Why I have an extra-terrestrial ancestor

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Abstract.
The paper concerns some paradoxes arising from the vagueness. For example, the *Heap paradox*, the *Bold Man paradox* and *Poincaré's paradox* of the indiscernibility are considered. The solutions proposed by fuzzy logic in the framework of approximate reasoning theory are exhibited.

Riassunto. Il lavoro considera alcuni paradossi che nascono da nozioni vaghe. Ad esempio si esaminano il *paradosso del mucchio di grano*, il *paradosso dell'uomo calvo* ed il *paradosso sugli indiscernibili* di Poincaré . Si espongono le soluzioni proposte dalla logica fuzzy nell'ambito della teoria dei ragionamenti approssimati.

1 Introduction

The aim of this note, popular in nature, is to generate some interests in fuzzy logic and approximate reasoning theory. To do this we show that fuzzy logic is an useful tool to give a solution to some famous paradoxes as the Heap paradox, the Bald Man paradox and Poincaré paradox of the indiscernibility (see Goguen [1968/69], Höhle [1991], Gerla [2008]). Also, we suggest new paradoxes. These paradoxes show that difficulties related with the vagueness arise also in two basic fields of the scientific activity, physics and evolution theory, and not only in the everyday activities.

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2 Heap paradox and Goguen's solution

Denote by $Small(n)$ the sentence "a heap with n stones is small". Then the formula
(a) $Small(1)$
is true since an heap with only one stone is small. Moreover, it is natural to admit that all the formulas
(b) $Small(n) \rightarrow Small(n+1)$
are true, where n is any numeral. This since if you add one stone to a small heap, it remains small. Then, it is evident that, given any numeral n, we can prove that the formula $Small(n)$ is true: a paradox. Denote by MP the modus ponens rule, i.e. the rule enabling us to derive B from two formulas as A and $A \rightarrow B$. Then we can formalize the Heap paradox as follows.

Proposition 2.1 *Given any numeral m, from (a) and the formulas in (b) we can prove Small(m). Then, every heap is small.*

Proof. Assume (a) and the formulas in (b). Then by MP:
from $Small(1)$ and $Small(1) \rightarrow Small(2)$, we may state $Small(2)$;
from $Small(2)$ and $Small(2) \rightarrow Small(3)$, we may state $Small(3)$;
...
from $Small(n)$ and $Small(n) \rightarrow Small(n+1)$, we may state $Small(n+1)$.
In such a way, whatever the integer m is fixed, we may prove $Small(m)$. □

Such a reasoning is correct, (a) is true and the conclusion is false. Then, the only possibility looks to deny a formula in the schema (b) and therefore to admit the existence of an integer m such that $Small(m) \rightarrow Small(m+1)$ is false. In turn, this means that $Small(m)$ is true and $Small(m+1)$ is false and therefore that we have to admit the existence of a "*critical*" number m such that:
while a heap with m stones is small, by adding just one stone this heap becomes not small!
Many peoples accept such a point of view and consider the Heap paradox as a proof of the fact that a vague notion as *small* has an exact hidden boundary. Such a claim is extended to any vague notion. Namely, it is claimed that both the crisp and the vague predicates have sharp boundaries but it is a characteristic of a vague predicate that, due to the limitations of the human being, such a boundary is unknowable. In other words, vagueness is an epistemological phenomenon related to ignorance (Sorensen [2001]). So, as in some ideal world there is the set of odd numbers which is clearly distinct from the set even numbers, in the same world there is the set of all the small numbers clearly distinct from the set of all the numbers which are not small.
A different approach is proposed in fuzzy logic (see Goguen [68/69]) where one claims that in presence of vague predicates the notion of an inference rule has to be modified. This means

that in a rule we have to specify how (a constraint on) the truth degree of a conclusion depends on the available (constraints on the) truth degrees of the premises. For example, in accordance with Goguen proposal, it is reasonable to extend the classical modus ponens rule by assuming that,
IF you know that α is true at least at degree λ_1
AND $\alpha \to \beta$ is true at least at degree λ_2,
THEN you can conclude that β is true at least at degree $\lambda_1 \otimes \lambda_2$,
where $\otimes$ is a suitable binary operation in [0,1] and $\lambda_1$and λ_2 are elements in [0,1]. In accordance, any proof of α furnishes a constraint on the truth value of α (a precise and general formalization of approximate reasoning theory can be found in Pavelka [1979] and Gerla [2000]). In Goguen [1968/69] the operation $\otimes$ coincides with the usual product.

