SIMILARITIES AND FUZZY ORDERS IN APPROXIMATE REASONING

by
C. Crisconio and G. Gerla

Abstract. A general approach to fixed point theory is proposed as a tool for logic programming. Such an approach extends both fixed point theory in ordered sets and fixed point theory in metric spaces. Fuzzy set theory is the natural framework of the paper. In particular, we use the notions of similarity and fuzzy order.

Keywords. Logic programming, fixed points, similarities, fuzzy orders.

1. Introduction

Fixed point theory for operators in a lattice is a basic tool for formal logic. Indeed, usually a logical apparatus enables us to define an "*immediate consequence operator*" T in a lattice L whose elements represents "*pieces of information*". Given $x \in L$, the last fixed point of T greater or equal to x represents the theory generated by x (as an example, see [12] for crisp logic and [4] for fuzzy logic). In particular, fixed point theory is very useful in logic programming, where L is the power set $\mathcal{P}(B_P)$, B_P being the Herbrand base of a program P, and T is the immediate consequence operator T: $\mathcal{P}(B_P) \rightarrow \mathcal{P}(B_P)$ associated with the program P. The fixed points of T are the Herbrand models for P. Now if the logic under consideration is monotone (in particular if the program P is positive), T is a monotone operator, and therefore it is possible to apply the fixed point theorem of Knaster and Tarski. Nevertheless, when the immediate consequence operator T is not monotone, for instance if T is associated with a program containing negation, fixed point theorems for ordered set appear to be insufficient. In such case, it can be helpful to apply fixed point techniques in a metric space which are derived from Banach-Caccioppoli's theorem (see [7]). Another reason suggesting the opportunity to refer to metric spaces comes from fuzzy logic. Indeed, the process leading to a fuzzy set of consequences from a fuzzy set of hypotheses happens in a continuous environment. Such a process cannot finish by giving the exact output. Rather is an endless approximation of the ideal output. From here the need arises to define someway the notion of "approximation". This is possible only in a metric setting.

On the other hand, fixed point theory in ordered sets and fixed point theory in metric spaces can be unified. Indeed, if one introduces the notion of quasi-metric space, lacking in symmetric property, it is possible to demonstrate a theorem simultaneously generalising the fixed point theorem of Knaster and Tarski for ordered structures and the theorem of Banach-Caccioppoli for metric spaces (see [7], [8], [9]).

In this paper, we expose several results in this direction. Also, we emphasize the possibility of interpreting such results in terms of fuzzy set theory, extending a duality, proved by Valverde in [11], between some metric notions and the similarities.

2. Preliminaries

Let M be a set; we call *fuzzy subset* of M or, more simply, *fuzzy set* any function f: $M \rightarrow [0,1]$. A *fuzzy relation* is a fuzzy subset of a cartesian product. Given two fuzzy sets f and g, we set f ¯ g provided that $f(x) \leq g(x)$ for every x χ M. Moreover a fuzzy set f: $M \rightarrow [0,1]$ is called *crisp* if $f(x)\ \chi\ \{0, 1\}$ for every $x\ \chi\ M$; we identify the fuzzy set with the subset of M via the characteristic functions. Given a fuzzy set f, for every $\lambda\ \chi\ [0,1]$, the subsets

$$C(f, \lambda) = \{x\chi\ M \,/\, f(x) \geq \lambda\} \quad \text{and} \quad O(f, \lambda) = \{x\chi\ M \,/\, f(x) > \lambda\}$$

are called the *closed λ-cut* and the *open λ-cut* of f, respectively.

Definition 2.1 Let M be a non-empty set and $\square$: $[0,1]^2 \rightarrow [0,1]$ be a binary operation. Then $\square$ is called a *triangular norm* (briefly, a *t-norm*) provided the following conditions hold:

(i) $\square$ is associative;
(ii) $\square$ is commutative;
(iii) $\square$ is order-preserving in both variables;
(iv) $x \square 1 = x\ \ \ldots x\ \chi\ [0, 1]$.

A t-norm is called *continuous* provided that it preserves the least upper bounds.

Let $\square$ be a t-norm, $x\ \chi\ [0, 1]$ and $n\ \chi\ \square$. Then we define $x^{(n)}$ by

$$x^{(n)} = \begin{cases} x & \text{if } n = 1 \\ x \square x \square \ldots\ldots \square x \ \ n\text{-times} & \text{if } n > 1. \end{cases}$$

Definizione 2.2 A t-norm $\square$ is called *Archimedean* if, for any pair $x, y\ \chi\ [0, 1]$, an integer n exists such that $x^{(n)} < y$.

