Logics with Approximate Premises

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M. Ying proposed a propositional calculus in which the reasoning may be approximate by allowing the antecedent clause of a rule to match its premise only approximately. The aim of this note is to relate Ying's proposal to an extension principle for closure operators proposed by the authors. In this way it is possible to show that, in a sense, Ying's apparatus can be reduced to a fuzzy logic as defined by Pavelka [J. Pavelka, "On fuzzy logic I. Many valued rules of inference," *Zeitschrift fur Math. Logik und Grundlagen Math.*, **25**, 45–52 (1979)]. © 1998 John Wiley & Sons, Inc.

I. INTRODUCTION

In Ref. 1 M. Ying proposed a propositional calculus in which the reasoning may be approximate by allowing the antecedent clause of a rule to match its premise only approximately. Ying gives the following definitions. Let $\mathbb{F}$ be the set of formulas of the classical propositional calculus and let $R : \mathbb{F} \times \mathbb{F} \to [0,1]$ be a similarity relation. Then R is extended to a fuzzy relation between sets of formulas by setting, for every X and Y subsets of $\mathbb{F}$,

$$\hat{R}(X,Y) = \mathrm{Inf}_{y \in Y} \mathrm{Sup}_{x \in X} R(x,y) \tag{1.1}$$

The number $\hat{R}(X,Y)$ is a multivalued valuation of the claim that every element in Y is similar to a suitable element in X. If R is the identity relation then $\hat{R}$ is the (characteristic function of the) inclusion relation. The relation $\hat{R}$ enables us to define a suitable fuzzy consequence relation:

$$\mathrm{Con}(X,\alpha) = \mathrm{Sup}\left\{\hat{R}(X \cup \mathbb{A}, Y) \mid Y \vdash \alpha\right\} \tag{1.2}$$

where $\mathbb{A}$ is a fixed set of tautologies. More precisely, Ying considered only similarity relations induced in a natural way from a similarity relation on the set of propositional variables.

The aim of this note is to relate Ying's proposal to an extension principle proposed in Refs. 2, 3, and 4. This enables us to show that, in a sense, the deduction apparatus of Ying can be reduced to a fuzzy logic in Hilbert style as defined by J. Pavelka in Ref. 5.

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2. PRELIMINARIES

Let $\mathbb{U}$ be the interval $[0, 1]$ and set, for every λ and λ' in $\mathbb{U}$,

$$\lambda \vee \lambda' = \max\{\lambda, \lambda'\} \quad \text{and} \quad \lambda \wedge \lambda' = \min\{\lambda, \lambda'\}.$$

Then, $(\mathbb{U}, \wedge, \vee)$ is a complete, completely distributive lattice whose minimum is 0 and maximum 1. Let S be a set, then a *fuzzy subset* is any map from S in U. We denote by $\mathfrak{F}(S)$ the class of the fuzzy subsets of S. The basic notions of set theory are extended to the fuzzy subsets as follows. The *inclusion relation* is defined by setting, for every pair s and s' of fuzzy subsets

$$s \subseteq s' \Leftrightarrow s(x) \leq s'(x) \quad \text{for every } x \in S.$$

If $s \subseteq s'$, we say that s is *contained* in s' or that s is *a part of* s'. The *union* $s \cup s'$ and the *intersection* $s \cap s'$ of two fuzzy subsets s and s' are defined by

$$(s \cup s')(x) = s(x) \vee s'(x) \quad \text{and} \quad (s \cap s')(x) = s(x) \wedge s'(x),$$

respectively. More generally, given a family $(s_i)_{i \in I}$ of fuzzy subsets of S, the union $\bigcup_{i \in I} s_i$ and the intersection $\bigcap_{i \in I} s_i$ are defined by

$$\left(\bigcup_{i \in I} s_i\right)(x) = \text{Sup}\{s_i(x) \mid i \in I\} \quad \text{and} \quad \left(\bigcap_{i \in I} s_i\right)(x) = \text{Inf}\{s_i(x) \mid i \in I\}.$$

