Mathematical features of Whitehead's point-free geometry

In *HANDBOOK OF WHITEHEADIAN PROCESS THOUGHT pp. 507-19 & pp. 533-36, Michel Weber, Will Desmond, eds., Ontos Verlag, 2008*

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This paper is devoted to some mathematical considerations on the geometrical ideas contained in *PNK*, *CN* and, successively, in *PR*. Mainly, we will emphasize that these ideas give very promising suggestions for a modern point-free foundation of geometry.

1. Introduction

Recently the researches in point-free geometry received an increasing interest in different areas. As an example, we can quote computability theory, lattice theory, computer science. Now, the basic ideas of point-free geometry were firstly formulated by A. N. Whitehead in *PNK* and *CN* where the *extension relation* between events is proposed as a primitive. The points, the lines and all the "abstract" geometrical entities are defined by suitable abstraction processes. As a matter of fact, as observed in Casati and Varzi 1997, the approach proposed in these books is a basis for a "*mereology*" (i.e. an investigation about the *part-whole* relation) rather than for a point-free geometry. Indeed, the inclusion relation is set-theoretical and not topological in nature and this generates several difficulties. As an example, the definition of point is unsatisfactory (see Section 6). So, it is not surprising that some years later the publication of *PNK* and *CN*, Whitehead in *PR* proposed a different approach in which the primitive notion is the one of *connection relation*. This idea was suggested in de Laguna 1922.

The aim of this paper is not to give a precise account of geometrical ideas contained in these books but only to emphasize their mathematical potentialities. So, we translate the analysis of Whitehead into suitable first order theories and we examine these theories from a logical point of view. Also, we argue that multi-valued logic is a promising tool to reformulate the approach in *PNK* and *CN*.

In the following we refer to first order logic. If L is a first order language, α a formula whose free variables are among $x_1,\dots,x_n$ and I an interpretation of L with domain S, then we write $I \vDash \alpha\ [d_1,\dots,d_n]$ to say α is satisfied in I by the elements $d_1,...,d_n$. Given a relation $R \subseteq S^n$, we say that R is *defined by* α or that R is *the extension of* α in I, provided that $R = \{(d_1,...,d_n) : I \vDash \alpha\ [d_1,...,d_n]\}$. For example, the extension of the formula $\exists r(r \leq x \wedge r \leq y)$ is the overlapping relation.

2. A mathematical formulation of the inclusion based approach

In *PNK* and *CN* one considers as primitives the *events* and a binary relation named *extension.* Indeed, Whitehead says:

"The fact that event a extends over event b will be expressed by the abbreviation aKb. Thus 'K' is to be read 'extends over' and is the symbol for the fundamental relation of extension."

Moreover, Whitehead in *PNK* lists the following properties of the extension relation.

"Some properties of K essential for the method of extensive abstraction are,
i) aKb implies that a is distinct from b, namely, 'part' here means 'proper part':
ii) Every event extends over other events and is itself part of other events: the set of events which an event e extends over is called the set of parts of e:
iii) If the parts of b are also parts of a and a and b are distinct, then aKb:
iv) The relation K is transitive, i.e. if aKb and bKc, then aKc :
v) If aKc, there are events such as b where aKb and bKc:
vi) If a and b are any two events, there are events such as e where eKa and eKb. "

We adopt a slightly different notation which is related in a more strict way with the recent researches in point-free geometry. So, in accordance with *PR*, we utilize the word *"region"* instead of the one of *"event"*. Also, we call *inclusion relation* the converse of the extension relation and we refer to the partial order $\leq$ rather than to the strict partial order. In accordance, we can reformulate the list of properties proposed by Whitehead into a simple first order theory whose language $L_{\leq}$ contains only a binary relation symbol $\leq$.