The usual product is an example of Archimedean t-norm. The minimum is an example of non-Archimedean t-norm.

Given a first order language L, we can define a *fuzzy model* for L as a pair (D, I) where, for any n-ary relation name r, $R = I(r)$ is a fuzzy subset of D^n, i.e. an n-ary fuzzy relation. The constants and the name-functions are interpreted as usual. The logical connectives are interpreted by suitable operations in [0, 1]. In particular, usually $\square$ is interpreted by a triangular norm $\square$. Moreover $\rightarrow$ is interpreted in such a way that $|A \rightarrow B| = 1$ if and only if $|A| \square |B|$. Also we consider the unary connective c we interpret by the function C: $[0, 1] \rightarrow [0, 1]$ defined by setting $C(x) = 1$ if $x = 1$ $C(x) = 0$ otherwise. Also, in literature one considers some additional connectives called *modifiers* corresponding to words as "*very*", "*almost*", ... of the natural language. In this paper we are interested to a connective m ("*much*") we interpret by a power function x^c with $c > 1$. (see Zadeh, [13]). This enables to associate with any formula α, whose variables are among $x_1, \ldots., x_n$ and any $d_1, \ldots., d_n\ \chi\ D$, a value $|\alpha|^{d_1 \ldots d_n}\ \chi\ [0, 1]$ (see Hájek, [5]).

Assume that the language L contains a binary relation name r. Then we can consider the following axioms, which are basic one to define the notion of order and equivalence in classical set theory:

- $\ldots x\ \ r(x, x)$;
- $\ldots x\ \ldots y\ \ (r(x, y) \rightarrow r(y, x))$;
- $\ldots x\ \ldots y\ \ldots z\ \ (r(x, y) \square r(y, z) \rightarrow r(x, z))$;
- $\ldots x\ \ldots y\ \ (c(r(x, y)) \square c(r(y, x)) \rightarrow x = y)$.

It is evident that an interpretation $R = I(r)$ in a domain M satisfies the above axioms if and only if the following are satisfied:

(1)	$R(x, x) = 1$;	(*reflexivity*)
(2)	$R(x, y) = R(y, x)$;	(*symmetry*)
(3)	$R(x, y) \square R(y, z) \square R(x, z)$;	($\square$*-transitivity*)
(4)	$R(x, y) = 1$ and $R(y, x) = 1 \square x = y$.	(*antisymmetry*)

Definition 2.3 When conditions (1) and (3) are satisfied for any $x, y, z \chi M$, R is called a □-*fuzzy preorder*; when conditions (1), (3) and (4) are satisfied, R is called a □-*fuzzy order*; when conditions (1), (2) and (3) are satisfied, R is called a □-*similarity*. In particular, when □ is the t-norm of the minimum, we call R simply a *fuzzy preorder*, a *fuzzy order* or a *similarity*, respectively. We also observe that, if R is a □-fuzzy preorder, the position

$$x \square y \quad \square \quad R(x, y) = R(y, x) = 1$$

defines an equivalence's relation. In this case we say that x is *similar* with y. Then it is possible to consider the quotient $M' = M/\square = \{[x] / x \chi M\}$, where $[x] = \{y \chi M / R(x, y) = R(y, x) = 1\}$. Moreover, it is immediate to prove that the mapping

$$R': M' \% M' \rightarrow [0,1] \text{ such that } R'([x], [y]) = R(x, y)$$

is well defined and it is a □-fuzzy order on M'. By this identification is always possible to change from a □-fuzzy preorder relation to a □-fuzzy order one. In particular, if R is a □-similarity, in the quotient M' we still obtain a □-similarity R' such that $R'(x, y) = R'(y, x) = 1 \square x = y$.

If R is a □-fuzzy preorder, we define the *fuzzy interval* $[a, +\infty)$: $M \rightarrow [0,1]$ by setting $[a, +\infty)(x) = R(a, x)$. If R is a □-similarity, we write $[a]$ instead of $[a, +\infty)$ and we say that $[a]$ is a *fuzzy class*.

Definition 2.4 Given a map f: $M \rightarrow M$ and a fuzzy relation R we say that $x \chi M$ is a *fixed point* for f (w.r. to R) provided that $f(x) \square x$, i.e. $R(x, f(x)) = R(f(x), x) = 1$.

In the case that R is a □-fuzzy order, x is a fixed point if and only if $f(x) = x$.