In this way $\mathfrak{F}(S)$ becomes a complete lattice, i.e., the direct power with index set S of $(\mathbb{U}, \vee, \wedge)$, extending the lattice $(\mathfrak{P}(S), \cup, \cap)$. More precisely, if we call *crisp* a fuzzy subset s such that $s(x) \in \{0, 1\}$ for every $x \in S$, then we can identify the classical subsets of S with the crisp fuzzy subsets of S via the characteristic functions. As an example we identify the empty set $\varnothing$ with the map constantly equal to 0. Recall that a *closure operator* in S is any map $J : \mathfrak{P}(S) \to \mathfrak{P}(S)$ such that, for every X and Y in $\mathfrak{P}(S)$,

$$X \subseteq Y \Rightarrow J(X) \subseteq J(Y); \quad X \subseteq J(X); \quad J(J(X)) = J(X).$$

If we have also that

$$J(X \cup Y) = J(X) \cup J(Y) \quad \text{and} \quad J(\varnothing) = \varnothing,$$

then J is called a *topological closure operator*. We extend these notions as follows. A *fuzzy closure operator* in S is any operator $J : \mathfrak{F}(S) \to \mathfrak{F}(S)$ satisfying

(i) $s_1 \leq s_2 \Rightarrow J(s_1) \leq J(s_2)$ (monotony)
(ii) $s \leq J(s)$ (inclusion)
(iii) $J(J(s)) = J(s)$ (idempotence).

If we have also that

(iv) $J(s_1 \cup s_2) = J(s_1) \cup J(s_2)$ and $J(\varnothing) = \varnothing$,

then we say that J is a *fuzzy topological closure operator*. A *fixed point* of a fuzzy operator J is a fuzzy subset s such that $J(s) = s$. It is immediate that the set of fixed points of a (fuzzy) topological closure operator is the set of closed (fuzzy) subsets of a suitable (fuzzy) topology. Given two sets S_1 and S_2, a *fuzzy relation* between S_1 and S_2 is a fuzzy subset R of $S_1 \times S_2$, i.e., any map $R: S_1 \times S_2 \to \mathbb{U}$. A fuzzy relation in a set S is called a *similarity* if, for every $x, y, z \in S$, we have that:

(a) $R(x, x) = 1$ (reflexivity)
(b) $R(x, y) = R(x, z) \wedge R(z, y)$ (transitivity)
(c) $R(x, y) = R(y, x)$ (simmetry).

If condition (c) is skipped then we say that R is a *fuzzy preorder*. Obviously, the crisp similarity and the crisp fuzzy preorders coincide with the equivalence relations and the preorders, respectively. Every fuzzy preorder defines a fuzzy closure operator.

PROPOSITION 2.1. *Let R be a fuzzy preorder and define J by setting, for any $s \in \mathfrak{F}(S)$ and $x \in S$,*

$$J(s)(x) = \mathrm{Sup}\{R(x', x) \wedge s(x') \,|\, x' \in S\}. \tag{2.1}$$

Then J is a topological fuzzy closure operator we call the fuzzy closure operator associated with R.

Proof. Since $R(x, x) = 1$, we have that $J(s)(x) \geq s(x)$. It is immediate that J is order preserving. To prove that $J(J(s))(x) \leq J(s)(x)$ observe that by the transitivity

$$\begin{aligned} J(J(s))(x) &= \mathrm{Sup}_{x' \in S} R(x', x) \wedge J(s)(x') \\ &= \mathrm{Sup}_{x' \in S} R(x', x) \wedge (\mathrm{Sup}_{z \in S} R(z, x') \wedge s(z)) \\ &= \mathrm{Sup}_{x' \in S} \mathrm{Sup}_{z \in S} R(x', x) \wedge R(z, x') \wedge s(z) \\ &= \mathrm{Sup}_{x' \in S} \mathrm{Sup}_{z \in S} R(z, x') \wedge R(x', x) \wedge s(z) \\ &\leq \mathrm{Sup}_{Z \in S} R(z, x) \wedge s(z) \\ &= J(s)(x). \end{aligned}$$

This proves that J is a closure operator. The remaining part of the proposition is obvious. ■

Let R be the identity relation, i.e., assume that $R(x, y) = 1$ if $x = y$ and $R(x, y) = 0$ otherwise. Then J is the identity map. If s coincides with the crisp set X we have that:

$$J(X)(x) = \mathrm{Sup}\{R(x', x) \,|\, x' \in X\}.$$

If R is the preorder $\leq$, then

$$J(X) = \{x \in S \,|\, \exists x' \in X, x' \leq x\}.$$

If R is the equivalence relation $\equiv$, then

$$J(X) = \{x \in S \,|\, \exists x' \in X, x \equiv x'\}.$$

In this case $J(X)$ is called *upper approximation* of X and is denoted by $\overline{X}$. This notion is on the basis of rough set theory.

