Note: An extended version of this paper was published in *Fuzzy sets in approximate reasoning and information systems*, edited by J. C. Bezdek, D. Dubois and H. Prade, Kluwer Academic Publishers (1998).

CLOSURE OPERATORS IN FUZZY SET THEORY

Loredana Biacino
Giangiacomo Gerla

Abstract: In accordance with Tarski point of view, in this chapter the theory of closure operators is proposed as a unifying tool for fuzzy logics. Indeed, let F be the set of formulas of a given language. Then an abstract fuzzy logic is defined by a fuzzy semantics (i.e. a class of valuations of the formulas in F) and by a closure operator in the lattice of the fuzzy subsets of F (we call deduction operator). One proves that Pavelka's logic, similarity logic and graded consequence theory can be represented in this way.

3.1 INTRODUCTION.

Let U be a nonempty set and denote by $P(U)$ the lattice of all the subsets of U. Then a *closure operator* on U is any map J:$P(U)\to P(U)$ such that,

$$X\subseteq Y \Rightarrow J(X)\subseteq J(Y)\ ;\ X\subseteq J(X)\ ;\ J(J(X))=J(X),$$

for every X and Y subsets of U. The theory of closure operators is a very useful tool in several areas of classical mathematics and, in particular, in (crisp) mathematical logic. Indeed, in account of the fact that the deduction operator of a monotonic logic is a closure operator, A. Tarski, D. J. Brown, R. Suszko and other authors proposed a general approach in which a logic is seen as a pair (F,D) where F is the set of formulas in a given language and D:$P(F)\to P(F)$ is a closure operator (see, e.g., [Tarski 1956] and [Brown; Suszko 1973]).

In this chapter we will show that it is possible to extend such an abstract approach to fuzzy logic. To do this, we start from the notion of a closure operator in a complete lattice L (see, e.g., [Ward 1942]). Obviously, we are mainly interested in the case in which L is the lattice of the fuzzy subsets of the set of formulas of a given language. The resulting theory enables us to give a unified treatment of several different approaches to fuzzy logic (such as Pavelka's logic, similarity logic, graded consequence theory). It is worth noticing that the proposed extension of the theory of closure operators to the fuzzy framework is useful not only for fuzzy logic but also for many other branches of fuzzy set theory. Indeed, it gives an elegant and powerful way to treat notions such as those of fuzzy topologies, fuzzy subalgebras, necessity measures and envelopes (see, e.g., [Conrad 1980]), [Biacino; Gerla 1984], [Murali 1991], [Biacino; Gerla 1992], [Biacino 1993]).

3.2 CLOSURE OPERATORS IN A LATTICE

Let L be a complete lattice whose minimum and maximum we denote by 0 and 1, respectively. An operator $J:L\to L$ is called a *closure operator* if, for every $x,y\in L$,

(i) $x\geq y \Rightarrow J(x)\geq J(y)$ (monotony)
(ii) $x\leq J(x)$ (inclusion)
(iii) $J(J(x))=J(x)$ (idempotence).

We call an *almost closure operator* in L, in brief *a-c-operator*, an operator J satisfying (i) and (ii). A class C of elements of L is called a *closure system* if the meet of any family of elements of C is an element of C. Every closure system C is a complete lattice in which the meets are the same as in L but the join of a subset X of C is the meet of the set of elements of C that are greater or equal to every element of X. So a closure system is not a sublattice of L, in general; as an example, consider the class of closed subsets of an Euclidean space. Moreover, given a class $C\subseteq L$, we define the operator $\boldsymbol{J}(C):L\to L$ by setting,

$$\boldsymbol{J}(C)(x)=Inf\{y\in C \mid y\geq x\} \tag{3.1}$$

for every $x\in L$. If C is a closure system, then $\boldsymbol{J}(C)(x)$ belongs to C and it is called *the element of C generated by x*. Given an operator J, we set

$$\boldsymbol{C}(J)=\{x\in L \mid J(x)=x\}, \tag{3.2}$$

i.e., $\boldsymbol{C}(J)$ is the set of fixed points of J.

Proposition 3.2.1 Given any $C\subseteq L$, the operator $\boldsymbol{J}(C):L\to L$ defined in (3.1) is a closure operator. Given an a-c-operator J, the class $\boldsymbol{C}(J)\subseteq L$ defined in (3.2) is a closure system.

Proof. It is immediate that $\boldsymbol{J}(C)$ is a closure operator. To prove that $\boldsymbol{C}(J)$ is a closure system observe that, if $(x_i)_{i\in I}$ is any family of elements of $\boldsymbol{C}(J)$ then by the monotony of J, $J(Inf_{i\in I}x_i)\leq x_i$ for every $i\in I$. So we have $J(Inf_{i\in I}x_i)\leq Inf_{i\in I}x_i$ and, since the opposite inequality holds by (ii), it is $J(Inf_{i\in I}x_i)=Inf_{i\in I}x_i$. This proves tht $\boldsymbol{C}(J)$ is a closure system.

The class L^L of the operators in L is a complete lattice with respect to the order relation defined by setting, for every J_1 and J_2 in L^L, $J_1\leq J_2$ provided that $J_1(x)\leq J_2(x)$, for every $x\in L$. The joins and meets in such a lattice are given by setting

$$(J_1 \vee J_2)(x) = J_1(x) \vee J_2(x)\ \ ,\ (J_1 \wedge J_2)(x) = J_1(x) \wedge J_2(x),$$

for any $x \in L$. We have also the following proposition, whose proof is immediate.

Proposition 3.2.2 The class $CO(L)$ of the closure operators in L is a closure system in L^L. The class $CS(L)$ of the closure systems in L is a closure system in $P(L)$.

In accordance with the first part of Proposition 3.2.2, given an operator J, we denote by $\mathrm{c}(J)$ the closure operator generated by J, i.e., the meet of all the closure operators greater than or equal to J.

Proposition 3.2.3 Let J be an a-c-operator. Then

$$\mathrm{c}(J) = \boldsymbol{J}(\boldsymbol{C}(J)), \tag{3.3}$$

i.e., for any x in L, $\mathrm{c}(J)(x)$ is the least fixed point of J greater than or equal to x. In particular, if J is a closure operator, then $J = \boldsymbol{J}(\boldsymbol{C}(J))$.

Proof. Set $J' = \boldsymbol{J}(\boldsymbol{C}(J))$, then, J' is a closure operator. To prove that $J' \geq J$ observe that for every x, if $J'(x) = x'$ we have $x' \geq x$ and $J(x') = x \geq J(x)$, thus $J'(x) \geq J(x)$. Let H be a closure operator such that $H \geq J$. Then $J(H(x)) \leq H(H(x)) = H(x)$; on the other hand $J(H(x)) \geq H(x)$. This proves that $H(x) \in \boldsymbol{C}(J)$ and, since $H(x) \geq x$, that $H(x) \geq J'(x)$.