Theorem 2.2 *Approximate reasoning theory is able to give a formal representation of the heap argument preserving its intuitive content but avoiding its paradoxical character.*

Proof. Observe that everyone is convinced that *Small*(1) is true and the implications *Small*(n) $\to$ *Small*(n+1) are approximately true. Then by fuzzy logic we can "*respect*" this conviction by considering the *fuzzy set of axioms* obtained by assigning the truth value 1 to the formula *Small*(1) and a truth value near to 1, say 0.9, to the formulas *Small*(n) $\to$ *Small*(n+1). Also, in accordance with Goguen, we can interpret $\otimes$ by the usual product. So, we can formalize the Heap argument as follows:

- Since *Small*(1) [with degree 1]
  and *Small*(1) $\to$ *Small*(2) [with degree 0.9]
  we state *Small*(2) [at degree $1 \otimes 0.9 = 0.9$];
- Since *Small*(2) [with degree 0.9]
  and *Small*(2) $\to$ *Small*(3) [with degree 0.9]
  we state *Small*(3) [at degree $0.9 \otimes 0.9 = 0.9^2$] ;
  ...
- Since *Small*(n) [with degree 0.9^{n-1}]
  and *Small*(2) $\to$ *Small*(3) [with degree 0.9]
  we state *Small*(n+1) [at degree $0.9^{n-1} \otimes 0.9 = 0.9^n$].

Thus, if m is a very large number, we can prove *Small*(m) at degree 0.9^{m-1}, and this is not paradoxical. In fact, the resulting proof is valid at degree 0.9^{m-1} and this is a number very close to 0. □

Note. The fact that we can prove *Small*(m) at degree 0.9^{m-1} doesn't mean that this number is the truth degree of *Small*(m) but that it is greater or equal to 0.9^{m-1}. This since fuzzy logic

is a tool to find constraints on the truth values of the formulas and not necessarily their exact truth values. Different proofs give different constraints.
As an example, assume that we know that in the sequence P_1, P_2,...P_{1000} of points in a plane all the segments P_iP_{i+1} have the same length and that such a length is very small. Then, we can represent this information by claiming that all the formulas $Near(P_i,P_{i+1})$ are true at degree 0.9, say. Also, in accordance with the meaning of the predicate *Near*, it is natural to admit that the formulas $Near(P_i,P_{i+1}) \rightarrow (Near(P_{i+1},P_{i+2}) \rightarrow Near(P_i,P_{i+2}))$ hold true at degree 0.9. Then, by proceeding as in the Heap paradox, we can find a proof of $Near(P_1,P_{1000})$ with a degree δ very near to 0. Now, assume that further information is available, for example that each segment P_iP_{i+1} forms the angle $2\pi/1000$ with the segment $P_{i+1}P_{i+2}$. Then we can prove that the points are placed in a circle and that $P_{1000} = P_1$. As a consequence, since $Near(P_{999},P_{1000})$ hold true at degree 0.9, we have a proof of $Near(P_{999},P_1)$ and therefore of $Near(P_1,P_{999})$ with degree 0.9. Thus, we have two proofs of $Near(P_1,P_{999})$: the first one giving the truth value $\delta < 0.9$, the latter the truth value 0.9. Obviously no contradiction exists between these two claims provided that these values are interpreted as lower constraints to the exact truth value of $Near(P_1,P_{999})$.