DEFINITION 2.2. *Let R be a preorder relation. Then we say that a fuzzy subset s is closed with respect to R if it is a closed fuzzy subset of the fuzzy topological space associated with J, i.e., s is a fixed point of J.*

Obviously, the topological closure $J(s)$ of a fuzzy subset s is the smallest fuzzy subset containing s and closed with respect to R. The proofs of the following propositions are immediate.

PROPOSITION 2.3. *A fuzzy subset s is closed with respect to R if and only if*

$$s(x) \geq s(x') \wedge R(x', x) \tag{2.2}$$

for every x, x' *in* S. *In particular*

$$J(s)(x) \geq J(s)(x') \wedge R(x', x). \tag{2.3}$$

PROPOSITION 2.4. *Let R be a fuzzy preorder. Then*

$$R(x', x) \geq J(s)(x') \quad \textit{and} \quad R(x, x') \geq J(s)(x) \Rightarrow J(s)(x) = J(s)(x').$$

In particular

$$R(x, x') = R(x', x) = 1 \Rightarrow J(s)(x) = J(s)(x').$$

3. FUZZY LOGIC AND THE EXTENSION PRINCIPLE

In Ref. 5, J. Pavelka proposed a very general notion of fuzzy logic in Hilbert style. Namely, let $\mathbb{F}$ be a set whose elements we call *formulas*, then a *fuzzy deduction system* on $\mathbb{F}$ is a pair $\mathscr{S} = (a, \mathbb{R})$ where a is a fuzzy subset of $\mathbb{F}$, the fuzzy subset of *logical axioms*, and $\mathbb{R}$ is a set of fuzzy rules of inference. In turn, a *fuzzy rule of inference* is a pair $r = (r', r'')$, where

—r' is a partial n-ary operation on $\mathbb{F}$ whose domain we denote by Dom(r)
—r'' is an n-ary operation on $\mathbb{U}$ such that

$$r''(x_1, \dots, \mathrm{Sup}_{i \in I} y_i, \dots, x_n) = \mathrm{Sup}_{i \in I} r''(x_1, \dots, y_i, \dots, x_n) \tag{3.1}$$

So an inference rule r consists of a *syntactical component* r' that operates on formulas (in fact, it is a rule of inference in the usual sense) and an *evaluation component* r'' that operates on truth values to calculate how the truth value of the conclusion $r'(\alpha_1, \dots, \alpha_n)$ depends on the truth values of the premises $\alpha_1, \dots, \alpha_n$. A *proof* π of a formula α is a sequence $\alpha_1, \dots, \alpha_m$ of formulas such that $\alpha_m = \alpha$, together with a sequence of related "justifications." This means that, for every formula α_i, we have to specify whether

(i) α_i is assumed as a logical axiom; or
(ii) α_i is assumed as a proper axiom; or
(iii) α_i is obtained by a rule (in this case we have to indicate also the rule and the formulas from $\alpha_1, \dots, \alpha_{i-1}$ used to obtain α_i).

The justifications are necessary to valuate the proofs. Let $v : \mathbb{F} \to \mathbb{U}$ be any fuzzy set of formulas that we call *initial valuation* or *fuzzy set of proper axioms*. The meaning of v is that for every α we believe that α is true at least at degree $v(\alpha)$. Moreover, for every $i \leq m$ we denote by $\pi(i)$ the proof $\alpha_1, \ldots, \alpha_i$. Then *the valuation* $\mathrm{Val}(\pi, v)$ of π with respect to v is defined by induction on m by

$$\mathrm{Val}(\pi, v) = a(\alpha_m) \qquad \text{if } \alpha_m \text{ is assumed as a logical axiom}$$

$$\mathrm{Val}(\pi, v) = v(\alpha_m) \qquad \text{if } \alpha_m \text{ is assumed as a proper axiom}$$

$$\mathrm{Val}(\pi, v) = r''(\mathrm{Val}(\pi(i_1), v), \ldots, \mathrm{Val}(\pi(i_n), v)) \qquad \text{if } \alpha_m = r'(\alpha_{i_1}, \ldots, \alpha_{i_n}) \text{ with } i_1 < m, \ldots, i_n < m.$$