3. Distances.

Let M be a non-empty set and d: $M \% M \rightarrow [0,1]$ a mapping. Also, consider the following axioms for any $x, y, z \chi M$:

(d1) $d(x, y) = 0 \square x = y$;
(d'1) $d(x, x) = 0$;
(d2) $d(x, y) = d(y, x)$;
(d'2) $d(x, y) = 0$ and $d(y, x) = 0 \quad \square \quad x = y$;
(d3) $d(x, y) + d(y, z) \geq d(x, z)$;
(d'3) $d(x, y) \square d(y, z) \geq d(x, z)$.

Then

(M, d) is called a

- *generalized ultrametric space* if it satisfies (d'1) and (d'3);
- *ultrapseudometric space* if it satisfies (d'1), (d2) and (d'3);
- *ultrametric space* if it satisfies (d1), (d2) and (d'3);
- *quasi-ultrametric space* if it satisfies (d'2) and (d'3);

In the same way,

(M, d) is called a

- *generalized metric space* if it satisfies (d'1) and (d3);
- *pseudometric space* if it satisfies (d'1), (d2) and (d3);
- *metric space* if it satisfies (d1), (d2) and (d3);
- *quasi-metric space* if it satisfies (d'2) and (d3).

Then, by refering to the usual definition of metric space, the word "*generalized*" refers to the lack of the symmetric property. The word "*pseudo*" refers to the lack of the axiom

$d(x, y) = 0$ □ $x = y$. The word "*quasi*" refers to condition (d'2) we write instead of (d1) in the case in which the symmetric property is not required. Finally, the word "*ultra*" refers to the fact that (d3) is substituted by condition (d'3). Moreover, observe that (d'3) entails (d3). So, for example, any ultrametric space is a metric space.

The following proposition, whose proof is immediate, extends to fuzzy orders and quasi-ultrametric distances a connection between similarities and metrics proved in [11].

Proposition 3.1 *Let* □ *be the t- norm of the minimum, d: M % M* $\rightarrow$ [0,1] *a map, and set*

$$R(x, y) = 1 - d(x, y) \qquad \ldots\, x, y\ \chi\ M.$$

Then: (i) R is a similarity if and only if d is an ultrapseudometrics;

(ii) R is a fuzzy preorder if and only if d is a generalized ultrametrics;

(iii) R is a fuzzy order if and only if d is a quasi-ultrametrics.

Example. Recall that a *preorder* is a pair $(M, \leq)$ satisfying, for all p, q and r in M, $p \leq p$, and if $p \leq q$ and $q \leq r$ then $p \leq r$. A *partial order* is a preorder that moreover is antisymmetric, i.e. $p \leq q$ and $q \leq p$ □ $p = q$. Since any crisp preorder is a fuzzy preorder, by Proposition 3.1 we have that any preorder $(M, \leq)$ can be viewed as a generalized ultrametric space. Namely, the map d defined by setting

$$d(p, q) = \begin{cases} 0 & \text{if } p \leq q \\ 1 & \text{if } p - q. \end{cases}$$

is a generalized ultrametric (see also [1], [8]). In the same way, any partial order defines a quasi-ultrametric space.

As in the case of the fuzzy relations, if d is a generalized metric (ultrametric) space, the position

$$x□y \qquad □ \qquad d(x, y) = 0 \text{ and } d(y, x) = 0$$

defines an equivalence's relation. Then it is possible to divide the space into the classes $[x] = \{y\ \chi\ M \,/\, d(x, y) = d(y, x) = 0\}$. Moreover, it is immediate to prove that the mapping

$$d'\colon M/□ \,\%\, M/□ \rightarrow [0,1] \ \text{ such that } \ d'([x], [y]) = d(x, y)$$

is well defined and is a generalized metric (ultrametric) distance satisfying (d1) on the space of equivalence's classes. By this identification is possible to change from a pseudometric structure to a metric one. So, from Proposition 3.1 it follows that R' is a similarity if and only if d' is an ultrametric distance.

It is easy to extend to generalized metrics and □-fuzzy preorders the relation between pseudometrics and □-similarity exposed in [11].

Definition 3.2 Let □ be a continuous Archimedean t-norm; a continuous strictly decreasing function f: $[0, 1] \rightarrow [0, +\infty]$ with $f(1) = 0$ is called an *additive generator of* □ if

$$x □ y = f^{[-1]}(f(x) + f(y)) \qquad \text{for all } x, y \text{ in } [0, 1],$$

where the *pseudoinverse* $f^{[-1]}$ of f is so defined:

$$f^{[-1]}(x) = \begin{cases} f^{-1}(x) & \text{if } x\ \chi\ f([0, 1]) \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.3 *Let d: $M \% M \to [0,1]$ be a map and f: $[0, 1] \to [0, +\infty]$ be an additive generator of a continuous Archimedean t- norm $\square$. Consider the relation R_d defined by setting*

$$R_d(x, y) = f^{[-1]}(d(x,y)).$$

Then: (i) R_d is a $\square$-similarity if d is a pseudometrics;
(ii) R_d is a $\square$-fuzzy preorder if d is a generalized metrics;
(iii) R_d is a $\square$-fuzzy order if d is a quasi-metrics.