In accordance with the second part of Proposition 3.2.2, given a system C we denote by $\mathrm{c}(C)$ the closure system generated by C, i.e., the intersection of all the closure systems containing C.

Proposition 3.2.4 Given a class C of elements of L, we have

$$\mathrm{c}(C) = \boldsymbol{C}(\boldsymbol{J}(C)), \tag{3.4}$$

and therefore, $C = \boldsymbol{C}(\boldsymbol{J}(C))$ for every closure system C. Moreover,

$$\mathrm{c}(C) = \{Inf(X) \mid X \subseteq C\}. \tag{3.5}$$

Proof. (3.5) is immediate, to prove (3.4) observe that, since every element of C is a fixed point of $\boldsymbol{J}(C)$, $\boldsymbol{C}(\boldsymbol{J}(C))$ is a closure system containing C. Let C' be a closure system containing C, and x an element of $\boldsymbol{C}(\boldsymbol{J}(C))$. Then, since $x = \boldsymbol{J}(C)(x)$, x is a meet of elements of C and hence belongs to C'. Thus $\boldsymbol{C}(\boldsymbol{J}(C))$ is contained in C' and therefore $\boldsymbol{C}(\boldsymbol{J}(C)) = \mathrm{c}(C)$.

If C and C' are closure systems and J, J' closure operators, then:

$$C \subseteq C' \Leftrightarrow \boldsymbol{J}(C) \geq \boldsymbol{J}(C')\ \ ;\ J \leq J' \Leftrightarrow \boldsymbol{C}(J) \subseteq \boldsymbol{C}(J').$$

Since in Propositions 3.2.3 and 3.2.4 we proved that $J = \boldsymbol{J}(\boldsymbol{C}(J))$ and $C = \boldsymbol{C}(\boldsymbol{J}(C))$, this shows that (3.1) defines a lattice isomorphism from $CS(L)$ on $CO(L)$ and (3.2) defines the inverse isomorphism from $CO(L)$ on $CS(L)$.

Proposition 3.2.5 Let J be an a-c-operator and C a class. Then

$$\boldsymbol{C}(J) = \boldsymbol{C}(\mathrm{c}(J))\ \ ;\ \boldsymbol{J}(C) = \boldsymbol{J}(\mathrm{c}(C)). \tag{3.6}$$

Moreover, let be J_1, J_2 a-c-operators and C_1, C_2 classes. Then

$$\mathbf{c}(J_1) = \mathbf{c}(J_2) \Leftrightarrow \boldsymbol{C}(J_1) = \boldsymbol{C}(J_2)\ \ ;\ \mathbf{c}(C_1) = \mathbf{c}(C_2) \Leftrightarrow \boldsymbol{J}(C_1) = \boldsymbol{J}(C_2). \tag{3.7}$$

Proof. Every element of $\boldsymbol{C}(J)$ is a fixed point of $\boldsymbol{J}(C(J))$=c(J) and this proves that $\boldsymbol{C}(J)\subseteq\boldsymbol{C}(\mathrm{c}(J))$. Conversely, let x be a fixed point of c(J). Then $x=\boldsymbol{J}(\boldsymbol{C}(\boldsymbol{J}(x)))$ $=Inf\{y\in\boldsymbol{C}(J) \mid y\geq x\}$. But $\boldsymbol{C}(J)$ is a closure system, so $x\in\boldsymbol{C}(J)$. Thus $\boldsymbol{C}(J)\supseteq\boldsymbol{C}(\mathrm{c}(J))$ and therefore $\boldsymbol{C}(J)=\boldsymbol{C}(\mathrm{c}(J))$. The remaining part of the proposition is obvious.

3.3 ABSTRACT LOGICS

We define an *abstract deduction system* as a pair (L,D) where L is a complete lattice and D a closure operator in L. The elements in L are called *pieces of information* and D the *deduction operator*. A *theory* is defined as a fixed point of D, i.e., a piece of information τ such that $\tau\geq D(\tau)$. So, the theories are the deductively closed pieces of information. Obviously, τ is a theory iff a piece of information x exists such that $\tau=D(x)$. If this is the case, x is called a *system of axioms* for τ. The theory $D(0)$ is called the *system of tautologies* and is denoted by $Tau(D)$. Since $D(1)=1$, 1 is a theory, we call the *inconsistent theory*. A piece of information $x\in L$ is *consistent* provided that $D(x)$ is different from 1. We define an *abstract semantics* as a nonempty class M of elements of L such that $1\notin M$, and we call *models* the elements in M. If x is a piece of information and $m\in M$, then m is *a model of* x, in brief $m\vDash x$, provided that $x\leq m$. We say that x is *satisfiable* if a model of x exists and we denote by $Sat(M)$ the class of satisfiable pieces of information, i.e.,

$$Sat(M)=\{x\in L \mid m\in M \text{ exists such that } m\vDash x\}.$$

Two pieces of information admitting the same models are said to be *logically equivalent*. In accordance with Proposition 3.2.1, M induces a closure operator we call *logical consequence operator* and we denote by Con_M (or merely by Con). Then Con_M is defined by setting, given a piece of information x,

$$Con_M(x)=Inf\{m\in M \mid m\vDash x\}.$$

We define the system of tautologies of M as

$$Tau(M)=Inf\{m \mid m\in M\},$$

i.e., $Tau(M)=Con_M(0)$.

Definition 3.3.1 An *abstract logic* is an object like (L,D,M) where (L,D) is a deduction system and M a semantics such that $D=Con_M$.

For example, the classical first order logic is an abstract logic in which
- the pieces of information are the sets of formulas (systems of axioms)
- $D(x)$ is the set of formulas we can derive from x
- a theory τ is a set of formulas containing the logical axioms and closed under the inference rules
- a model is identified with the related set of true formulas (and therefore M with the class of complete theories)
- $Con_M(x)$ is the intersection of all the complete theories containing x.

Given any abstract deduction system (L,D) we can define an abstract logic in a trivial way by setting M equal to the class of consistent theories of D. In this sense any deduction system admits an abstract semantics. Nevertheless, such a semantics is unsatisfactory since we look for semantics containing only those theories that are complete systems of information, in a sense.

3.4 CONTINUITY FOR ABSTRACT LOGICS

The deduction operators of the crisp logics are compact, i.e., for every $\alpha \in D(X)$ a finite subset X_f of X exists such that $\alpha \in D(X_f)$. This is an immediate consequence of the fact that a proof involves only a finite number of hypotheses. Since the notion of a finite subset is not defined in a generic lattice L, we have to search for a different notion of compactness. A nonempty class T of elements in L is called *directed* if

$$x \in T,\ y \in T \Rightarrow \exists z \in T,\ x \leq z,\ y \leq z.$$

The chains are typical examples of directed classes. If $z=Sup(T)$ we say that z is *the limit of* T and we write $z=limT$. If J is an order-preserving operator and T is directed, then the image $J(T)=\{J(x) \mid x \in T\}$ is also directed, obviously.