Now, unlike the crisp deduction systems, in a fuzzy deduction system different proofs of a same formula α may give different contributions to the degree of validity of α. This suggests setting

$$\mathfrak{D}(v)(\alpha) = \mathrm{Sup}\{\mathrm{Val}(\pi, v) / \pi \text{ is a proof of } \alpha\}. \tag{3.2}$$

This formula defines, for every initial valuation v, a fuzzy subset $\mathfrak{D}(v)$ we call the *fuzzy set of formulas deduced from* v. Also, we call *deduction operator* the function $\mathfrak{D} : \mathfrak{F}(\mathbb{F}) \to \mathfrak{F}(\mathbb{F})$ so defined, i.e., the operator associating any fuzzy subset v of hypotheses with the fuzzy subset $\mathfrak{D}(v)$ of its consequences. The notion of "crisp deduction system," in Hilbert style, can be obtained by some obvious modifications of the notion of fuzzy deduction system. Namely, a *crisp rule of inference* is any partial operation in $\mathbb{F}$, that is a map $r : D \to \mathbb{F}$ such that $D \subseteq \mathbb{F}^n$, $n \in N$. A *crisp deduction system* is a pair $\mathscr{S} = (\mathbb{A}, \mathbb{R})$ such that $\mathbb{A}$ is a subset of $\mathbb{F}$, *the set of logical axioms*, and $\mathbb{R}$ a set of crisp inference rules. The notion of a *proof* π under the set X of hypothesis is immediate and we call *deduction operator* the operator $\mathfrak{D}$ defined by

$$\mathfrak{D}(X) = \{\alpha \in \mathbb{F} \mid \text{a proof of } \alpha \text{ exists whose hypothesis are in } X\}. \tag{3.3}$$

Sometimes we write $X \vdash \alpha$ to denote that $\alpha \in \mathfrak{D}(X)$, $\alpha_1, \ldots, \alpha_n \vdash \alpha$ to denote that $\alpha \in \mathfrak{D}(\{\alpha_1, \ldots, \alpha_n\})$ and $\vdash \alpha$ to denote that $\alpha \in \mathfrak{D}(\emptyset)$. In Ref. 4 a way to extend any crisp deduction system $\mathscr{S}$ into a fuzzy deduction system $\mathscr{S}^*$ is proposed and examined. Indeed, given an n-ary crisp rule of inference r, we say that the fuzzy rule $r^* = (r', r'')$ is the *canonical extension* of r if $r' = r$ and $r''(x_1, \ldots, x_n) = x_1 \wedge \cdots \wedge x_n$. Given a crisp deduction system $\mathscr{S} = (\mathbb{A}, \mathbb{R})$ we call *canonical extension* of $\mathscr{S}$ the fuzzy deduction system $\mathscr{S}^* = (a^*, \mathbb{R}^*)$ where

—a^* is the characteristic function of $\mathbb{A}$
—$\mathbb{R}^* = \{r^* \mid r \in \mathbb{R}\}$.

One proves that the deduction operator $\mathfrak{D}^*$ of $\mathscr{S}^*$ can be obtained by the formula

$$\mathfrak{D}^*(v)(\alpha) = \begin{cases} 1 & \text{if } \alpha \in \mathrm{Tau} \\ \mathrm{Sup}\{v(\gamma_1) \wedge \cdots \wedge v(\gamma_m) \mid \gamma_1, \ldots, \gamma_m \vdash a\} & \text{otherwise} \end{cases} \tag{3.4}$$

where Tau is the set of the *tautologies*, i.e., the formulas α such that $\vdash \alpha$. The following proposition is immediate (see Ref. 4).