Definition 2.1. We call *inclusion based point-free geometry* the first order theory defined by the following axioms:

(*i*) $\forall x(x \leq x)$ (reflexive)
(*ii*) $\forall x \forall y \forall z\ ((x \leq z \wedge z \leq y) \Rightarrow x \leq y)$ (transitive)
(*iii*) $\forall x \forall xy(x \leq y \wedge y \leq x \Rightarrow x = y)$ (anti-simmetric)
(*iv*) $\forall z \exists x \exists y(x < z < y)$ (there is no minimal or maximal region)
(*v*) $\forall x \forall y(x < y \Rightarrow \exists z\ (x < z < y)$ (dense)
(*vi*) $\forall x \forall y \exists z(x \leq z \wedge y \leq z)$ (upward-directed)
(*vii*) $\forall x \forall y(\forall x'(x' < x \Rightarrow x' < y) \Rightarrow x \leq y)$.

We call *inclusion space* any model of this theory.

Then, if we denote by $\leq$ the interpretation of $\leq$, an inclusion space is a structure $(S,\leq)$ where S is a nonempty set and $\leq$ an order relation with no minimal or maximal element which is dense, upward-directed and such that, for every region x, $x = sup\{x' \in S : x' < x\}$.

The existence of suitable mathematical models for the inclusion-based point-free geometry is a basic question, obviously (in spite of the fact that Whitehead looks understimate it). To argue how this should be done, recall that, in the case of non-Euclidean geometry, Poincaré, Klein and others authors proposed models which were defined from the usual Euclidean spaces. In the same way, we are justified in defining models of point-free geometry by starting from a n-dimensional (point-based) Euclidean space R^n (where R denotes the real numbers set). In accordance, we have to propose a suitable class of subsets of R^n to represent the notion of region. To do this, usually in literature one refers to the regular subsets of R^n.

Definition 2.2. We call *closed regular* any subset x of R^n such that $x = cl(int(x))$ where *cl* and *int* denotes the closure and interior operators, respectively. We denote by $RC(R^n)$ the class of all the closed regular subsets of R^n.

There are several reasons in favour of the notion of regular set. As an example, in accordance with our intuition, the closed balls and the cubes of tridimensional geometry are regular sets. Instead, points, lines and surfaces are not regular (in accordance with Whitehead's aim to define these geometrical notions by abstraction processes). Moreover the class $RC(R^n)$ defines a very elegant algebraic structure. Indeed it is a complete atom-free Boolean algebra with respect to the inclusion relation. More precisely, to obtain a model of Whitehead's axioms we have to refer to a suitable subclass of $RC(R^n)$. In fact, Axiom *iv*) says that the whole space R^n and the empty set $\varnothing$ are not seen as regions by Whitehead. Also, it is evident that Whitehead refers only to bounded regions. Otherwise, for example, the proposed notion of "point" (see Section 4) should be not able to exclude points which are "at the infinity". This leads to assume as natural models of the notion of region the bounded nonempty closed regular sets.

Theorem 2.3. Let *Re* be the class of all closed nonempty bounded regular subsets of R^n, then $(Re,\subseteq)$ is an inclusion space we call *the canonical inclusion space*.

Proof. Properties (i), (ii), (iii) and (vi) are trivial. To prove (iv), let $x \in Re$, then, since $int(x) \neq \emptyset$, an open ball with radius r and centre P contained in x exists. We denote by x_1 the closed ball with centre P and radius $r/2$. Then $x_1 \in Re$ and $x_1 \subset x$. Also, since x is bounded, a closed ball with centre P and radius r' containing x exists. Let x_2 the closed ball with centre P and radius $2 \cdot r'$. Then $x_2 \in Re$ and $x \subset x_2$.

To prove (v), assume that $x \subset y$ and let $P \in int(y)$ such that $P \notin x$. Then there is a closed ball y', $y' \subset y$ and $y' \cap x = \emptyset$. By setting $z = y' \cup x$ we obtain a bounded regular subset such that $x \subset z \subset y$. To prove *(vii),* assume that all the regular proper subsets of x are contained in y and that x is not contained in y. Then $int(x)$ is not contained in y, too. So there is a point $P \in int(x)$ and a real positive number r such that the closure of the ball with centre P and radius r is contained in $int(x)$ and disjoint from y: a contradiction.