3 Bald man paradox

Obviously, the core of Heap paradox lies in the vagueness of the predicate "*small*". As a matter of fact, it is easy to prove that any vague property which is "sufficiently graded" originates a paradox similar to Heap Paradox. To prove this, we represent the vague properties by the notion of fuzzy subset. A *fuzzy subset* of a set S is a map $s : S \rightarrow [0,1]$ where, given $x \in S$, the value $s(x)$ is interpreted as the membership degree of x to s. A fuzzy relation is a fuzzy subset of a Cartesian product (Zadeh [1975]).

Proposition 3.1 *Any vague property which is "sufficiently graded" originates a paradox similar to Heap Paradox.*

Proof. Let P be any vague property in a set D and assume that the extension of such a property is represented by a fuzzy subset $s : D \rightarrow [0,1]$ whose values go from 0 to 1 in a graded way. This means that, for any x in S the number $s(x)$ is the truth degree of the claim "*x satisfies P*". Also, we assume that a very small number ϵ and a sequence d_1, d_2, ... of elements of D exists such that
$s(d_1) = 1, \quad 0 < s(d_n) - s(d_{n+1}) < \epsilon, \quad \lim_{n\rightarrow\infty} s(d_n) = 0.$
Under these conditions $P(d_1)$ is true and all the implications $P(d_n) \rightarrow P(d_{n+1})$ look to be very plausible. If we accept these formulas as a system of hypotheses in classical logic, then we obtain a paradox similar to Heap Paradox. Indeed, given any integer n, after a suitable

number of applications of *MP* we can prove $P(d_n)$. On the other hand, from the hypothesis $\lim_{n\to\infty} s(d_n) = 0$, it follows that $P(d_n)$ looks to be false for a sufficiently large integer n. $\square$

As an example, if P denotes the vague predicate "*bald*", then we obtain the famous *Bald man paradox*. More precisely, assume that $Bald(m)$ denotes the claim "*a man with m hairs is bald*". Then, $Bald(1)$ is true and all the formulas $Bald(n) \to Bald(n{+}1)$ are plausible. So, given any integer m, we are able to prove $Bald(m)$. Observe that we can also consider paradoxes in *reverse form*, going on by "subtraction". As an example, let "*Big*" be the negation of "*Small*", i.e., let $Big(n)$ denote the claim "*a heap with n stones is big*". Then it is rather natural to admit that for a sufficiently large integer m, $Big(m)$ holds and that all the implications $Big(n)\to Big(n\text{-}1)$ are plausible. Thus, by using m-1 times M.P., we arrive to the paradoxical conclusion that $Big(1)$ holds.

Theorem 3.2 *Bald man paradox and the reverse form of Heap paradox cannot be solved by Gouguen's fuzzy logic based on the product.*

Proof. Assume that $Bald(n) \to Bald(n+1)$ holds to a degree $\lambda \neq 0$. Then, since $Bald(1)$ holds to degree 1, in Goguen's formalization $Bald(10.000)$ can be proved at degree λ^{m-1}. Since $\lambda^{m-1} \neq 0$, this contradicts the fact that $Bald(10.000)$ is totally false. Likewise, let m be a natural number such that $Big(m)$ is true to suitable degree $\mu \neq 0$ and assume that, given any natural number n, the formula $Big(n) \to Big(n-1)$ holds to a degree $\lambda \neq 0$. Then it is easy to see that $Big(1)$ can be proved to the degree $\mu \cdot (\lambda^{m-1})$. Since $\mu \cdot (\lambda^{m-1}) \neq 0$, this contradicts the fact that $Big(1)$ is totally false. $\square$

Such a fact suggests to consider an operation different from the usual product in extending *MP*. The more natural candidate is the *Lukasiewicz triangular norm* defined by setting, for every $x, y \in [0,1]$, $x \otimes y = max\{x + y - 1, 0\}$. In fact, define the power λ^n as usual, i.e. by setting $\lambda^0 = 1$ and $\lambda^n = \lambda \otimes \lambda^{n-1}$. Then, by referring to the reverse form of the Heap paradox, it is easy to see that a suitable integer m exists such that $\mu \otimes \lambda^{m-1} = 0$. So, it is not contradictory to assume that the truth value μ of $Big(m)$ is different from zero.