PROPOSITION 3.1. *If $\mathscr{L}$ is the classical propositional calculus, then for every fuzzy subset v and $\alpha, \beta \in \mathbb{F}$*

(j) $\mathfrak{D}^*(v)(\alpha) = 1$ *for every tautology* α
(jj) α *logically equivalent to* $\beta \Rightarrow \mathfrak{D}^*(v)\alpha) = \mathfrak{D}^*(v)(\beta)$
(jjj) $\mathfrak{D}^*(v)(\alpha \wedge \beta) = \mathfrak{D}^*(v)(a) \wedge \mathfrak{D}^*(v)(\beta)$.

4. SIMILARITY LOGIC AND THE EXTENSION PRINCIPLE

In this section we will prove that the logic of the approximate premises proposed by Ying can be reduced to the canonical extension of a crisp logic as defined in the previous section. We refer ourselves to a similarity relation R in order to agree with Ying's definitions and since this is a meaningful case. However, all the results can be proved under the hypothesis that R is a fuzzy preorder. Let $\mathscr{S} = (\mathbb{A}, \mathbb{R})$ be a crisp deduction system and let R denote a similarity relation in $\mathbb{F}$. The idea is that, for every pair α and β of formulas, $R(\alpha, \beta)$ represents the degree at which α can be considered similar to β. Extending (1.1), we associate R with a fuzzy relation $\hat{R}: \mathfrak{F}(\mathbb{F}) \times \mathfrak{P}(\mathbb{F}) \to \mathbb{U}$, by putting

$$\hat{R}(s, B) = \mathrm{Inf}_{\alpha \in B} \mathrm{Sup}_{\beta \in \mathbb{F}} R(\beta, \alpha) \wedge s(\beta), \tag{4.1}$$

i.e.,

$$\hat{R}(s, B) = \mathrm{Inf}_{\alpha \in B} J(s)(\alpha). \tag{4.2}$$

The number $\hat{R}(s, B)$ gives the extent at which each formula of B is similar to a formula of s. Obviously, the map $N: \mathfrak{P}(\mathbb{F}) \to \mathbb{U}$ defined by setting $N(X) = \hat{R}(s, X)$ for every $X \in \mathfrak{P}(\mathbb{F})$, is the possibility measure whose distribution function is $J(s)$. Note that if R is the crisp equivalence relation $\equiv$ then,

$$\hat{R}(s, B) = \mathrm{Inf}_{\alpha \in B} \mathrm{Sup}\{ s(\beta) \mid \beta \equiv \alpha \}.$$

If R is the identity,

$$\hat{R}(s, B) = \mathrm{Inf}_{\alpha \in B} s(\alpha).$$

If s is the crisp set A, then

$$\hat{R}(A, B) = 1 \Leftrightarrow \overline{B} \subseteq \overline{A}.$$

In particular if R is the (crisp) identity, then $\hat{R}$ is the (characteristic function of the) inclusion relation.

In the following we extend (1.2):

DEFINITION 4.1. *Let $\mathscr{S} = (\mathbb{A}, \mathbb{R})$ be a crisp deduction system and R a similarity relation in $\mathbb{F}$. The fuzzy consequence relation* Con: $\mathfrak{F}(\mathbb{F}) \times \mathbb{F} \to \mathbb{U}$ *associated with R and $\mathscr{L}$ is defined by*

$$\mathrm{Con}(s, \alpha) = \mathrm{Sup}\left\{ \hat{R}(s \cup \mathbb{A}, B) \mid B \subseteq \mathbb{F}, B \vdash \alpha \right\} \tag{4.3}$$

where s is a fuzzy set of formulas and α a formula.

Obviously, due to the compactness of the relation $\vdash$, the set B in (4.3) can be assumed to be finite. The meaning of $\mathrm{Con}(s,\alpha)$ is immediate, it represents the degree at which we can prove α by using formulas that are similar to formulas in s or in $\mathbb{A}$. Note that if R is the identity relation, then

$$\mathrm{Con}(s,\alpha) = \mathfrak{D}^*(s)(\alpha) \tag{4.4}$$

where $\mathfrak{D}^*$ is the deduction operator of the canonical extension of $\mathscr{S}$. Consequently $\mathfrak{D}^*$, as operating on s and α, can be viewed as a particular fuzzy consequence relation. In order to extend (4.4), set

$$H(s) = J(s \cup \mathbb{A}) = J(s) \cup J(\mathbb{A}). \tag{4.5}$$

It is immediate that H is a closure operator. We interpret $H(s)$ as the fuzzy subset of formulas that are similar either to a formula of s or to a logical axiom.