Notice that in literature inclusion spaces are obtained also from the class $RO(R^n)$ of *open regular subsets,* i.e. the sets x such that $x = int(cl(x))$ (see for example Pratt 2006). The arising model is isomorphic with $(Re, \subseteq)$.

Proposition 2.4. Denote by Re' the class of bounded nonempty open regular subsets. Then the structure $(Re', \subseteq)$ is isomorphic with the structure $(Re, \subseteq)$ and therefore is an inclusion space.

Proof. We observe only that the map $int : RC(R^n) \rightarrow RO(R^n)$ is an order-isomorphism between $(RC(R^n), \subseteq)$ and $(RO(R^n), \subseteq)$. Moreover, x is bounded if and only if $int(x)$ is bounded.

Finally, observe that perhaps the class Re is still too large. Indeed Whitehead's intuition refers to the *connected* (in a topological sense) elements in Re. Also, for example, in Pratt 2006 several possible subsets of Re are considered.

3. A mathematical formulation of the connection based approach

Some years later the publication of *PNK*, and *CN*, Whitehead in *PR* proposed a different idea in which the primitive notion is the one of *connection relation*:

"...the terms 'connection' and 'connected' will be used ...The term 'region' will be used for the relata which are involved in the scheme of 'extensive connection'. Thus, in the shortened phraseology, regions are the things which are connected." (PR, Chapter II, Section I)

Now, Whitehead is not interested to formulate the properties of this relation as a system of axioms and to reduce them at a logical minimum.

> *"No attempt will be made to reduce these enumerated characteristic to a logical minimum from which the remainder can be deduced by strict deduction. There is not a unique set of logical minima from which the rest can be deduced. There are many such sets. The investigation of such sets has great logical interest, and has an importance which extends beyond logic. But it is irrelevant for the purpose of this discussion." (PR, Chapter II, Section I)*

So a very long list of "assumptions" is proposed. As an example in Chapter II Whitehead exposed as much as 31 assumptions! Nevertheless, it is possible to try to reduce these assumption into a sufficiently small set of axioms. As an example, in Gerla and Tortora 1992 one proves that the first 12 assumptions are equivalent to the following first order theory. We refer to a language L_C with a binary relation symbol C to represent the connection relation and we write $x \leq y$ to denote the formula $\forall z(zCx \Rightarrow zCy)$ and $x<y$ to denote the formula $(x\leq y)\wedge(x\neq y)$.

Definition 3.1. We call *connection theory* the first order theory whose axioms are:

$C1$ $\forall x\forall y(xCy \Rightarrow yCx)$ (symmetry)
$C2$ $\forall x\exists y(x<y)$ (there is no maximum for $\leq$)
$C3$ $\forall x\forall y\exists z(zCx\wedge zCy)$
$C4$ $\forall x(xCx)$
$C5$ $\forall z(zCx \Leftrightarrow zCy) \Rightarrow x=y$
$C6$ $\forall z\exists x\exists y((x\leq z)\wedge(y\leq z)\wedge(\neg xCy))$

where $x\leq y$ denotes the formula $\forall z(zCx \Rightarrow zCy)$ and $x<y$ the formula $(x\leq y)\wedge(x \neq y)$. We call *connection space* any model of such a theory.

We denote by $\boldsymbol{C}$ the interpretation of the relation symbol C and by $\leq$ the interpretation of $\leq$. So we write $(S,\boldsymbol{C})$ to denote a connection space. It is easy to prove that in any connection space $\leq$ is an order relation. As in the case of inclusion spaces, we can define "canonical" connection spaces.