Conversely, let R: $M \% M \to [0,1]$ be a map, $\square$ be a continuous Archimedean t-norm, and f be an additive generator of $\square$. Consider the function d_R defined by setting

$$d_R(x, y) = f(R(x, y))$$

Then: (i') d_R is a pseudometrics if R is a $\square$-similarity;
(ii') d_R is a generalized metrics if R is a $\square$-fuzzy preorder;
(iiii) d_R is a quasi-metrics if R is a $\square$-fuzzy order.

Proof. (*i*) Reflexivity and simmetry of R_d follows immediatly from definitions. Moreover, to prove the $\square$-transitivity it is enough to take x, y, z in M such that $d(x,y)$ and $d(y,z)$ χ $f([0, 1])$. In the opposite case, the inequality (3) is trivially verified. So,

$$\begin{aligned} R_d(x, y) \square R_d(y, z) &= f^{-1}(d(x,y)) \square f^{-1}(d(y,z)) = \\ &= f^{[-1]}(f(f^{-1}(d(x,y))) + f(f^{-1}(d(y,z)))) = \\ &= f^{[-1]}(d(x,y) + d(y,z)) \leq f^{[-1]}(d(x, z)) = R_d(x, z), \end{aligned}$$

because $f^{[-1]}$ is strictly decreasing.
(*ii*) The proof is analogue to the previous one.
(*iii*) We have to prove the antisymmetry of R_d. So, let x, y χ M such that $R_d(x, y) = 1$ and $R_d(y, x) = 1$. From this conditions follows that $f^{[-1]}(d(x, y)) = 1 = f^{[-1]}(d(y, x))$, and therefore $f^{-1}(d(x,y)) = 1 = f^{-1}(d(y,x))$. Then $d(x,y) = 0 = d(y,x)$ and so $x = y$ for the antisymmetry of d.
(*i'*) Reflexivity and simmetry of d_R follows immediatly from definitions. Before to prove the transitivity of d_R, we observe that

$$f(f^{[-1]}(x)) = \begin{cases} x & \text{if } x \ \chi \ f([0, 1]) \\ f(0) & \text{otherwise} \end{cases}$$

where $f(0)$ is the maximum of the function.
From $\square$-transitivity of R_d it follows that

$$f(R(x, y) \square R(y, z)) \geq f(R(x, z)),$$

because f is strictly decreasing. Then

$$f[f^{[-1]}(f(R(x, y) \square f(R(y, z)))] \geq f(R(x, z)).$$

Now, if $f(R(x, y)) + f(R(y, z))$ χ $f([0, 1])$, we obtain that

$$f(R(x, y)) + f(R(y, z)) \geq f(R(x, z),$$

and then the thesis. Otherwise, $f(R(x, y)) + f(R(y, z)) \geq f(0) \geq f(R(x, z)$.
(*ii'*) Proof is analogue to the previous one.
(*iii'*) We have to prove the antisymmetry of d_R. So, let x, y χ M such that $d_R(x, y) = 0$ and $d_R(y, x) = 0$. Then $f(R(x, y)) = 0 = f(R(y, x))$, hence $R(x, y) = 1 = R(y, x))$ because f is injective. From the antisymmetry of R it follows that $x = y$. $\square$

4. Fixed point theorem and fuzzy order

A sequence $(x_n)_{n□□}$ in M is called *forward Cauchy* if, for each $0 \leq \varepsilon < 1$, there exists a natural number N such that $R(x_n, x_m) \geq \varepsilon$ whenever $m \geq n \geq N$.

Notice that, when R is a fuzzy order, a sequence is forward Cauchy if and only if for each $1 > \varepsilon > 0$, there exists a natural number N such that $R(x_{n+1}, x_n) \geq \varepsilon$ for all $n \geq N$.

Furthermore, a sequence forward Cauchy $(x_n)_{n□□}$ *converges to* ℓ in M, and we write $\lim x_n = \ell$, if $R(\ell, x) = \lim R(x_n, x)$ for all x χ M. As usual, ℓ is called *limit* of $(x_n)_{n□□}$. Since the proposed convergence depends on the convergence defined in □, it inherits lots of properties of the convergence in □. As an example, if $(x_n)_{n□□}$ is a sequence forward Cauchy and ℓ is a limit of the sequence, an extract sequence of $(x_i)_{i□□}$ converges to the same limit ℓ.