THEOREM 4.2. *Let s be a fuzzy set of formulas. Then*

$$\mathrm{Con}(s,\cdot) = (\mathfrak{D}^* \circ H)(s). \tag{4.6}$$

Proof. We have that $\hat{R}(s \cup \mathbb{A}, B) = \mathrm{Inf}_{u \in B} H(s)(u)$ and therefore

$$\begin{aligned}\mathrm{Con}(s,q) &= \mathrm{Sup}\{H(s)(\alpha_1) \wedge \cdots \wedge H(s)(\alpha_n) \mid \alpha_1,\ldots,\alpha_n \vdash q\}\\ &= \mathfrak{D}^*(H(s))(q).\end{aligned}$$ ■

In particular, we obtain the following representation of Ying's consequence relation expressed by formula (1.2).

COROLLARY 4.3. *Let $\mathscr{S}$ be the classical propositional calculus, X a set of formulas and α a formula. Then*

$$\mathrm{Con}(X,\alpha) = \mathfrak{D}^*(J(X) \cup J(\mathbb{A}))(\alpha).$$

Note that even if both $\mathfrak{D}^*$ and H are closure operators, the composition $\mathfrak{D}^* \circ H$ only satisfies properties (i) and (ii) but is not a closure operator, in general. Nevertheless, it is possible to prove that, by confining ourselves to the similarity relations considered by M. Ying,[1] such a composition is a closure operator (see Ref. 6).

5. HOW PAVELKA'S DEDUCTION CAN BE REDUCED TO SIMILARITY DEDUCTION

Corollary 4.3 shows that Ying's approach can be reduced to fuzzy logic in Pavelka's sense. Conversely, let $\mathscr{S}$ be a crisp deduction system and $\mathfrak{D}^*$ the deduction operator of the canonical extension $\mathscr{S}^*$. We will see that, given an initial valuation of the formulas v, there are many ways to define a similarity relation and a crisp set of proper axioms, A, such that

$$\mathrm{Con}(A,\alpha) = \mathfrak{D}^*(v)(\alpha).$$

Obviously, (4.4) entails that, if R is the identity, then $\mathfrak{D}^*(v)(\alpha)$ coincides with $\mathrm{Con}(v, \alpha)$. So, the difficulty arises from the requirement that Con is to be applied to a crisp set of proper axioms as in the original Ying's definition.

Suppose that $\mathscr{S}$ is the classical calculus and therefore that the connective $\Leftrightarrow$ is defined. Then given an initial valuation v, we define the fuzzy relation R in the following way:

$$R(x, y) = \mathfrak{D}^*(v)(x \Leftrightarrow y). \tag{5.1}$$

PROPOSITION 5.1. *The relation R defined by* (5.1) *is a similarity relation such that, given any tautology τ,*

$$R(\alpha, \tau) = \mathfrak{D}^*(v)(\alpha), \tag{5.2}$$

for every formula α. Moreover, R is compatible with the logical equivalence, i.e.,

$$x \equiv x', \quad y \equiv y' \Rightarrow R(x, y) = R(x', y') \quad \textit{and} \quad x \equiv x' \Rightarrow R(x, x') = 1.$$

Proof. In virtue of Proposition 3.1, we have, for every $x, y, x \in \mathbb{F}$

$$\begin{aligned} R(x, x) &= \mathfrak{D}^*(v)(x \Leftrightarrow x) = \mathfrak{D}^*(v)(\tau) = 1; \\ R(x, y) &= \mathfrak{D}^*(v)(x \Leftrightarrow y) = \mathfrak{D}^*(v)(y \Leftrightarrow x) = R(y, x); \\ R(x, y) \wedge R(y, z) &= \mathfrak{D}^*(v)(x \Leftrightarrow y) \wedge \mathfrak{D}^*(v)(y \Leftrightarrow z) \\ &= \mathfrak{D}^*(v)(x \Leftrightarrow y \wedge y \Leftrightarrow z) \leq \mathfrak{D}^*(v)(x \Leftrightarrow z) \\ &= R(x, z). \end{aligned}$$