Theorem 3.2. (Gerla and Tortora 1996) Define in *Re* the relation $\boldsymbol{C}$ by setting

$$X\boldsymbol{C}Y \Leftrightarrow X\cap Y \neq \varnothing.$$

Then $(Re,\boldsymbol{C})$ is a connection space we call *canonical connection space*. Moreover, the associated order relation coincides with the usual set theoretical inclusion.

An analogous definition can be given by referring to the regular open sets and by putting XCY if and only if $cl(X)\cap cl(Y) \neq \emptyset$.

4. Abstractive classes

In order to define the points and the lines and other "abstract" entities, Whitehead in *PNK* considers the following basic notion.

Definition 4.1. Given an inclusion space, we call *abstractive class* any totally ordered class G of regions such that
i) G is totally ordered
ii) no region exists which is contained in all the regions in G.
We denote by AC the set of abstractive classes.

The idea is that an abstractive class represents an "abstract object" which is the limit (the intersection, in a sense) of the regions in the abstractive class. Condition *ii*) means that these objects have a dimension lower than the one of the regions. As an example, in the canonical Euclidean plane an abstractive class is intended to represent either a point or a line. Now, it is possible that two different abstractive classes represent the same object. For example, let G_1 be the class of closed balls with centre in P and let G_2 be the class of closed squares with the same centre. Then while $G_1 \neq G_2$, our intuition says that both G_1 and G_2 represent P. To face such a question, we define a pre-order relation and a corresponding equivalence relation.

Definition 4.2. The *covering* relation $\leq_c$ is defined by setting, for any G_1 and G_2 in AC,

$$G_1 \leq_c G_2 \Leftrightarrow \forall x \in G_2\ \exists y \in G_1\ x > y.$$

The covering relation $\leq_c$ is a pre-order in AC, i.e. it is reflexive and transitive, and therefore it defines an equivalence.

Proposition 4.3. Define in AC the relation

$$G_1 \equiv G_2 \Leftrightarrow G_1 \leq_c G_2 \text{ and } G_2 \leq_c G_1.$$

Then $\equiv$ is an equivalence relation.

We can consider the quotient $AC/\equiv$ and the partial order relation $\leq_c$ in $AC/\equiv$ defined by setting, for every pair $[G_1]$, $[G_2]$ of elements in $AC/\equiv$,

$$[G_1] \leq_c [G_2] \Leftrightarrow G_1 \leq_c G_2.$$

At this point it is possible to give the notion of geometrical element. Since the definition in *PNK*, and *CN* is uselessly complicate, we refer to the equivalent definition adopted in *PR*.

Definition 4.4. We call *geometrical element* any element of the quotient $AC/\equiv$, i.e. any complete class of equivalence modulo $\equiv$. Also, we call *point* any geometrical element which is minimal in the ordered set $(AC/\equiv, \leq_c)$.

Analogous definitions can be given by referring to the connection spaces provided that we modify the notion of abstractive class by involving the topological notion of non-tangential inclusion.

Definition 4.5. Two connected regions are called *externally connected* if they do not overlap. A region y is *non-tangentially included* in a region x, if
(*j*) y is included in x,
(*jj*) no region exists which is externally connected with both x and y.

If we denote by xOy the formula $\exists r(r\leq x \wedge r\leq y)$ expressing the overlapping relation, we can represent the non-tangential inclusion in a very simple way.

Proposition 4.6. The non-tangential inclusion is the relation $\ll$ defined by the formula $\forall z(zCy \Rightarrow zOx)$.

Proof. We have to prove that, under the hypothesis $y\leq x$, the conditions
a) every region z which is externally connected with y is not externally connected with x,
b) if a region z is connected with y, then z overlaps x,
are equivalent. Assume *a*) and observe that, in account of the inclusion $y \leq x$, any region z which is connected with y is also connected with x. Assume that z is connected with y. Then if z overlaps y it is trivial that z overlaps x. Otherwise, z is externally connected with y and therefore, by *a*), it is not externally connected with x. So since z is connected with x, then z overlaps x.
Conversely, assume *b*). Then since z overlaps x entails that z is connected with x, $y\leq x$ by definition. Assume that z is externally connected with y. Then, since z is connected with y, it overlaps x. Thus, z is not externally connected with x.