The structure (M, R) is called *complete* if every forward Cauchy sequence is equipped with a limit.

Proposition 4.1 *Let R be a □-fuzzy preorder. Then, two limits of a given sequence are similar. Also, if R is a □-fuzzy order, then limits are unique.*

Proof. Assume that $\lim x_n = \ell$ and $\lim x_n = \ell'$. Then, by definition, $R(\ell, x) = \lim R(x_n, x)$ and $R(\ell', x) = \lim R(x_n, x)$ for all x χ M. In particular, by setting $x = \ell$, $1 = (\ell, \ell) = \lim R(x_n, \ell) =$ $= R(\ell', \ell)$ and, by setting $x = \ell'$, $1 = R(\ell', \ell') = \lim R(x_n, \ell') = R(\ell, \ell')$. Then $1 = R(\ell', \ell) = R(\ell, \ell')$ and ℓ is similar with ℓ'. Trivially, when R is a □-fuzzy order, limits are unique. □

For instance, assume that R is a partial order $\leq$; so a sequence $(x_n)_{n□□}$ is forward Cauchy if and only if □N …$n \geq N$, $x_n \leq x_{n+1}$, i.e. if and only if is "eventually chain". Moreover, the statement $\lim x_n = \ell$ is equivalent to

…x χ M ($\ell \leq x$ □ □m …$n \geq m$, $x_n \leq x$).

In particular, if x_n is order-preserving, then $\lim x_n = \ell$ if and only if $\ell = \mathrm{Sup}\{x_n / n$ χ $N\}$.

Let M be a set, R be a □-fuzzy relation, and f: $M \to M$ be a mapping. The following definitions are the duals of well-known notions in metric space theory.

Definition 4.2 Let R be a □-fuzzy preorder; f is called *continuous* if from $\lim x_n = \ell$ it follows $\lim f(x_n) = f(\ell)$, for every forward Cauchy sequence $(x_n)_{n□□}$ in M.

Obviously, when M is a partial ordered set, f is continuous if and only if it preserves upperbounds of chain.

Definition 4.3 We say that f is *non-expansive* if $R(f(x), f(y)) \geq R(x, y)$ for all x, y χ M, i.e. if the following formula …x …y $(r(x, y) \to r(f(x), f(y)))$ holds.

Observe that if R is a □-similarity, then a non-expansive map is a function "compatible" with the □-similarity R. If R is a □-fuzzy preorder, then a non-expansive map is in a sense a order-preserving map.

Definition 4.4 We say that f is *contractive* if $c > 1$ exists such that $(R(f(x), f(y)))^c \geq R(x, y)$, i.e. the formula …$x$ …y $(r(x, y) \to m(r(f(x), f(y))))$ holds, where m is a linguistic modifier "*much*".

In other terms, a contraction is a map that increases the similarity-degree between elements.

Theorem 4.5 *Let R be a □-fuzzy preorder such that (M, R) is complete and let f: M → M be a non-expansive continuous map such that R(x, f(x)) = 1 for a suitable x χ M. Then f has a fixed point.*

Proof. Consider the sequence $(x, f(x), f^2(x),....)$. Since f is non-expansive, such sequence is forward Cauchy. Indeed, trivially we have that

$$1 = R(x, f(x)) \leq R(f(x), f^2(x)) \leq ... \leq R(f^n(x), f^{n+1}(x)),$$

and so $R(f^n(x), f^m(x)) = 1$ for each $m \geq n$. Moreover, from completeness it follows that there is a limit ℓ of the sequence $(f^n(x))_{n\square\square}$. Also, since f is continuous, we have that $f(\ell)$ is a limit of $(f^{n+1}(x))_{n\square\square}$, and therefore of $(f^n(x))_{n\square\square}$. Since R is a □-fuzzy preorder, limits are similar, i.e. $R(f(\ell), \ell) = 1 = R(\ell, f(\ell))$. Then ℓ is a fixed point for f. □

Theorem 4.6 *Assume that □ is a t-norm greater or equal to the usual product, R is a □-fuzzy order and f is continuous and contractive. Then f has a unique fixed point.*