Definition 3.4.1 An operator J is called *continuous* if it is order preserving and, for every directed class T,

$$J(limT)=limJ(T). \tag{3.8}$$

A continuous closure operator is also called an *algebraic* closure operator. One proves that if L is the lattice of all the subsets of a given set, then J is continuous iff J is compact.

Definition 3.4.2 A class C of elements of L is called *inductive* if the limit of every directed family of elements in C belongs to C. An inductive closure system is called *algebraic*.

The following proposition, whose proof we omit, shows that the notion of algebraic closure system is the natural counterpart of the one of algebraic closure operator.

Proposition 3.4.3 Given a nonempty class C,

C is an algebraic closure system $\Leftrightarrow$ $\boldsymbol{J}(C)$ is an algebraic closure operator.

Given a closure operator J

J is algebraic $\Leftrightarrow$ $\boldsymbol{C}(J)$ is an algebraic closure system.

The continuity is a necessary condition for a deduction operator D works well. Indeed, if we are able to approximate an information x with a partial information y such that $y \leq x$, then $D(y)$ has to be a suitable approximation of $D(x)$.

Definition 3.4.4 An abstract deduction system (L,D) (more generally, an abstract logic) is called *continuous* provided that D is continuous.

3.5 STEP-BY-STEP DEDUCTION SYSTEMS

Usually a deduction operator D is obtained by starting from a suitable set A of logical axioms and a suitable set of inference rules. Namely, denote by $J(X)$ the set of formulas that can be obtained by one application of the inference rules to formulas in X, and set

$$H(X)=J(X)\cup A\cup X,$$

i.e., $\alpha\in H(X)$ provided that

- either α is obtained by applying an inference rule to formulas in X,
- or α is a logical axiom
- or α is an element in X.

Also, define H^n by induction on n, by setting $H^1=H$ and $H^{n+1}=H\circ H^n$. Then it is immediate that H is an almost closure operator, D is the closure operator generated by H and that $D(X)=\cup H^n(X)$. Obviously, $H^n(X)$ represents the set of formulas we can obtain from X by an n-step inferential process.

To extend such an approach to abstract logics we at first examine how to obtain the closure operator generated by a continuous a-c-operator.

Proposition 3.5.1 Let H be a continuous a-c-operator. Then the set $\boldsymbol{C}(H)$ of fixed points of H is an algebraic closure system and the closure operator c(H) generated by H is continuous.

Proof. Let T be a directed subclass of $\boldsymbol{C}(H)$. Then, since H is continuous

$$H(Sup(\{x \mid x\in T\}))=Sup(\{H(x) \mid x\in T\})=Sup(\{x \mid x\in T\})$$

and, hence, $Sup(\{x \mid x\in T\})\in\boldsymbol{C}(H)$. This proves that $\boldsymbol{C}(H)$ is inductive. Thus, since by Proposition 3.2.3 $c(H)=\boldsymbol{J}(\boldsymbol{C}(H))$, by Propositions 3.4.3 we can conclude that $c(H)$ is algebraic.

If H is a continuous a-c-operator the following famous theorem enables us to calculate the closure operator $c(H)$ generated by H.

Theorem 3.5.2 (Fixed-point Theorem) Let H be a continuous a-c-operator. Then

$$\mathrm{c}(H)=Sup_{n\in N}H^n\ . \tag{3.9}$$

Proof. We have to prove that, for every $x\in L$, $Sup_{n\in N}H^n(x)$ is the least fixed point of H greater than or equal to x. Now, since $H(x)\geq x$ we have also that $H^{n+1}(x)\geq H^n(x)$ for every n, and hence the family $(H^n(x))_{n\in N}$ is directed. Since H is continuous

$$H(Sup_{n\in N}H^n(x))=Sup_{n\in N}H^{n+1}(x)=Sup_{n\in N}H^n(x)$$

and $Sup_{n\in N}H^n(x)$ is a fixed point for H greater than or equal to x. If y is any fixed point such that $y\geq x$, then for every $n\in N$, $y=H^n(y)\geq H^n(x)$ and hence $y\geq Sup_{n\in N}H^n(x)$. This proves that $Sup_{n\in N}H^n(x)=\mathrm{c}(H)(x)$.

In accordance with the above considerations, we propose the following definition.

Definition 3.5.3 A *step-by-step deduction system* is an object like (L,J,a) where
- J is a continuous operator (the *immediate consequence* operator)
- a is an element of L (the *system of logical axioms*).

Let J be a continuous operator and define H by setting

$$H(x)=J(x)\vee a\vee x \qquad (3.10)$$

for every $x\in L$. Then it is easily seen that H is a continuous a-c-operator. $H(x)$ corresponds to the set of formulas that either are immediate consequences of x, or are logical axioms or are hypotheses (i.e. elements in x).

Definition 3.5.4 Let (L,J,a) be a step-by-step-deduction system, define H by (3.10) and denote by D the closure operator generated by H. Then (L,D) is called *the deduction system associated with* (L,J,a).

The following proposition is an obvious consequence of the Fixed-point Theorem.

Proposition 3.5.5 Let (L,J,a) be a step-by-step-deduction system and (L,D) the deduction system associated with (L,J,a). Then D is algebraic and

$$D(x)=Sup_{n\in N}H^n(x). \qquad (3.11)$$

Moreover, τ is a theory of (L,D) iff

$$\text{(i) } \tau\geq J(\tau) \quad \text{and} \quad \text{(ii) } \tau\geq a. \qquad (3.12)$$

Then, while for every integer n, $H^n(x)$ represents the information available in n-steps, $Sup_{n\in N}H^n(x)$ represents the whole information we can derive from x.

3.6 LOGICAL COMPACTNESS

A closure operator D is said to be *logically compact*, in brief *l-compact*, provided that the related class of consistent pieces of information is inductive. A semantics M is called *logically compact* provided that the related logical consequence operator C is logically compact, i.e., the class Sat(M) of satisfiable pieces of information is inductive. Note that logical compactness is different from continuity. Indeed, while the deduction operator of a fuzzy logic in Hilbert style is always continuous, there are very interesting examples of such logics whose deduction operator is not logically compact (as an example, see the logic of the necessities examined in [Biacino; Gerla 1992]).

Proposition 3.6.1 Let D be a deduction operator. Then the following are equivalent
(a) D is continuous and logically compact
(b) the class of consistent theories is inductive.

Proof. (a)$\Rightarrow$(b). Let T be a directed family of consistent theories and denote by τ its limit. Then, since D is continuous, τ is a theory and, since D is logically compact, τ is consistent.
(b)$\Rightarrow$(a). In order to prove that D is continuous we prove that the class of theories is inductive. Let T be any directed family of theories and denote by τ its limit. Then, if all the elements in T are consistent, we have that τ is a consistent theory. If the inconsistent theory 1 belongs to T, then $\tau=1$. So, in any case τ is a theory. Let $(x_i)_{i\in I}$ be a directed family of consistent pieces of information and denote by x its limit. Then $(D(x_i))_{i\in I}$ is a directed family of consistent theories and hence its limit τ is a consistent theory. Since $x\leq\tau$, this proves that x is consistent. Thus, D is logically compact.