The relation $\ll$ is on the basis of the notion of abstractive class.

Definition 4.7. An *abstractive class* is a set G of regions such that
j) G is totally ordered by the non-tangential inclusion,

jj) no region exists which is contained in all the regions in *G*.

The *geometrical elements* and, in particular, the *points* are defined as in Definition 4.4. The reference to the non-tangential inclusion will be motivated in Section 6.

5. Ovals to define geometrical notions

The question of defining the basic notion of straight segment arises. Now Whitehead in Chapter III of *PR* criticizes Euclid's definition "A straight line is any line which lies evenly with the points on itself" since "evenly" requires definition and since "nothing has been deduced from it". In alternative, a good definition "should be such that the uniqueness of the straight segment between two points can be deduced from it". In accordance, an attempt of giving an adequate definition in terms of the "extensive notions" is proposed. More precisely Whitehead assumes that in the space of the regions we can isolate a class of regions whose elements are called *ovals.* The underlying idea is perhaps that the ovals are suitable convex regions of an Euclidean space (a set x of points is *convex* if for every P and Q in x the segment PQ is contained in x). The interest of the convex sets lies in the fact that the straight segment PQ is the intersection of all the convex sets containing P and Q. Obviously, Whitehead lists suitable properties for the class of ovals. As an example "Any two overlapping regions of the ovate class have a unique intersect which also belongs to that ovate class". It is an open question to translate these properties into a suitable system of axioms. The following is a reformulation, in mathematical terms, of Whitehead's definition if straight segment.

Definition 5.1. We call *convex* a geometrical element represented by an abstractive class whose elements are ovals. The *straight segment* between two points P and Q is the convex geometrical element containing P and Q and which is minimal with respect to this condition.

Whitehead proves that there is one straight segment between two points. We conclude this section by emphasizing that Whitehead's addition of the notion of oval as a primitive to the one of connection is a necessary step from a mathematical point of view. In fact, taking in account of the topological nature of the notion of connection, we have that all the notions we can define in a canonical model are invariant with respect to the topological transformations. As a consequence, there is no possible definition of straight segment based on the connection relation.

6. Mathematical motivations for the passage from *PNK* and *CN* to *PR*

Surely there are philosophical motivations on the basis of Whitehead's passage from the inclusion-based approach proposed in the books *PNK* and *CN* to the connection-based approach proposed in *PR*. In such a section we will argue that, in any case, there are also mathematical reasons (we do not know whether Whitehead were completely aware of this or not). The first one is related with the definition of point. Indeed, consider in R^2 the abstractive classes G_{-0}, G_0 and G_{+0} defined by sequences of balls with radius $1/n$ and centre in $(-1/n,0)$, $(0,0)$ and $(1/n,0)$, respectively. Then, since G_{-0}, G_0 and G_{+0} are not equivalent, they represent different geometrical elements. As a matter of fact, the class G_0 covers both the classes G_{-0} and G_{+0}. So, since G_0 is not minimal, it cannot represent a point. Obviously should be intriguing to imagine an universe in which an Euclidean point as $P = (0,0)$ splits in three different "points" $P_{-0} = [G_{-0}]$, $P_0 = [G_0]$, $P_{+0} = [G_{+0}]$ (as a matter of fact into a cloud of infinite points). A similar phenomenon occurs in non-standard analysis. However, this is surely far from the aim of Whitehead.

Instead these difficulties do not occur in the case of the canonical connection spaces. In fact the sequences G_{-0} and G_{+0} (differently from G_0) are not abstractive classes since they are not ordered with respect to the non-tangential inclusion. As a matter of fact, we can prove the following proposition giving a strong reason in favour of the connection-based approach.