4 Theseus's ship and Poincaré paradoxes

The just exposed paradoxes are based on a vague monadic predicate. A similar class of paradoxes is related with vague binary relations, for example the approximate identities. As an example, the so called "paradox" of Poincaré refers to indistinguishability by emphasizing that, in spite of common intuition, this relation is not transitive (see Poincaré [1902] and Poincaré

[1904]). In fact, it is possible that we are not able to distinguish d_1 from d_2, d_2 from d_3, . . . ,d_{m-1} from d_m and, nevertheless, that we have no difficulty in distinguishing d_1 from d_m.
A similar paradox is the ancient Theseus's ship paradox. Theseus and his men are sailing the Mediterranean in a ship. Each day a sailor replaces the worn-out wooden planks in the ship so that after five years every plank has been replaced. Are Theseus and his men still sailing in the same ship that was launched five years earlier? "Yes" most will answer. But suppose that the removed wooden planks are still sufficiently good and that therefore they are not destroyed but taken ashore. Also, suppose that at the end of five years, a ship is rebuild with these planks exactly in the same manner as the original Theseus's ship. Is this ship on shore Theseus's ship? Or is the ship sailing the Mediterranean?
Such a problem is related with the more troubling problem of the persistence of personal identity. How do I know that I am that person who I was yesterday, or last year, or twenty-five years ago? Why would an old high school friend say that I am G. Gerla, even though thousands of things about me have changed since high school ? Probably no cells are in common between my organism at this time and the organism that responded to the name G. Gerla forty years ago.
Now an immediate solution of these paradoxes would merely conclude that our intuition about the transitivity of the indiscernibility relation is wrong. A different solution is proposed by fuzzy logic in which the paradoxical behavior of the indistinguishability is avoided by assuming that such a relation is a fuzzy relation satisfying a weak notion of transitivity (Gerla [2008]). More precisely, if I denotes such a relation, we assume the formula
$I(x, y) \wedge I(y, z) \rightarrow I(x, z)$
at a degree λ different from 1 and the formulas
$I(d_i, d_{i+1})$
at degree 1. Then in fuzzy logic we can prove $E(d_1, d_3)$ only at degree λ^2 and, more in general, the formula $E(d_1, d_m)$ only at degree λ^{m-1}. On the other hand, such a value is very close to zero.

5 Constraints and negative information

Consider the following form of Heap paradox. Again, we assume that a heap containing just one stone is small, i.e., that
a) $Small(1)$ is true
Moreover, we express directly our opposition to accept the existence of a *critical* number n such that a heap with n stones is small and a heap with $n+1$ stones is not small. Indeed, we claim that
b) $Small(n) \wedge \neg Small(n + 1)$ *is "impossible"*.
Then,
- since $Small(1)$ holds and $Small(1) \wedge \neg Small(2)$ is impossible, $Small(2)$ holds;

- since $Small(2)$ holds and $Small(2) \wedge \neg Small(3)$ is impossible, $Small(3)$ holds;
...
- since $Small(n-1)$ holds and $Small(n-1) \wedge \neg Small(n)$ is impossible, $Small(n)$ holds.

In such an argument the presence of the expression "*impossible*" means that we admit negative information as a premise in an inferential step. Notice that this is not usual in logic since inference rules are preferred in which the premises are true formulas. Clearly, in any logic with a good negation the distinction is not essential because we can translate the claim that a formula is false into the equivalent claim that its negation is true. In spite of that, it is an interesting task to formalize directly those forms of reasoning starting from negative information. Now, the better way to express negative information is to consider signed formulas. We call *signed formula* a pair (A,λ) where $\lambda \in \{0,1\}$. The pair (A,0) represents the claim "*A is false*", the pair (A,1) the claim "*A is true*". The signed formulas enable us to formalize the just exposed argument by the classical "*inference rule*"

$$\frac{(A,1)\ ,\ (A \wedge \neg B,0)}{(B,1)}$$

and by assuming the signed formulas:
a) $(Small(1),1)$
b) $(Small(n) \wedge\ \neg Small(n+1),0)$.
Indeed,

