = [< p_n >_{n\in N}]$ be any element in $\bar{M}$. Then, we can consider for any $n \in N$, a point P_n in M' such that $P_n \in Pt(p_n)$. Then, since by (6.1) $\lim_{n\to\infty} \underline{d}(p_n, P) = 0$ and

$$\bar{\delta}(P_n, P) \leq \underline{d}(P_n, p_n) + \underline{d}(p_n, P) = \underline{d}(p_n, P),$$

we have that $lim_{n\to\infty}\bar{\delta}(P_n, P) = 0$. Thus every element of $\bar{M}$ is a limit of a sequence of elements of M' and therefore by the completeness of $(\bar{M}, \bar{\delta})$, the space $(\bar{M}, \bar{\delta}\ )$ is the metric completion of (M', δ').

In accordance with Theorem 4.4, the metric space associated with a quasi-metric space of regions is complete. The question arises whether the associated nested metric space satisfies some completeness property.

Definition 9.4. Let (M, δ) be a metric space, then we say that (M, δ) is *weakly complete* if any nested sequence of non-empty regularly closed subsets with vanishing diameters has a non-empty intersection. We say that a metric space (M', δ') is a *weak* completion of (M, δ) if (M', δ') is weakly complete and (M, δ) is dense in (M', δ').

Theorem 9.5. *Let (M, δ) a metric space. Then the nested metric space (M', δ') associated with $(Re(M), e_\delta)$ is a weak completion of (M, δ).*

Proof. By miming Theorem 8.4 we have that (M, δ) is isometric to a dense subspace of (M', δ'). Also, observe that any regularly closed subset x' of M' is the closure in (M', δ') of some $x \in Re(M)$. Let $< x'_n >_{n\in N}$ be any nested sequence of elements in $R_e(M')$ with vanishing diameters and let $x_n \in Re(M)$ such that its closure in (M', δ') is x'_n. Then $< x_n >_{n\in N}$ is a nested representing sequence and therefore it determines a point P in M' which belongs to the closure x'_n of x_n in (M', δ').

The definition of point by the nested-representing sequences refers only to the inclusion relation between regions and to the diameter of a region. So, the question arises whether a possible approach to point-free geometry can be based on these two notions as primitives. A reasonable proposal should be the following one. We start from a structure $(R, \leq, D)$ where $\leq$ is a partial order and $D : R \to R^+$ a map. In this structure we define the notion of nested-representing sequence as in Definition 9.1. In the set Pr of nested-representing sequences we can set

$$d(< x_n >_{n\in N}, < y_n >_{n\in N}\ \}) = \inf\{D(x) : xOx_n \text{ and } xOy_n \text{ for any } n \in N\}$$

where O is the overlapping relation defined by setting xOy provided that a region z exists contained in both x and y. By imposing suitable properties on the diameter D it should be possible to prove that (Nr, d) is a pseudo-metric space and therefore to define a metric space as in Section 4. With regard to this idea, observe that the partial order and the diameter induced by a quasi-metric do not exhaust the information carried by the quasi-metric. Namely the following proposition holds true.

Proposition 9.6. *Let R be the set of real numbers. Then there are two canonical quasi-metric spaces in R^2 which are not isometric but define the same diameter and the same inclusion relation.*

Proof. Let (R^2, d) be the Euclidean metric space and let δ be the taxi-metric, i.e. set

$$\delta((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

Then the balls in such a space are the squares whose sides have the direction of the diagonals. It can be shown easily that (R^2, δ) is topologically equivalent to (R^2, d). Nevertheless, these spaces are not isometric. Indeed, in (R^2, δ) the four points (-1,0), (1,0), (0,-1), (0,1) define a square whose diagonals are equal to the sides and in (R^2, d) such a point configuration cannot exist. Consider the Hausdorff excesses defined in these spaces by the class of taxi-balls. It is evident that they are not isometric. Also, given a closed taxi-ball of radius ε, both its Euclidean-diameter and taxi-diameter are equal to $2{\cdot}\varepsilon$. Moreover, in both cases the partial order associated with the Hausdorff excess is the usual inclusion relation.

10. Open questions and future works

As it was proved by L. M. Blumenthal in [2], given an integer $n \in N$, it is possible to add to the theory of metric spaces MS a suitable set of axioms ES to obtain a theory $T = MS \cup ES$ for the Euclidean $n-$dimensional metric space, i.e. a theory whose models coincide with the metric space of the Euclidean space whose dimension is n. Obviously, the axioms in T refer to the points and the distance between points as primitives. Now, assume as primitives the regions and a distance between regions. Then it is an open question to look for a system of axioms ES to add to the axioms d1-d8 in order to obtain a theory T whose models are the atom-free quasi-metric spaces of regions whose associated metric space (M, δ) is an Euclidean metric space. Such a theory should be a point-free approach to Euclidean geometry in accordance with the Whitehead's ideas.

Further, it should be interesting to study the category whose objects are the quasi-metric spaces of regions and whose morphisms are the non expansive maps.

Finally, in accordance with the recent literature, it is an important task to explore the computability dimension of the proposed notions.

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