Proposition 6.1. Consider a canonical connection space $(Re,\mathbf{C})$ in an Euclidean space R^n. Then the points in $(Re,\mathbf{C})$ defined by the abstractive classes "coincide" with the usual points in R^n (i.e. with the elements of R^n).

Another reason is related with the strenght of the two approaches. Indeed, the following theorem holds true.

Theorem 6.2. It is not possible to define the connection relation in a canonical inclusion space $(Re,\subseteq)$. So, the connection-based approach is strictly more potent than the inclusion-based one.

Proof. Theorem 3.2 shows that in a canonical connection space the inclusion relation is definable by the formula $\forall z(zCx \Rightarrow zCy)$ involving only the connection relation. Then the connection-based approach is either equivalent or more potent than the inclusion-based one. Consider an automorphism $f : Re \to Re$, i.e. a map such that

$$d_1 \subseteq d_2 \Leftrightarrow f(d_1) \subseteq f(d_2).$$

Then from a general result in model theory we have that

$$(Re,\subseteq) \vDash \alpha\,[d_1,d_2] \quad \Leftrightarrow \quad (Re,\subseteq) \vDash \alpha\ [f(d_1),f(d_2)] \qquad (6.1)$$

for any formula α whose free variables x_1 and x_2 and for any d_1,d_2 in Re. In particular, if α is able to define the connection relation $\boldsymbol{C}$, then

$$d_1\ \boldsymbol{C}\ d_2 \Leftrightarrow f(d_1)\ \boldsymbol{C}\ f(d_2) \qquad (6.2)$$

for any automorphism f. Consider the case $n = 2$, set

$$r_y = \{(x,y)\in R^2 : x = 0\}\ ;\ P^< = \{(x,y)\in R^2 : x<0\}\ ;\ P^> = \{(x,y)\in R^2 : x>0\}$$

and define the map $g : R^2 \to R^2$ by setting

$$g((x,y)) = (x,y+1) \quad \text{if } x\in r_y\cup P^>$$
$$g((x,y)) = (x,y) \qquad \text{otherwise.}$$

We can visualize this map as a *cut* of the Euclidean plane along the y-axis r_y and a vertical translation of the half-plane $r_y\cup P^>$. Now, if $X \in Re$, then $g(X)$ is not regular, in general. Nevertheless, we have that $int(g(X)) \neq \emptyset$ and therefore that $reg(g(X))$ is a regular bounded non-empty subset of R^2. In fact, since $int(X) \neq \emptyset$, either $int(X)\cap P^> \neq \emptyset$ or $int(X)\cap P^< \neq \emptyset$ and therefore either $g(int(X)\cap P^>)$ or $g(int(X)\cap P^>)$ is a non-empty open set contained in $g(X)$. We claim that the map $f : Re \to Re$ defined by setting

$$f(X) = reg(g(X))$$

is an automorphism. In fact, it is evident that $X\subseteq Y$ entails $f(X) \subseteq f(Y)$. To prove the converse implication assume that $f(X) \subseteq f(Y)$ and, by absurdity, that X is not contained in Y. Then $int(X)$ is not contained in Y and a closed ball B exists such that $B \subseteq int(X)$ and $B\cap Y = \emptyset$. Also, it is not restrictive to assume that B is either completely contained in $P^>$ or completely contained in $P^<$ and therefore that $f(B) = g(B)$. Now, since g is injective and $B\cap Y = \emptyset$, we have that $g(B)\cap g(Y) = \emptyset$ and therefore $int(g(B))\cap g(Y) = \emptyset$. On the other hand

$$int(g(B)) \subseteq g(B) = f(B) \subseteq f(X) \subseteq f(Y) \subseteq r_y\cup g(Y).$$

Therefore, $int(g(B))\subseteq r_y$, an absurdity. This proves that f is an automorphism. On the other hand, for example, two closed balls which are tangent in the same point in r_y are connected while their images are not connected. This contradicts (6.2).