- **from** ($Small$(1), 1) **and** ($Small$(1) $\wedge \neg$ $Small$(2), 0) **we prove** ($Small$(2), 1),
- **from** ($Small$(2), 1) **and** ($Small$(2) $\wedge \neg$ $Small$(3), 0) **we prove** ($Small$(3), 1),
...
- **from** ($Small$(n-1), 1) **and** ($Small$(n-1)) $\wedge \neg$ $Small$(n), 0) **we prove** ($Small$(n), 1).

Now, as observed in the note at the end of Section 2, in fuzzy logic it would be misleading to consider the available information $v(\alpha) \in [0,1]$ on a formula α as the truth value of α. Indeed, such a number is a constraint on its actual truth value, namely a constraint like "*the truth value of* α *is greater than or equal to* $v(\alpha)$". Now, it is rather natural to admit also that the available information about an unknown fuzzy world can be expressed by constraints of a more general kind. For example, constraints as "*the truth value of* α *is between 0.3 and 0.5*", "*the truth value of* β *is less than 0.7*". The general form of such information is "*the truth value of* α *belongs to X*", where X is a subset of U (see Gerla [1999]). The consideration of the constraint enables us give a solution to the negative form of Heap paradox as follows. Indeed, observe that everyone is convinced that the formulas $Small(n) \wedge \neg$ $Small(n+1)$ are very far from the truth but, in general, not completely false. In other words, the truth values of these formulas are either equal to 0 or, in any case, "*very near*" to 0. This can be expressed in a precise way by enlarging the notion of signed formula and, in accordance, by changing the notion of fuzzy inference rule and fuzzy reasoning. Indeed we call *signed formula* a pair (A,X)

where A is a formula and X an interval in [0,1]. Moreover, we propose the rule

$$\frac{(A,[a,b])\ ,\ (A\wedge\neg B,[c,d])}{(B,[e,f])}$$

where [e,f] is the closed interval generated by the set $\{\lambda \in$[0,1] : there is $x \in$[a,b] such that $x \otimes (1-\lambda) \in [c,d]\}$. By definition, this rule is sound with respect to Łukasiewicz semantics. Indeed, we refer to the set of possible truth values for B given the information that the truth value of A is in [a,b] and the truth value of $A \wedge \neg B$ is in [c,d]. In particular, we have that

$$\frac{(A,[\lambda,1])\ ,\ (A\wedge\neg B,[0,\mu])}{(B,[0\vee(\lambda-\mu),1])}$$

Indeed, let x and y be the truth values of the formulas A and B, respectively. Then
$x \geq \lambda$ and $(x+(1-y)-1)\vee 0 \leq \mu \Leftrightarrow x-y \leq \mu \Leftrightarrow y \geq \lambda-\mu$.
In such a case the rule says that:
IF A is approximately true (to degree λ),
AND $A \wedge \neg B$ is approximately false (to degree μ),
THEN B is approximately true (to degree $\lambda - \mu$),
Coming back to the paradox, assume the signed formulas,
(*Small*(1),[1,1]) ,
(*Small*(n) $\rightarrow$ *Small*(n+1), [0,0.0001]).
Then we can consider the following approximate reasoning:
from (*Small*(1),[1,1])
and *Small*(1)$\wedge\neg$*Small*(2),[0,0.0001]),
we state (*Small*(2),[0.9999,1]);
since (*Small*(2),[0.9999,1]) **and** (*Small*(2)$\wedge\neg$*Small*(3),[0,0.0001]),
we state (*Small*(3),[0.9998,1]);
...
since (*Small*(10000),[0.0001,1]) **and** (*Small*(10000)$\wedge\neg$*Small*(10001),[0,0.0001]),
we state (*Small*(10001),[0,1]).
Thus, as in Section 3, for any integer n we can give a correct proof π of the formula *Small*(n) but, for $n >$ 10.000, no information about the truth value of *Small*(n) is furnished by π.