The interest of the logically compact semantics is expressed in the following propositions.

Proposition 3.6.2 If M is logically compact, then every satisfiable piece of information admits a maximal model. Equivalently, every element of M is contained in a maximal element in M.

Proof. Assume that x is satisfiable, then, since Sat(M) is inductive, the class $C=\{y\in\text{Sat}(M) \mid y\geq x\}$ is inductive. By Zorn's Lemma, a maximal element z of C exists. Since z is satisfiable, $m\in M$ exists such that $m\geq z$. Since $m\in C$, by the maximality of z we can conclude that $z=m$. This proves both that z belongs to M and that z is a maximal element in M.

In the same way one proves that:

Proposition 3.6.3 If D is logically compact every consistent piece of information is contained in a maximal theory.

3.7 BASIC NOTIONS IN FUZZY SET THEORY

Denote by $\vee$ and $\wedge$ the maximum and the minimum operations in [0,1] and by $\neg$ the unary operation defined by setting $\neg(x)=1-x$ for $x\in[0,1]$. Then $([0,1],\vee,\wedge,\neg)$ is a complete lattice with an involution. Given a set U, we denote by $(F(U),\cup,\cap,c)$ the direct power of $([0,1],\vee,\wedge,\neg)$ with index set U and we call *fuzzy subsets* the elements of $F(U)$. The operations $\cup$, $\cap$ and c are called *union*, *intersection*, *complement*, respectively. Then a fuzzy subset is a map $A:U\rightarrow[0,1]$ from U to [0,1] and, if A, B are fuzzy subsets of U,

$$(A\cup B)(x)=A(x)\vee B(x) \;;\; (A\cap B)(x)=A(x)\wedge B(x) \;;\; A^c(x)=1-A(x).$$

Notice that we prefer the notation A^c instead of the prefix notation $c(A)$. A fuzzy subset A is called *crisp* provided that either $A(x)=0$ or $A(x)=1$. If X is a subset of U, then we denote by X the characteristic function of X, too, i.e. we denote by X the map defined by setting $X(x)=1$ if $x\in X$ and $X(x)=0$ if $x\notin X$. The following proposition summarizes the main properties of the class of fuzzy subsets.

Proposition 3.7.1 $(F(U),\cup,\cap,c)$ is a complete lattice with an involution extending the Boolean algebra $(P(U),\cup,\cap,c)$. More precisely, the map associating every $X\in P(U)$ with the related characteristic function is a complete monomorphism from $(P(U),\cup,\cap,c)$ into $(F(U),\cup,\cap,c)$.

Consequently, we can identify the subsets of U with the crisp fuzzy subsets. For every $\lambda\in[0,1]$, we denote by U^λ the fuzzy subset constantly equal to λ. Obviously, in such a way U^1 is (the characteristic function of) U and U^0 is (the characteristic function of) the empty set. Given a fuzzy subset A of U, for every $\lambda\in[0,1]$ the subsets

$$A_\lambda=\{x\in U \mid A(x)\geq\lambda\} \;;\; A_{>\lambda}=\{x\in U \mid A(x)>\lambda\}$$

are called *the closed λ-cut* and the *open λ-cut* of A, respectively. In the propositions below we summarize some basic properties of the cuts.

Proposition 3.7.2 Let A and B be two fuzzy subsets, $(A_i)_{i\in I}$ a family of fuzzy subsets and $\lambda\in[0,1]$. Then

(a) $A_0=U$	(b) $\lambda\leq\lambda' \Rightarrow A_\lambda\supseteq A_{\lambda'}$
(c) $A\subseteq B \Rightarrow A_\lambda\subseteq B_\lambda$	(d) $A_\lambda=\bigcap_{x<\lambda}A_{>x}$
(e) $(A\cup B)_\lambda=A_\lambda\cup B_\lambda$	(f) $(A\cap B)_\lambda=A_\lambda\cap B_\lambda$
(g) $A_\lambda=\bigcap_{x<\lambda}A_x$	(h) $(\bigcap A_\mathrm{i})_\lambda=\bigcap(A_\mathrm{i})_\lambda$
(a') $A_{>1}=\varnothing$	(b') $\lambda\leq\lambda' \Rightarrow A_{<\lambda}\supseteq A_{<\lambda'}$
(c') $A\subseteq B \Rightarrow A_{>\lambda}\subseteq B_{>\lambda}$	(d') $A_{>\lambda}=\bigcup_{x>\lambda}A_x$
(e') $(A\cup B)_{>\lambda}=A_{>\lambda}\cup B_{>\lambda}$	(f') $(A\cap B)_{>\lambda}=A_{>\lambda}\cap B_{>\lambda}$
(g') $A_{>\lambda}=\bigcup_{x>\lambda}A_{>x}$	(h') $(\bigcup A_i)_{>\lambda}=\bigcup(A_i)_{>\lambda}$.

For every $\lambda\in[0,1]$ and $X\subseteq U$, we denote by $\lambda\wedge X$ the fuzzy subset $U^\lambda\cap X$, that is

$(\lambda\wedge X)(x)=\lambda$ if $x\in X$;
$(\lambda\wedge X)(x)=0$ otherwise.

Dually, we define by $\lambda\vee X$ by setting

$(\lambda\vee X)(x)=1$ if $x\in X$
$(\lambda\vee X)(x)=\lambda$ otherwise.

The following proposition shows that a fuzzy subset is characterized both by its closed cuts and its open cuts.

Proposition 3.7.3 For every fuzzy subset A

$$A=\bigcup\lambda\wedge A_\lambda \tag{3.13}$$

$$A=\bigcup\lambda\wedge A_{>\lambda} \tag{3.14}$$

and, dually,

$$A=\bigcap\lambda\vee A_\lambda \tag{3.15}$$

$$A=\bigcap\lambda\vee A_{>\lambda}. \tag{3.16}$$

Proof. It is sufficient to observe that it is possible to rewrite the above equalities as follows

$$A(x)=Sup\{\lambda\in[0,1] \mid x\in A_\lambda\} \tag{3.17}$$
$$A(x)=Sup\{\lambda\in[0,1] \mid x\in A_{>\lambda})\} \tag{3.18}$$
$$A(x)=Inf\{\lambda\in[0,1] \mid x\notin A_\lambda\} \tag{3.19}$$
$$A(x)=Inf\{\lambda\in[0,1] \mid x\notin A_{>\lambda}\} \tag{3.20}$$

where $x\in U$.