Proof. It is enough to prove the thesis for the t-norm of the product ∃. Indeed, if □ is a t-norm greater or equal to the product, then the □-transitivity implies the ∃-transitivity. Therefore, any □-fuzzy order is a ∃-fuzzy order. Let x_0 be an element of M, and consider the sequence $(x_n)_{n\square\square}$ defined as follows: $x_1 = f(x_0)$, $x_2 = f(x_1)$, ... $x_{n+1} = f(x_n)$. We have to prove that the sequence is forward Cauchy. Observe by hypotheses that c χ]0, 1[exists such that:

$$R(x_1, x_2) = R(f(x_0), f(x_1)) \geq (R(x_0, x_1))^c = (R(x_0, f(x_0)))^c$$

$$R(x_2, x_3) = R(f(x_1), f(x_2)) \geq (R(x_1, x_2))^c \geq (R(x_0, x_1))^{c^2} = (R(x_0, f(x_0)))^{c^2}$$

$$...$$

$$R(x_n, x_{n+1}) = R(f(x_{n-1}), f(x_n)) \geq (R(x_{n-1}, x_n))^c \geq (R(x_0, x_1))^{c^n} = (R(x_0, f(x_0)))^{c^n}$$

So, if $0 \leq \varepsilon < 1$, we have that $R(x_n, x_{n+r}) \geq R(x_n, x_{n+1})$ ∃ $R(x_{n+1}, x_{n+2})$ ∃ ...... ∃ $R(x_{n+r-1}, x_{n+r}) \geq$ $\geq (R(x_0, f(x_0)))^{c^n}$ ∃ $(R(x_0, f(x_0)))^{c^{n+1}}$ ∃........∃ $(R(x_0, f(x_0)))^{c^{n+r-1}}$ $= (R(x_0, f(x_0)))^{c^n + c^{n+1} + ... + c^{n+r-1}}$.

Observe that $c^n + c^{n+1} + ... + c^{n+r-1} = \dfrac{c^n - c^{n+r}}{1-c} = \dfrac{c^n(1-c^r)}{1-c} \leq \dfrac{c^n}{1-c}$. Since $c^n \to 0$ if $n \to \infty$, we have that $\lim(c^n + c^{n+1} + ... + c^{n+r-1}) = 0$ for all r χ □, provided that n is enough large. Therefore $\lim(R(x_0, f(x_0)))^{c^n + c^{n+1} + ... + c^{n+r-1}} = 1$, and then $(x_n)_{n\square\square}$ is forward Cauchy. From completeness it follows that there is a limit ℓ of the sequence. Because of f is continuous, $f(\ell) = \lim f(x_n) = \lim x_{n+1} = \ell$, and then ℓ is a fixed point for f. Suppose ℓ_1 is another fixed point. Then $R(\ell, \ell_1) = R(f(\ell), f(\ell_1)) \geq (R(\ell, \ell_1))^c$, hence necessarily $R(\ell, \ell_1) = 1$. In the same way we have $R(\ell_1, \ell) = 1$, so from antisymmetry it follows $\ell = \ell_1$. □

Note. In particular, Theorem 4.6 is true for the t-norm of the minimum. Indeed, it is immediate to prove that the minimum is the maximum t-norm.

5. Examples

In order to apply the proposed notions to programming logic, we need to have metric and quasi-metrics defined in a power set $\mathtt{P}(M)$. In this section we will show some interesting examples in such a direction. In accordance with the nomenclature in fuzzy set theory, we call *fuzzy inclusion* a fuzzy order in $\mathtt{P}(M)$ extending the classical crisp order. In the following λ: $M \rightarrow [0,1]$ is any fixed fuzzy subset of M we interpret as the fuzzy subset of elements which are "*relevant*".

We define the map μ: $\mathtt{P}(M) \rightarrow [0,1]$ such that $\mu(\Leftrightarrow) = 0$ and, if $\Leftrightarrow \neq X \ \bar{} M$,

$$\mu(X) = \mathrm{Sup}\{\lambda(x) \,/\, x \chi X\}. \tag{5.1}$$

We interpret $\mu(X)$ as the truth-degree of the claim "*there is a relevant element in X*". Moreover, we associate with μ the map d_λ: $\mathtt{P}(M) \times \mathtt{P}(M) \rightarrow [0,1]$ as follows:

$$d_\lambda(X, Y) = \mu(X \Delta Y), \tag{5.2}$$

where $X \Delta Y$ denotes the symmetric difference of X, Y, i.e. $X \Delta Y = (X / Y)\ 4\ (Y / X)$.

Proposition 5.1 *The map* d_λ: $\mathtt{P}(M) \times \mathtt{P}(M) \rightarrow [0,1]$ *defined by* (5.2) *is an ultrametric distance*.