6 Darwin evolution theory, classical mechanics and relativity theory

In this section we consider two paradoxes related with Darwin evolution theory and relativity theory, respectively. Denote
- by $Ub(x)$ the claim "*x is an human being*" and

- by $Anc(x', x, n)$ the claim "x' *is an ancestor of* x and x' *was born* n *years before* x *at least*", where x and x' vary in the class of life forms and n in the set of natural numbers. Then, we can consider the following formulas
(a) $Anc(x'', x', p) \wedge Anc(x', x, q) \rightarrow Anc(x'', x, p + q)$
(b) $Ub(x) \rightarrow \exists x'(Ub(x') \wedge Anc(x', x, 10))$.
We have that while (a) holds by definition, (b) is more questionable. In fact the opinion of many religious men is that both $\exists x' Anc(x', Adam, 10)$ and $\exists x' Anc(x', Eve, 10)$ are false. Then, since both *Ub(Adam)* and *Ub(Eve)* are true, (b) has to be rejected. Instead a non-creationist point of view leads to admit (b) since it is sufficient to assume that x' is the mother of x. This since it is a characteristic of the human species that every man has a mother, that this mother is an human being and that it is not possible for a woman to give birth before ten years. This suggests to call *non-creationist theory* the theory defined by (a) and (b).

Theorem 6.1 *Assume the non-creationistic theory. Then I have an extraterrestrial ancestor. As a consequence, Darwin evolution theory is wrong.*

Proof. Let *Gianni* be my name. Then for any natural number n it is possible to prove, by induction on n, the formula

$$\exists x'(Ub(x') \wedge Anc(x', Gianni, 10 \times n)) \tag{6.1}$$

In fact, in the case $n = 1$, since $Ub(Gianni)$ is true, (6.1) follows from (b) by the particularization rule and *MP*. Assume that (6.1) is satisfied by n and let *Carlo* be such that $Ub(Carlo) \wedge Anc(Carlo, Gianni, 10 \times n)$ and *Luigi* such that $Ub(Luigi) \wedge Anc(Luigi, Carlo, 10)$. Then, we can prove also the formula

$$Anc(Luigi, Carlo, 10) \wedge Anc(Carlo, Gianni, 10 \times n).$$

Since by the particularization rule (a) entails

$$Anc(Luigi, Carlo, 10) \wedge Anc(Carlo, Gianni, 10 \times n) \rightarrow Anc(Luigi, Gianni, 10 \times (n + 1)),$$

by *MP* we obtain the formula $Anc(Luigi, Gianni, 10 \times (n + 1))$ and therefore the formula

$$\exists x'(Ub(x') \wedge Anc(x', Gianni, 10 \times (n + 1)))$$

This proves (6.1) completely. To prove the theorem, observe that (6.1) entails that an ancestor of mine existed exactly $10 \times 10^{100.000}$ years ago. Since the earth is less than $10 \times 10^{100.000}$ years old, such an ancestor was extraterrestrial. □

Obviously the claim in Theorem 5.1 is not a paradox. In fact there is no logical contradiction in the claim that I have an extraterrestial ancestor. The paradox lies in the possibility of confuting