It is possible to identify the fuzzy subsets with the continuous chains as was proposed in [Negoita; Ralescu 1975]. Consider the class $P(U)^{[0,1]}$ whose elements are the families of subsets of U with index set [0,1]. Such a class can be view as the direct power of the complete lattice $P(U)$ with index set [0,1]. Therefore $P(U)^{[0,1]}$ is a complete lattice whose order relation is defined by setting, for every $(A_\lambda)_{\lambda\in[0,1]}$ and $(B_\lambda)_{\lambda\in[0,1]}$ in $P(U)^{[0,1]}$

$$(A_\lambda)_{\lambda\in[0,1]}\leq(B_\lambda)_{\lambda\in[0,1]} \Leftrightarrow A_\lambda\subseteq B_\lambda \text{ for every } \lambda\in[0,1].$$

Further, the join and the meet of these two families are $(A_\lambda\cup B_\lambda)_{\lambda\in[0,1]}$, and $(A_\lambda\cap B_\lambda)_{\lambda\in[0,1]}$, respectively. One similarly defines the infinite joins and meets. We are interested in a particular class of elements of $P(U)^{[0,1]}$.

Definition 3.7.4 We call a *chain* in U any order-reversing family $(C_\lambda)_{\lambda\in[0,1]}$ of subsets of U such that $C_0=U$ and we denote by $Ch(U)$ the set of chains in U. We say that a chain $(C_\lambda)_{\lambda\in[0,1]}$ is *continuous* if

$$C_\lambda=\bigcap_{x<\lambda}C_x \tag{3.21}$$

for every $\lambda\in[0,1]$. We denote by $CCh(U)$ the class of continuous chains.

The family of closed cuts of a given fuzzy set is an example of continuous chain.

Proposition 3.7.5 The class $Ch(U)$ of chains of subsets of U is a closure system in $P(U)^{[0,1]}$. Let $(C_\lambda)_{\lambda\in[0,1]}$ be any family of subsets of U. Then the chain $(C^*_\lambda)_{\lambda\in[0,1]}$ generated by $(C_\lambda)_{\lambda\in[0,1]}$ can be obtained by setting $C^*_0=U$ and, for $\lambda\neq0$,

$$C^*_\lambda=\bigcup_{x\geq\lambda}C_x \tag{3.22}$$

Proof. The first part of the proposition is obvious. It is immediate that $(C^*_\lambda)_{\lambda\in[0,1]}$ is a chain containing $(C_\lambda)_{\lambda\in[0,1]}$. Let $(A_\lambda)_{\lambda\in[0,1]}$ be a chain containing $(C_\lambda)_{\lambda\in[0,1]}$. Then, given $\lambda\in[0,1]$, for every $\mu\geq\lambda$, $A_\lambda\supseteq A_\mu\supseteq C_\mu$ and therefore $A_\lambda\supseteq C^*_\lambda$.

Proposition 3.7.6 The class $CCh(U)$ of the continuous chains is a closure system in $Ch(U)$ (hence in $P(U)^{[0,1]}$). Let $(C_\lambda)_{\lambda\in[0,1]}$ be a chain and let $(C'_\lambda)_{\lambda\in[0,1]}$ be the continuous chain generated by $(C_\lambda)_{\lambda\in[0,1]}$. Then, for every $\lambda\in[0,1]$

$$C'_\lambda=\bigcap_{x<\lambda}C_x. \tag{3.23}$$

Proof. Let I be a set and, for every $i\in I$, let $(C_\lambda^i)_{\lambda\in[0,1]}$ be a continuous chain. Then the intersection of such a family of continuous chains is the chain $(C_\lambda)_{\lambda\in[0,1]}$ defined by $C_\lambda=\bigcap_{i\in I}C_\lambda^i$. Since

$$\bigcap_{x<\lambda}C_x=\bigcap_{x<\lambda}(\bigcap_{i\in I}C_x^i)=\bigcap_{i\in I}(\bigcap_{x<\lambda}C_x^i)=\bigcap_{i\in I}C_\lambda^i=C_\lambda\ ,$$

$(C_\lambda)_{\lambda\in[0,1]}$ is a continuous chain. Suppose $(C_\lambda)_{\lambda\in[0,1]}$ is a chain. Then, since $C_\mu\supseteq C_\lambda$ for every $\lambda>\mu$, we have $C'_\lambda\supseteq C_\lambda$. In order to prove that $(C'_\lambda)_{\lambda\in[0,1]}$ is a continuous chain, observe that

$$\bigcap_{x<\lambda}C'_x=\bigcap_{x<\lambda}(\bigcap_{y<x}C_y)=\bigcap_{y<\lambda}C_y=C'_\lambda.$$

Finally, let $(A_\lambda)_{\lambda\in[0,1]}$ be a continuous chain containing $(C_\lambda)_{\lambda\in[0,1]}$, then

$$A_\lambda=\bigcap_{x<\lambda}A_x\supseteq\bigcap_{x<\lambda}C_x=C'_\lambda.$$

The following definition enables us to associate any family of subsets of U with a fuzzy subset of U.

Definition 3.7.7. Let $(C_\lambda)_{\lambda\in[0,1]}$ be any family of subsets of U and set

$$A=\bigcup\lambda\wedge C_\lambda. \tag{3.24}$$

Then A is said to be the *fuzzy subset associated with* $(C_\lambda)_{\lambda\in[0,1]}$.

The proof of the following propositions is matter of routine.

Proposition 3.7.8 Let $(C_\lambda)_{\lambda\in[0,1]}$ be any family of subsets of U and A the associated fuzzy subset. Then, both the chain $(C^*_\lambda)_{\lambda\in[0,1]}$ and the continuous chain $(C'_\lambda)_{\lambda\in[0,1]}$ generated by $(C_\lambda)_{\lambda\in[0,1]}$ define the same fuzzy subset A. Moreover,

$$A_{>\lambda}=\bigcup_{x>\lambda}C_x\subseteq C^*_\lambda\subseteq C'_\lambda=A_\lambda. \tag{3.25}$$

In particular, if $(C_\lambda)_{\lambda\in[0,1]}$ is a chain, the fuzzy subset associated with $(C_\lambda)_{\lambda\in[0,1]}$ by Equation (3.24) coincides with the fuzzy subset associated with $(C'_\lambda)_{\lambda\in[0,1]}$.

Proposition 3.7.9 Let $(C_\lambda)_{\lambda\in[0,1]}$ be any chain of subsets of U and define A by (3.24). Then we have also that

$$A=\bigcap\lambda\vee C_\lambda. \tag{3.26}$$

Moreover,

$$A_{>\lambda}=\bigcup_{x>\lambda}C_x\subseteq C_\lambda\subseteq\bigcap_{x<\lambda}C_x=A_\lambda. \tag{3.27}$$

If $(C_\lambda)_{\lambda\in[0,1]}$ is continuous, then, for every $\lambda\in[0,1]$, $A_\lambda=C_\lambda$.

The following theorem shows that we can identify the lattices $F(U)$ and $CCh(U)$ (see [Negoita; Ralescu 1975]).