Proof. (d1) and (d2) follow from the definition; to prove condition (d'3), observe that $X \Delta Z \ \bar{} ((X \Delta Y)\ 4\ (Y \Delta Z))$. As a matter of fact, if $x \chi X / Z$, then, in the case $x \chi Y$ we have $x \chi Y \Delta Z$, in the case $x \varpi Y$ we have $x \chi X \Delta Y$. If $x \chi Z / X$, we proceed in a similar way. Obviously,

$$\sup\{\lambda(x) \,/\, x \chi X \Delta Z\} \leq \sup\{\lambda(x) \,/\, x \chi (X \Delta Y)\ 4\ (Y \Delta Z)\} =$$
$$= \sup\{\lambda(x) \,/\, x \chi X \Delta Y\} \ \square\ \sup\{\lambda(x) \,/\, x \chi Y \Delta Z\},$$

so (d'3) is verified. $\square$

We interpret $d_\lambda(X, Y)$ as the truth-degree of the claim "*X and Y differs for a relevant element*". The associated similarity $R_\lambda(X, Y) = 1 - d_\lambda(X, Y)$ is the truth-degree of the claim "*X and Y contains the same relevant elements*"

Proposition 5.2 *Let* μ *be the defined possibility by* (5.1) *and* d: $\mathtt{P}(M) \times \mathtt{P}(M) \rightarrow [0,1]$ *be a mapping such that*

$$d(X, Y) = \mu(X - Y). \tag{5.3}$$

Then, d is a quasi-ultrametrics (*and not an ultrametrics, since it is lacking in symmetry*).

Proof. Actually, (d'1) and (d'2) follow from the definition, and the proof of (d'3) is as the one in proposition 5.1. $\square$

Assume that λ: $M \rightarrow [0,1]$ satisfies $\sum_{x \in M} \lambda(x) = 1$. Then we can define the mapping η: $\mathtt{P}(M) \rightarrow [0,1]$ such that $\eta(\Leftrightarrow) = 0$ and, if $\Leftrightarrow \neq X \ \bar{} M$,

$$\eta(X) = \sum\{\lambda(x) \,/\, x \chi X\}. \tag{5.4}$$

This map is called a *finitely additive probability with density* λ. Moreover, we associate with η the map d_λ: $\mathtt{P}(M) \times \mathtt{P}(M) \rightarrow [0,1]$ as follows:

$$d^\lambda(X, Y) = \eta(X \Delta Y). \tag{5.5}$$

Proposition 5.3 η *is an ultrametric distance.*

Proposition 5.4 *The map* d: P(M) × P(M) → [0,1] *such that*

$$d_\lambda(X, Y) = \inf\{\alpha \ \chi \ [0, 1] \ / \ X \ \rho \ C(\lambda, \alpha) \ 3 \ Y\} \tag{5.6}$$

is a quasi-ultrametric distance.

Proof. (d'1) and (d'2) are obvious. To prove that $d_\lambda(x, y)$ □ $d_\lambda(y, z) \geq d_\lambda(x, z)$ it is enough to prove this condition:

if □ $\delta > 0$ such that $d_\lambda(x, y) < \delta$ and $d_\lambda(y, z) < \delta$ then $d_\lambda(x, z) \leq \delta$.

In fact, $d_\lambda(x, y)$ □ $d_\lambda(y, z) = \inf\{\delta \ / \ d_\lambda(x, y) < \delta$ and $d_\lambda(y, z) < \delta\} \geq \inf\{\delta \ / \ d_\lambda(x, z) \leq \delta\}$. So, from $d_\lambda(x, y) < \delta$ it follows that exists α_1 χ [0, 1] such that $\alpha_1 < \delta$ and X ρ $C(\lambda, \alpha_1)$ 3 Y, and from $d_\lambda(y, z) < \delta$ it follows that exists α_2 χ [0, 1] such that $\alpha_2 < \delta$ and Y ρ $C(\lambda, \alpha_2)$ 3 Z. Taking a α = α_1 □ α_2, we have that X ρ $C(\lambda, \alpha)$ 3 Z with $\alpha < \delta$; then (d'3) is verified. □

Observe that such definition extends one given in a Seda's framework of programming logic ([9]). Indeed, given a map n: M → □, consider for every subset X of M the finite set

$$I(X, \alpha) = \{x \ \chi \ X \ / \ n(x) \leq \alpha\}.$$

Then, the map d: P(M) × P(M) → [0,1] such that

$$d(X, Y) = \inf\{2^{-\alpha} \ / \ I(X, \alpha) \ \rho \ I(Y, \alpha)\} \tag{5.7}$$

is the quasi-ultrametric distance defined in [9]. Also, if we consider the fuzzy set λ:M → [0, 1] such that $\lambda(x) = 2^{-n(x)}$, we have $d_\lambda(X, Y) = d(X, Y)$. Actually,