a theory on factual reality by a purely logical argumentation. Plausibly, the paradox originates from the vagueness of the notion "*human being*" in the scale of the evolution of the species. On the other hand such a vagueness is a presupposition of evolution theory. So, perhaps a formal solution of such a paradox by the fuzzy logic apparatus should be possible.
Another paradox is related with the "*correspondence principle*", concerning the connection between a new physical theory and the corresponding old one. Such a principle claims that an empirically confirmed theory will not disappear as false with the appearance of a new, more universal theory, but it preserves its importance as *special case* or the *limiting* of the new theory. This contradicts Kuhn's or Feyerabend's conception of incommensurability claiming that a scientific revolution generates not only a new theory, but it changes the whole world of the given discipline. Also, this is related with Popper question: can we explain how one theory can be *closer to the truth*, or has *greater verisimilitude* than another? (for a geometrical approach to this question see Gerla [1992] and Gerla [2006]). The most widely known correspondence principle was enunciated by Niels Bohr in 1923 and is related with quantum mechanics and classical mechanics. We have also a correspondence principle for general relativity to special relativity and special relativity to Newtonian mechanics. By referring, for example, to this last case, the correspondence principle starts from the observation that Newton's laws and classical mechanics do a very good job of explaining and predicting motion at everyday speeds. Thus we would hope that when small speeds are involved, the equations in special relativity should reduce to the Newtonian form. So, in a sense, no contradiction exists between the two theories. This is all O.K., but it is very difficult to give a precise and logical meaning to this principle. Indeed, denote by CT the classical mechanics and by RT special relativity theory. Moreover, assume that g is a physical quantity and that, since the velocities are very small
- in TR we can prove the claim $\alpha_R =$ "g is 0.567662220", and
- in CT we can prove the claim $\alpha_C =$ "g is 0.567662221".
Then, from $TR \cup CT$ we derive that $0.567662220 = 0.567662221$, a contradiction. This means that the two theories are mutually excluding and therefore, since we accept that RT is the correct theory, we have reject CT completely. This contradicts the correspondence principle and our feeling that, since the sentences α_R and α_C are approximately equivalent and α_R is totally correct,α_C is approximately true. On the other hand, in classical logic the expressions "approximately equivalent sentences" and "approximately true sentence" have no meaning. Thus, we have the following paradox:

Theorem 6.2 *Consider a world with small velocities. Then classical mechanics is a very useful theory in spite of the fact that nearly all its theorems are false.*

We can try to improve such an unsatisfactory situation by admitting that our language is approximate and therefore that our claims are not completely accurate. Indeed, as claimed by C. S. Peirce, "*It is easy to be certain. One has only to be sufficiently vague*". For example, we can decide to confine ourselves only to eight digit approximation of the real numbers. In such a case both α_R and α_C coincides with the claim "g is 0.56766222" whose correct meaning

is that "*g is a value between* 0.56766222 and 0.56766223". In other words, by admitting an approximate language, we can obtain precise (i.e., Boolean) truth values. Then, we can try for a relaxed form of the inclusion principle: *Assume that the velocities are small and that our language is sufficiently approximate. Then, the theorems in CT coincide with the theorems in TR.* In any case my feeling is that also such a version of the inclusion principle is wrong and this since the notion "*small velocities*" is vague and therefore originates paradoxes similar to Heap paradox. As an example, denote by $Small(x, y)$ the claim
"*the velocity of y with respect to x is small*"
and by $Galilean(x, y)$ the claim
"*assume that our language is sufficiently approximate, then y is connected with x by a Galilean transformation.*"
Then, the relaxed inclusion principle is expressed by the formula

$$Small(x, y) \rightarrow Galilean(x, y).$$

Now, consider a finite sequence $R_1, R_2, ..., R_n$ of solid bodies such that the velocity of each R_{i+1} with respect to R_i is small and that, nevertheless, the velocity of R_n with respect to R_1 is near to light velocity. In brief, assume that all the formulas $Small(R_i, R_{i+1})$ are valid and that $Small(R_1, R_n)$ is false. Also, since the product of two Galilean transformation is a Galilean transformation, assume the implication

$$Galilean(x, y) \wedge Galilean(y, z) \rightarrow Galilean(x, z). \tag{6.2}$$

Then
from $Small(R_1, R_2)$ **and** $Small(R_1, R_2) \rightarrow Galilean(R_1, R_2)$ **we obtain** $Galilean(R_1, R_2)$
from $Small(R_2, R_3)$ **and** $Small(R_2, R_3) \rightarrow Galilean(R_2, R_3)$ **we obtain** $Galilean(R_2, R_3)$
from $Galilean(R_1, R_2), Galilean(R_2, R_3)$ **and** (5.2) **we obtain** $Galilean(R_1, R_3)$
. . .
from $Galilean(R_1, R_{n-1}), Galilean(R_{n-1}, R_n)$ **and** (5.2) **we obtain** $Galilean(R_1, R_n)$

But this contradicts the basic assumption in *RT* that R_n is related with R_1 by a Lorentz transformation. Thus we have that *RT* contradicts *CT* also in a relaxed and imprecise language.