Consider now a tautology τ, since $\alpha \Leftrightarrow \tau$ is logically equivalent to α,

$$R(\alpha, \tau) = \mathfrak{D}^*(v)(\alpha \Leftrightarrow \tau) = \mathfrak{D}^*(v)(\alpha).$$

To prove the remaining part of the proposition assume that $x \equiv x'$ and $y \equiv y'$. Then $(x \Leftrightarrow y) \equiv (x' \Leftrightarrow y')$, and, since $\mathfrak{D}^*(v)$ is compatible with the logical equivalence, $\mathfrak{D}^*(v)(x \Leftrightarrow y) = \mathfrak{D}^*(v)(x' \Leftrightarrow y')$. By setting $y = x'$ and $y' = x'$ in such an equality, one obtains that $R(x, x') = \mathfrak{D}^*(v)(x' \Leftrightarrow x') = 1$. ■

Note that the similarity relations considered by Ying are not compatible with the logical equivalence.

THEOREM 5.2. *Let v be an initial valuation and let R be the similarity relation associated to it by* (5.1). *Then if A is a nonempty set of tautologies, we have, for every formula α*

$$\mathrm{Con}(A, \alpha) = \mathfrak{D}^*(v)(\alpha) \tag{5.3}$$

Proof. At first observe that by (5.2), for every set B of formulas

$$\hat{R}(A, B) = \mathrm{Inf}_{\beta \in B}\mathrm{Sup}_{\alpha \in A} R(\alpha, \beta) = \mathrm{Inf}_{\beta \in B}\mathfrak{D}^*(v)(\beta).$$

Then

$$\begin{aligned}\mathrm{Con}(A, \alpha) &= \mathrm{Sup}\{\hat{R}(A, B) \mid B \vdash \alpha\}\\ &= \mathrm{Sup}\{\mathfrak{D}^*(v)(\alpha_1) \wedge \cdots \wedge \mathfrak{D}^*(v)(\alpha_n) \mid \alpha_1, \dots, \alpha_n \vdash \alpha\}\\ &= \mathfrak{D}^*(v)(\alpha)\end{aligned}$$

and the theorem is proven. ∎

Unfortunately, the relation R defined by (5.1) depends on the particular valuation v, and this cannot be avoided, obviously. If the t-norm we consider in $\mathbb{U}$ is the Lukasievich t-norm $x \otimes y = \max(0, x + y - 1)$, a way to define a similarity relation, starting from an initial valuation v, is the following:

$$R(x, y) = 1 - |v(x) - v(y)| \tag{5.4}$$

We have that:

$$\begin{aligned}R(x, x) &= 1; \quad R(x, y) = R(y, x);\\ R(x, y) \otimes R(y, z) &= \max(0, 1 - |v(x) - v(y)| + 1 - |v(y) - v(z)| - 1)\\ &= \max\{0, 1 - (|v(x) - v(y)| + |v(y) - v(z)|)\}\\ &\leq 1 - |v(x) - v(z)|\\ &= R(x, z)\end{aligned}$$

and so R is a similarity relation.

THEOREM 5.3. *Let v be an initial valuation and let R be the associated similarity relation defined by* (5.4). *Then, if* $A = \{x \in \mathbb{F} \mid v(x) = 1\}$ *is nonempty, we have that, for every* $p \in \mathbb{F}$,

$$\mathrm{Con}(A, p) = \mathfrak{D}^*(v)(p). \tag{5.5}$$

Proof. Observe that, if $B = \{\alpha_1, \dots, \alpha_n\}$,

$$\hat{R}(A, B) = \mathrm{Inf}_{i=1,\dots,n}\mathrm{Sup}\{R(q, \alpha_i) \mid q \in A\}.$$

Since for $q \in A$ we have $R(q, \alpha_i) = 1 - |1 - v(\alpha_i)| = v(\alpha_i)$, it is

$$\begin{aligned}\mathrm{Con}(A, p) &= \mathrm{Sup}\{\hat{R}(A, B) \mid B \vdash p\}\\ &= \mathrm{Sup}\{v(\alpha_1) \wedge \cdots \wedge v(\alpha_n) \mid \alpha_1, \dots, \alpha_n \vdash p\}.\end{aligned}$$ ∎

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