Note that analogous results were proved in a series of basic papers of I. Pratt. Anyway, in these papers Pratt one refers to a different notion of canonical space in which also unbounded regions are admitted (and this is far from Whitehead's ideas).

7. Multi-valued logic to reformulate Whithehead's inclusion-based approach

We have just argued about the inadequateness of the inclusion-based approach to point-free geometry. Nevertheless, in our opinion, we can get around this inadequateness by reconsidering this approach in the framework of multi-valued logic (see Gerla and Miranda 2004). Indeed, consider the first two axioms in Definition 2.1 in a language L_{Incl} with a predicate symbol *Incl*:

$$\textbf{A1}\quad \forall x(Incl(x,x)) \quad ; \quad \textbf{A2}\quad \forall x\forall y\forall z((Incl(x,z)\wedge Incl(z,y)) \Rightarrow Incl(x,y)).$$

But, differently from Section 2, interpret these axioms in a multi-valued logic. For example, we can consider the *product* logic (see for example Hájek 1998) whose set of truth values is [0,1] and in which

- the conjunction is interpreted by the usual product in [0,1],
- the implication by the operation $\rightarrow$ defined by setting $x\rightarrow y = 1$ if $x\leq y$ and $x\rightarrow y = y/x$ otherwise,
- the equivalence by the operation $\leftrightarrow$ defined by setting $x\leftrightarrow y = 1$ if $x = y$ and $x\leftrightarrow y = (x\wedge y)/(x\vee y)$ otherwise,
- the universal quantifier by the greatest lower bound.

In such a case an *interpretation* of L_{Incl} is a pair $I = (S,incl)$ such that S is a nonempty set and $incl : S^2\rightarrow [0,1]$ is a fuzzy relation to interpret *Incl.* As in the classical case, given a formula α whose free variables are among $x_1,\ldots,x_n$ and $d_1,\ldots,d_n$ in S, the truth value $Val(I,\alpha,d_1,\ldots,d_n) \in [0,1]$ of α in $d_1,\ldots,d_n$ is defined. This enables us to associate α with its *extension in* I, i.e. the n-ary fuzzy relation $I(\alpha) : S^n \rightarrow [0,1]$ defined by setting

$$I(\alpha)(d_1,\ldots,d_n) = Val(I,\alpha,d_1,\ldots,d_n)$$

for every $d_1,\ldots,d_n$ in S. Also, $(S,incl)$ is a model of *A1* and *A2* if and only if

$$\textbf{a1}\quad incl(x,x) = 1 \quad ; \quad \textbf{a2}\quad incl(x,y)\cdot incl(y,z) \leq incl(x,z),$$

for every x, y, $z \in S$. In order to express the anti-symmetric property, we assume that in our logic there is a modal operator *Cr* such that $Cr(\alpha)$ means "*α is completely true*" and that this operator is interpreted by the function cr : $[0,1]\rightarrow [0,1]$ such that $cr(x) = 1$ if $x = 1$ and $cr(x) = 0$ otherwise. Then we can consider the axiom

$$\textbf{A3}\quad Cr(Incl(x,y))\wedge Incl(y,x)) \rightarrow x = y.$$

A fuzzy interpretation $(S, incl)$ satisfies *A3* if and only if

$$\textbf{a3}\quad (incl(x,y) = incl(y,x) = 1) \Rightarrow x = y.$$

Definition 7.1. Denote by $x\leq y$ the formula $Cr(Incl(x,y))$ and by $\leq$ its extension in a given interpretation. Then $\leq$ is called *the crisp inclusion associated with incl*. Denote by $Pl(x)$ the formula $\forall x'(x'\leq x \Rightarrow Incl(x,x'))$ and by pl its extension. Then the fuzzy set pl expresses the *pointlikeness property*.