7 The induction principle is false.

Recall that the *induction principle* claims that, given a formula $P(x)$ expressing a property defined in the set of natural numbers, the formula
$P(1) \rightarrow (\forall n(P(n) \rightarrow P(n+1))) \rightarrow \forall n P(n)$,
is true. We can restate the Heap paradox by such a principle. Indeed, assume the following two formulas
(a) $Small(1)$

(b) $\forall n(Small(n) \rightarrow Small(n+1))$
Also, as a consequence of the induction principle, admit the formula
(c) $Small(1) \rightarrow (\forall n(Small(n) \rightarrow Small(n+1))) \rightarrow \forall n\ Small(n)$,
Then, from (a), (b) and (c) and two applications of *MP* we can prove the formula $\forall n\ Small(n)$ and therefore, by particularization, $Small(m)$ for any natural number m. This means that we obtain the same paradoxical conclusion as in Section 2. Nevertheless, this argument is totally different since we arrive to $Small(m)$ in a three steps reasoning and not in an m-1 steps reasoning.

Proposition 7.1 *If we admit the induction principle, fuzzy logic is not able to solve the Heap paradox.*

Proof. To try to solve such a form of the paradox by fuzzy logic, assume that (a) is completely true, that (b) is valid at degree 0.9 and that (c) is completely true. Also, let $\otimes$ be the operation used to extend *MP*. Then $\forall n\ Small(n)$ can be proved at degree $1 \otimes 0.9 = 0.9$ and therefore, in particular, any formula *Small(m)* can be proved at degree 0.9. Such a conclusion is as much paradoxical as the conclusion that any formula *Small(m)* is true. $\square$

From Proposition 6.1 it follows that in order to avoid the inconsistency of fuzzy logic we have to reject the induction principle completely. As a matter of fact we can prove the following theorem.

Theorem 7.2 *The induction principle is false in any fuzzy logic in which both $\otimes$ and the interpretation $\rightarrow$ of the implication are continuous functions.*

Proof. Assume that the induction principle is valid with a degree $\lambda \neq 0$. Then, since $\lambda \otimes 1 = \lambda > 0$, by the continuity hypothesis $\mu \neq 1$ exists such that $\lambda \otimes \mu > 0$. Since $\rightarrow$ is a continuous function, and $\{(x,y) \in [0,1] \times [0,1] : x = y\}$ is a compact set contained in the open set $\{(x,y) \in [0,1] \times [0,1] : x \rightarrow y > \mu\}$, a positive real number ϵ exists such that
$y - x < \epsilon \Rightarrow x \rightarrow y > \mu$.
Let P be a vague predicate in a set S as in the proof of Proposition 2.3. Namely, we assume that P is interpreted by a fuzzy set $s : S \rightarrow [0,1]$ such that a sequence $d_1, d_2, ...$ of elements of S exists such that
$s(d_1) = 1\ ;\ 0 < s(d_n) - s(d_{n+1}) < \epsilon\ ;\ \lim_{n\to\infty} s(d_n) = 0$.
Under these conditions $P(d_1)$ is true and the truth degree of the formula $\forall n(P(d_n) \rightarrow P(d_{n+1}))$ is greater or equal to μ. Then $\forall n P(d_n)$ can be proved at degree $1 \otimes \lambda \otimes \mu = \lambda \otimes \mu$, and therefore, by particularization, each $P(d_m)$ can be proved at degree $\lambda \otimes \mu \neq 0$. This contradicts the hypothesis $\lim_{n\to\infty} s(d_n) = 0$. $\square$

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