Theorem 3.7.10 The correspondence $h:F(U)\to CCh(U)$ defined by setting, for every $A\in F(U)$

$$h(A)=(A_\lambda)_{\lambda\in[0,1]} \tag{3.28}$$

is a lattice isomorphism between $F(U)$ and $CCh(U)$. Moreover, the inverse map $h^{-1}:CCh(U)\to F(U)$ associates every continuous chain $(C_\lambda)_{\lambda\in[0,1]}$ with the fuzzy subset A defined by (3.24).

3.8 ABSTRACT FUZZY LOGIC

Let F be a set whose elements we call *formulas*. We call an *abstract crisp deduction system*, (*crisp semantics*, *crisp logic*) any abstract deduction system (semantics, logic) in the lattice of the subsets of F. Then, *a crisp semantics* is any class M of subsets of formulas such that $F\notin M$. Also, given a set X of formulas and an element M of M, M is a model of X provided that $X\subseteq M$, and we define the set $\boldsymbol{C}(M)(X)$ of the logical consequences of X by

$$\boldsymbol{C}(M)(X)=\bigcap\{M\in M \mid M\supseteq X\}.$$

An *abstract crisp logic* is an object like (F,D,M) such that $D=\boldsymbol{C}(M)$ and the elements of the set

$$Tau(\mathcal{M})=\bigcap\{M \mid M\in\mathcal{M}\},$$

are called *tautologies*. Also, the complement operation c enables us to define the set

$$Contr(\mathcal{M})=\bigcap\{M^c \mid M\in\mathcal{M}\}$$

whose elements we call *contradictions*. Then, a contradiction is a formula that is false in any model. We can justify the above definitions as follows. Consider a classical logic, then we can identify the class of possible models with the class $\mathcal{M}$ of the complete theories. Indeed, we associate each model M with the complete theory $T_M=\{\alpha\in F \mid \alpha$ is true in $M\}$ and, conversely, for every complete theory T a model M exists such that $T_M=T$. It is also immediate that M is a model of a set X of formulas iff $X\subseteq T_M$ and that the set $C(\mathcal{M})(X)$ of logical consequences of X is the intersection of all the complete theories containing X.

Given a set U, a fuzzy closure operator (system) in U is any closure operator (system) in the lattice $\mathcal{F}(U)$. We call *an abstract fuzzy deduction system*, (*fuzzy semantics, abstract fuzzy logic*) any abstract deduction system (semantics, logic, respectively) in the lattice of the fuzzy subsets of F. Then, a fuzzy semantics is a class $\mathcal{M}$ of fuzzy subsets of formulas such that $U^1\notin\mathcal{M}$. The meaning of such a condition is obvious; no world in which every formula is true exists. The elements in $\mathcal{M}$ are named *fuzzy models*. Examples of fuzzy semantics are obtained by setting $\mathcal{M}$ equal to the class of the truth-functional valuations of the formulas in a multivalued logic. Another example is furnished by probability logic in which a model is a fuzzy set of formulas P such that, for every $\alpha,\beta\in F$

$P(\alpha\vee\beta)=P(\alpha)+P(\beta)$ (if α is inconsistent with β)

$P(\alpha)=P(\beta)$ (if α is logically equivalent to β)

$P(\alpha)=1$ (if α is logically true).

Sometimes we call an *initial valuation* (or a *piece of fuzzy information* or a *fuzzy system of axioms*) any fuzzy subset V of formulas. We interpret V as an incomplete information about an unknown world M, namely, since M is a model of V provided that $V\subseteq M$, the information carried on by V is that, given any formula α,

"the actual truth value of α is at least $V(\alpha)$".

From this point of view, an initial valuation V is not a fuzzy subset since the values $V(\alpha)$ are not truth degrees but constraints on the possible truth degrees. This is in accordance with the classical inferential processes where the available information is expressed by a set T of formulas (the assumptions) arising from a partial knowledge of a world M and the information carried on by T is that *"at least the formulas in T are true in M"*. Obviously, the logical consequence operator C is defined by setting, given an initial valuation V,

$$C(V)=\bigcap\{M\in\mathcal{M} \mid M\supseteq V\}.$$

The meaning of $C(V)(\alpha)$ is still *"α is true at least at degree $C(V)(\alpha)$"*, but we have also that

$C(V)(\alpha)$ is the best possible valuation we can draw from the information V.

Notice that, while $C(V)(\alpha)=1$ entails that α is true in any model of V, $C(V)(\alpha)=0$ does not mean that α is false but that the available information V says nothing that supports α. On the other hand, this happens in the classical logic, too. Indeed, assume that T is a set of sentences expressing our knowledge about an unknown world M and that α is a formula that is not a logical consequence of T. Then we cannot conclude that α is false in M but only that we are not able to prove α. In other words $C(V)(\alpha)$ carries on only positive information about α. The negative information about α is represented by the number $C(V)(-\alpha)$(provided that the language under consideration is equipped with a negation -).

Also, recall that *the fuzzy subset of tautologies* is defined by

$$Tau(\mathcal{M})=\bigcap\{M \mid M\in\mathcal{M}\}.$$

The meaning of such a fuzzy subset is that, given any formula α, α is true in any model M at least at degree $Tau(\mathcal{M})(\alpha)$. We define a *tautology* as any formula α such that $Tau(\mathcal{M})(\alpha)=1$, i.e., $M(\alpha)=1$ for every $M\in\mathcal{M}$. We can also define *the fuzzy subset of contradictions* $Contr(\mathcal{M})$ as the intersection of all the complements of the elements in $\mathcal{M}$, i.e.,

$$Contr(\mathcal{M})=\bigcap\{M^c \mid M\in\mathcal{M}\}$$

It is immediate that

$$Tau(\mathcal{M})(\alpha)+Contr(\mathcal{M})(\alpha)\leq 1$$

and that

$$Contr(\mathcal{M})(\alpha)=1-Sup\{M(\alpha) \mid M\in\mathcal{M}\}.$$

The meaning of the fuzzy subset $Contr(\mathcal{M})$ is that, given any formula α, α is true in any model M at most at degree $1-Contr(\mathcal{M})(\alpha)$. As a consequence, the interval $[Tau(\mathcal{M})(\alpha),1-Contr(\mathcal{M})(\alpha)]$ represents the *a-prior information* about a formula α. We define a *contradiction* as any formula α such that $Contr(\mathcal{M})(\alpha)=1$, i.e., α is a contradiction provided that $M(\alpha)=0$ for every $M\in\mathcal{M}$.

We conclude this section by examining the notion of continuity for fuzzy operators. Recall that a classical closure operator J is continuous iff J is *compact*, i.e., for every subset X of U

$$J(X)=\bigcup\{X_f \mid X_f \text{ is a finite part of } X\}. \tag{3.29}$$

Also, this is equivalent to saying that for every $X\subseteq U$ and $x\in U$

$$x\in J(X) \Leftrightarrow \text{a finite subset } X_f \text{ exists such that } x\in J(X_f). \tag{3.30}$$

In order to extend this definition to fuzzy closure operators, we call *finite* any fuzzy subset whose support is finite.