$$d_\lambda(X, Y) = \inf\{\alpha \ \chi \ [0, 1] \ / \ X \ \rho \ \{x \ / \ 2^{-n(x)} \geq \alpha\} \ 3 \ Y\} =$$
$$= \inf\{\alpha \ \chi \ [0, 1] \ / \ X \ \rho \ \{x \ / \ log_2 2^{-n(x)} \geq log_2(\alpha)\} \ 3 \ Y\} =$$
$$= \inf\{\alpha \ \chi \ [0, 1] \ / \ X \ \rho \ \{x \ / \ n(x) \leq -log_2(\alpha)\} \ 3 \ Y\} =$$
$$= \inf\{2^{-\alpha} \ / \ X \ \rho \ \{x \ / \ n(x) \leq \alpha\} \ 3 \ Y\} = d(X, Y).$$

Let P be a non-positive program and B_L be the Herbrand base of P. To apply Theorems 4.5 and 4.6 to the immediate consequence operator T: P(B_L) → P(B_L), we can introduce the distance defined in Proposition 5.4. So we consider the quasi-ultrametric d: P(B_L) × P(B_L) → [0,1] such that

$$d(X, Y) = \inf\{2^{-\alpha} \ / \ I(X, \alpha) \ \rho \ I(Y, \alpha)\},$$

that makes the structure (P(B_L), d) complete (see [8]). Therefore, from Proposition 3.1 it follows that the map R: P(B_L) × P(B_L) → [0,1] such that

$$R(X, Y) = 1 - d(X, Y)$$

is a fuzzy order, and (P(B_L), R) is complete. So it is possible to prove the theorem which guarantees the existence of the least Herbrand model of a non-positive program, under certain hypotheses on the immediate consequence operator.

Theorem 5.6 *If the immediate consequence operator* T: P(B_L) → P(B_L) *is non-expansive and continuous, and if a subset* X *exists in* B_L *such that* $R(X, T(X)) = 1$, *then* T *has a fixed point. If* T *is continuous and contractive, then the fixed point is unique.*

Proof. It follows from Theorem 4.5 and Theorem 4.6. □

References

[1] M.M. BONSANGUE, F. VAN BREUGEL, J.J.M.M. RUTTEN: *Generalized Ultrametric Spaces: Completion, Topology, and Powerdomains via the Yoneda Embedding*, Technical Report CS-R9560, Centrum voor Wiskunde en Informatica, Amsterdam, September, (1995). [Available via FTP at **ftp.cwi.nl** in directory pub/CWIreports/AP/CS-R9560.ps.gz].

[2] M. FITTING: *Metric Methods: Three Examples and a Theorem*, J. Logic Programming **21** (3) (1994) 113-127.

[3] F. FORMATO: *On Similarity and Its Application to Logic Programming*, Ph. D. Thesis, Università di Napoli Federico II, (1988).

[4] G. GERLA: *Fuzzy Logic: Mathematical Tools for Approximate Reasoning*, Kluwer Ac. Publ., (2001).

[5] P. HAJEK: *Metamathematics of Fuzzy Logic*, Kluwer Ac. Publ., Dordrecht, (1998).

[6] J.W. LLOYD: *Foundations of Logic Programming*, Springer-Verlag, Berlin, (1984).

[7] S. PRIESS-CRAMPE, P.RIBENBOIM: *Logic Programming and Ultrametric Spaces*, Rend. Mat., Serie VII, **19**, Roma (1999), 155-176.

[8] J.J.M.M. RUTTEN: *Elements of Generalized Ultrametric Domain Theory*, Technical Report CS-R9507, Centrum voor Wiskunde en Informatica, Amsterdam, (1995). [Available via FTP at **ftp.cwi.nl** in directory pub/CWIreports/AP/CS-R9507.ps.gz].

[9] A.K. SEDA: *Quasi-metrics and the Semantics of Logic Programs*, Fundamenta Informaticae **29** (2) (1997), 97-117.

[10] M.B. SMITH: *Quasi Uniformities: Reconciling Domains with Metric Spaces.* In: M. Main, A. Melton, M. Mislove and D. Schmidt (Eds.), Mathematical Foundations of Programming Language Semantics. Lecture Notes in Computer Science, Vol. 298, Springer-Verlag, (1987), 236-253.

[11] L. VALVERDE: *On the Structure of F-Indistinguishability Operators.* Fuzzy Sets and Systems, **17** (1985), 313-328.

[12] R. WÓJCICKI: *Theory of Logical Calculi: Basic Theory of Consequence Operations*, Kluwer Ac. Publ., Dordrecht, (1988).

[13] L.A. ZADEH: *The Concept of a Linguistic Variable and its Application to Approximate Reasoning*, Part II, Information Science **8** (1975), 301-357.