Trivially, the crisp inclusion is defined by,

$$x \leq y \Leftrightarrow incl(x,y) = 1 \tag{7.1}$$

and the pointlikeness property is defined by,

$$pl(x) = inf\{incl(x,x') : x' \leq x\}. \tag{7.2}$$

Such a property is a graded counterpart of the definition

"x is a point provided that every part of x coincides with x".

The next axiom says that if the regions x and y are (approximately) points, then the graded inclusion is (approximately) symmetric.

A4) $Pl(x) \wedge Pl(y) \rightarrow (Incl(x,y) \leftrightarrow Incl(y,x))$.

Then such an axiom is satisfied if and only if

a4) $$pl(x) \cdot pl(y) \leq (incl(x,y) \leftrightarrow incl(y,x)) \tag{7.3}$$

or, equivalently,

$$pl(x) \cdot pl(y) \cdot incl(x,y) \leq incl(y,x).$$

Definition 7.2. We call *graded inclusion space* any model of *A1*, *A2*, *A3*, *A4* .

In any graded inclusion space we can define a notion of point as follows.

Definition 7.3. Given a graded inclusion space $(S,incl)$, we call *nested abstraction process* any order-reversing sequence $<p_n>_{n\in N}$ of regions such that

$$lim_{n\rightarrow\infty} pl(p_n) = 1.$$

We denote by Nr the class of the nested abstraction processes.

We can give to the set Nr a structure of pseudo-metric space.

Proposition 7.4. Let $(S,incl)$ be a graded inclusion space such that $Nr \neq \emptyset$, then the map $d : Nr \times Nr \rightarrow R^+$ obtained by setting

$$d(<p_n>_{n\in N}, <q_n>_{n\in N}) = - lim_{n\rightarrow\infty} Log(incl(p_n,q_n)), \tag{7.4}$$

defines a pseudo-metric space (Pr,d).

As it is usual in the theory of pseudo-metric spaces, we can associate (Pr,d) with a metric space.

Proposition 7.5. The relation $\equiv$ in Nr defined by setting $<p_n>_{n\in N} \equiv <q_n>_{n\in N}$ if $d(<p_n>_{n\in N}, <q_n>_{n\in N}) = 0$ is an equivalence relation. In the quotient $Nr/\equiv$ we can define a metric d by setting

$$d([<p_n>_{n\in N}], [<q_n>_{n\in N}]) = d(<p_n>_{n\in N}, <q_n>_{n\in N}).$$

We call *points* the elements in $Nr/\equiv$, i.e. the complete equivalence classes

$$[<p_n>_{n\in N}] = \{<q_n>_{n\in N} \in Nr : d(<p_n>_{n\in N}, <q_n>_{n\in N}) = 0\}.$$

Observe that the "pathological" abstractive classes G_{-0} , G_0 and G_{+0} defined in Section 6 are equivalent nested abstraction processes and therefore they represent the same point.

There is no difficulty to define canonical spaces in the Euclidean space R^n. In fact if δ denotes the usual distance in R^n and x, y are nonempty bounded subsets of R^n, then we define the *excess function e* by setting,

$$e(x,y) = sup_{P\in x}\ inf_{Q\in x}\ \delta(P,Q)\ . \tag{7.5}$$

Theorem 7.6. Let Re be the class of all nonempty bounded closed regular subsets of R^n and define $incl : Re\times Re \to[0,1]$ by setting

$$incl(x,y) = 10^{-e(x,y)}. \tag{7.6}$$

Then $(R^n,incl)$ is a graded inclusion space we call *canonical graded inclusion space*. In such a space $pl(x) = 10^{-|x|}$.

It is possible to see that in these spaces the inclusion and the connection relations are definable by the two formulas $Cr(Incl(x,y))$ and $Cr(\exists z(Pl(z)\wedge(Incl(z,x)\wedge Incl(z,y)))$, respectively. Moreover the points coincides with the usual points in the Euclidean metric space R^n. This suggests that the notion of graded inclusion space looks to be a good candidate to reformulate Whitehead's point-free geometry as proposed in *PNK* and *CN.*

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