Definition 3.8.1 A fuzzy operator J is called *compact* if, for every fuzzy set A

$$J(A)=\bigcup\{J(A_f) \mid A_f \text{ finite and } A_f\subseteq A\}. \tag{3.31}$$

J is called *p-compact* provided that J is order-preserving and, for every fuzzy subset A and $x\in U$, a finite fuzzy subset A_f of A exists such that

$$J(A)(x)=J(A_f)(x). \tag{3.32}$$

The definition of p-compactness was proposed in [Pavelka 1979]. The following proposition whose proof we omit shows that compacity, p-compacity and continuity are not equivalent notions for fuzzy operators.

Proposition 3.8.2 Let J be a fuzzy operator. Then

(i) J p-compact $\Rightarrow$ J compact ;

(ii) J continuous $\Rightarrow$ J compact ;

(iii) J continuous does not imply J p-compact ;

(iv) J p-compact does not imply J continuous ;

(v) J compact does not imply J p-compact ;
(vi) J compact does not imply J continuous.

Nevertheless, an interesting characterization of the continuous closure operators in terms of finite fuzzy subsets was established in [Murali 1991]. In the following, given two fuzzy subsets A and B, $A \ll B$ means that $A(x)<B(x)$ for every $x \in \mathrm{Supp}(A)$.

Proposition 3.8.3 A fuzzy operator J is continuous iff, for every fuzzy subset A,

$$J(A)=\cup\{J(A_f) \mid A_f \text{ is finite and } A_f \ll A\} \qquad (3.33)$$

Proof. At first we prove a basic property of the relation $\ll$, namely that if T is a directed class, then

$$A_f \text{ finite fuzzy subset and } A_f \ll \cup\{A' \mid A' \in T\} \Rightarrow \exists A' \in T \text{ s.t. } A_f \subseteq A'.$$

Indeed, if $\mathrm{Supp}(A_f)=\{x_1,...,x_n\}$, then for i=1,...,n, from $A_f(x_i)<Sup\{A'(x_i) \mid A' \in T\}$ it follows that $A_i \in T$ exists such that $A_f(x_i)<A_i(x_i)$. So it is sufficient to consider any element A' in T containing $A_1,..., A_n$. Suppose (3.33), then it is immediate that J is order-preserving. Let T be any directed class of fuzzy subsets. Then we must prove that $J(\cup\{A' \mid A' \in T\}) = \cup\{J(A') \mid A' \in T\}$. Set $A=\cup\{A' \mid A' \in T\}$, then, on account of the above implication,

$$J(A)= \cup\{J(A_f) \mid A_f \text{ is finite and } A_f \ll A\} \subseteq \cup\{J(A') \mid A' \in T\}.$$

Since the converse inclusion is immediate, this proves that J is continuous.
Conversely, assume that J is continuous. Then, since $\{A_f \mid A_f$ is finite and $A_f \ll A\}$ is a directed family whose union is A, (3.33) is immediate.

3.9. PAVELKA'S LOGIC

This section is devoted to expose Pavelka's approach to fuzzy logic, i.e., an approach in Hilbert style (see also Chapter 2 in this book). We define a *crisp Hilbert deduction system*, in brief an *H-system*, as a pair $S=(A,R)$ such that A is a subset of the set of formulas F, the *set of logical axioms*, and R is a set of crisp inference rules. In turn, a *crisp inference rule* is any partial operation in the set of formulas F, i.e., any map $r:D \to F$ where $D \subseteq F^n$, $n \in N$. We write $Dom(r)$ to denote the domain D of r. A *proof* π of a formula α under the hypothesis $\gamma_1,...,\gamma_h$ is any sequence $\alpha_1...\alpha_m$ of formulas such that $\alpha_m=\alpha$ and

- either $\alpha_i \in \{\gamma_1,...,\gamma_h\}$,
- or $\alpha_i \in A$
- or $\alpha_i=r(\alpha_{i(1)},...,\alpha_{i(n)})$, $i(1)<i$, ... , $i(n)<i$.

Given a set X of formulas, we write $X \vdash \alpha$ to denote that a proof of α exists whose hypotheses are contained in X. These notions enable us to define an operator D by setting

$$D(X)=\{\alpha \in F \mid \text{a proof of } \alpha \text{ exists whose hypotheses are in } X\}. \qquad (3.34)$$

Obviously, $X \vdash \alpha$ iff $\alpha \in D(X)$. The proof of the following proposition is immediate.

Proposition 3.9.1 The operator D associated with an H-system (A,R) by (3.34) is an algebraic closure operator. So, every crisp H-system (A,R) defines a crisp abstract deduction system (F,D).

By extending the above definitions, we define a *fuzzy H-system* as a pair (A,R) where A is a fuzzy subset of F, *the fuzzy subset of logical axioms*, and R is a set of fuzzy rules of inference. In turn, a *fuzzy rule of inference* is a pair $r=(r',r'')$, where

- r' is a partial n-ary operation on F, i.e. a crisp inference rule
- r'' is an n-ary operation on [0,1] preserving the least upper bound in each variable, i.e.,

$$r''(x_1,\ldots,Sup_{i\in I}y_i,\ldots,x_n) = Sup_{i\in I}r''(x_1,\ldots,y_i,\ldots,x_n) \tag{3.35}$$

So an inference rule r consists of a *syntactical component* r' that operates on formulas and a *valuation component* r'' that operates on truth values to calculate how the truth value of the conclusion depends on the truth values of the premises (see [Zadeh 1975]). Condition (3.35) entails that r'' is order-preserving with respect to any component. Such a condition is also called *continuity condition* since it enables us to prove that the deduction operator associated with a fuzzy H-system is continuous. We indicate an application of an inference rule r by

$$\frac{\alpha_1,\ldots,\alpha_n}{r'(\alpha_1,\ldots,\alpha_n)} \quad ; \quad \frac{\lambda_1,\ldots,\lambda_n}{r''(\lambda_1,\ldots,\lambda_n)}$$

whose meaning is that:

IF *you know that the formulas* $\alpha_1,\ldots,\alpha_n$ *are true at least to the degree* $\lambda_1,\ldots,\lambda_n$,
THEN *you can conclude that* $r'(\alpha_1,\ldots,\alpha_n)$ *is true at least to the degree* $r''(\lambda_1,\ldots,\lambda_n)$.

Examples of fuzzy inference rules can be obtained by assuming that r' is the classical modus ponens, i.e., the function associating with any pair of formulas of type $\alpha\to\beta$ and α the formula β, and assuming that r'' is any continuous T-norm (in this case, $Dom(r)=\{(\alpha,\alpha\to\beta) \mid \alpha\in F, \beta\in F\}$).

A *proof* π is a sequence $\alpha_1,\ldots,\alpha_m$ of formulas, together with a sequence of related "justifications". This means that, given any formula α_i, we must specify whether

(i) α_i is assumed as a logical axiom; or
(ii) α_i is assumed as a proper axiom; or
(iii) α_i is obtained by a rule (in this case we must indicate also the rule and the formulas from $\alpha_1,\ldots,\alpha_{i-1}$ used to obtain α_i).

Differently from the crisp case, the justifications are necessary since different justifications of the same formula give rise to different valuations. Indeed, let $V:F\to[0,1]$ be any initial valuation. Then the *valuation* $Val(\pi,V)$ *of a proof* π *with respect to* V is defined by induction on the length m of π by setting

$\mathrm{Val}(\pi,V)=A(\alpha_m)$ if α_m is assumed as a logical axiom
$=V(\alpha_m)$ if α_m is assumed as a proper axiom
$=r''(\mathrm{Val}(\pi(i(1)),V),\ldots,\mathrm{Val}(\pi(i(n)),V))$ if $\alpha_m=r'(\alpha_{i(1)},\ldots,\alpha_{i(n)})$

where, for any $i\leq m$, $\pi(i)$ denotes the proof $\alpha_1,\ldots,\alpha_i$. If α is the formula proven by π, the meaning we assign to $\mathrm{Val}(\pi,V)$ is that

given the information V, *the proof* π *assures that* α *holds at least to degree* $Val(\pi,V)$.

Now, it could happen that another proof π' of α exists such that Val(π',V)>Val(π,V). This happen, for instance, if the assumptions used in π' are more true than the assumptions used in π. In other words, unlike the usual Hilbert systems, in a fuzzy H-system different proofs of a same formula α can give different contributions to the degree of validity of α. This suggests that, given a fuzzy set of axioms V (the available fuzzy information), in order to evaluate α we must refer to the whole set of proofs of α and to calculate the number $Sup\{\mathrm{Val}(\pi,V) \mid \pi \text{ is a proof of } \alpha\}$.

Definition 3.9.2. The deduction operator of an H-system S is the operator D:$\mathcal{F}(F)\rightarrow\mathcal{F}(F)$ defined by setting,

$$D(V)(\alpha)=Sup\{\mathrm{Val}(\pi,V) \mid \pi \text{ is a proof of } \alpha\} \qquad (3.36)$$

for every initial valuation V and every formula α.

The meaning of $D(V)(\alpha)$ is still

given the information V, we may prove that α holds at least at degree $D(V)(\alpha)$,

but we have also that

$D(V)(\alpha)$ is the best possible valuation we can draw from the information V.

Pavelka proved the following proposition.

Proposition 3.9.3 Let S be an H-system, then the operator D defined by (3.36) is a compact fuzzy closure operator. Consequently, every H-system is associated with a compact fuzzy abstract logic.

We show also that S defines a step-by-step deduction system in a natural way and therefore that D is continuous.

Proposition 3.9.4 Let S=(A,R) be an H-system and define J by

$$J(V)(\alpha)=Sup\{r''(V(\alpha_1),...,V(\alpha_n)) \mid (r',r'')\in R \text{ and } r'(\alpha_1,...,\alpha_n)=\alpha\}, \qquad (3.37)$$

for every $V\in\mathcal{F}(F)$ and $\alpha\in F$. Then ($\mathcal{F}(F)$,J,A) is a step-by-step deduction system whose associated deduction operator D coincides with the one associated with S.

Proof. In order to prove that J is continuous, observe that, since every valuation component r'' is an order-preserving map, J is order preserving, too. As a consequence, given a directed family T of fuzzy subsets and V=lim$T$, $J(V)\supseteq\bigcup\{J(X) \mid X\in T\}$=lim$J(T)$. Conversely, given a formula α, by observing that for every S_1,..., $S_n\in T$ an element $X\in T$ exists such that $S_1\subseteq X$,..., $S_n\subseteq X$, we have

$$\begin{aligned}J(V)(\alpha)&=Sup\{r''(V(\alpha_1),...,V(\alpha_n)) \mid (r',r'')\in R \text{ and } r'(\alpha_1,...,\alpha_n)=\alpha\}\\&=Sup\{r''(S_1(\alpha_1),..., S_n(\alpha_n)) \mid (r',r'')\in R,\ r'(\alpha_1,...,\alpha_n)=\alpha \text{ and } S_1,..., S_n\in T\}\\&=Sup\{r''(X(\alpha_1),..., X(\alpha_n)) \mid (r',r'')\in R\ ,\ r'(\alpha_1,...,\alpha_n)=\alpha \text{ and } X\in T\}\\&=\bigcup\{J(X)(\alpha) \mid X\in T\}.\end{aligned}$$

Define H as in (3.10), i.e.,

$$H(V)=J(V)\cup V\cup A. \qquad (3.38)$$

Let D be the deduction operator associated with S, we have to prove that D is the closure operator generated by H, i.e.,

$$D(V)=c(H)(V)=\cup H^n(V). \tag{3.39}$$

To this purpose we have to prove that D has the same fixed points of H and therefore of c(H). Indeed, V is a fixed point of H iff $V\supseteq A$ and $V\supseteq J(V)$, i.e., V is closed with respect to the inference rules, i.e. $r''(V(\alpha_1),...,\tau(\alpha_n))\leq V(r'(\alpha_1,...,\alpha_n))$ for every $(r',r'')\in R$ and $\alpha_1,...,\alpha_n$ in $Dom(r')$. Now, under these conditions, there is no difficulty to prove, by induction on the length of the proofs, that Val$(\pi,V)\leq V(\alpha)$ for every formula α and π proof of α. In turn this is equivalent to say that $D(V)=V$.

3.10 LOGICAL COMPACTNESS AND ULTRAPRODUCTS

The relation $\ll$, defined in Section 3.8, enables us also to characterize the logical compactness for fuzzy operators.

Proposition 3.10.1 A fuzzy closure operator J is logically compact iff, for every initial valuation V,

$$V \text{ consistent} \Leftrightarrow \text{every finite } V_f\ll V \text{ is consistent.} \tag{3.40}$$

Proof. Assume that J is logically compact. Then it is immediate that V consistent implies that every finite fuzzy subset V_f such that $V_f\ll V$ is consistent. In order to prove the converse implication, assume that every finite fuzzy subset V_f such that $V_f\ll V$ is consistent, then, since V is the inductive limit of the class $\{V_f \mid V_f\ll V, V_f \text{ finite}\}$, V is consistent. This proves (3.40). Conversely, assume (3.40) and let H be an inductive class of consistent fuzzy subsets. We have to prove that V=limH is consistent. Now, for every finite fuzzy set V_f such that $V_f\ll V$ an element $A\in$H exists such that $V_f\subseteq A$. Since A is consistent, V_f is consistent too, and by (3.40) V is consistent.

We can rewrite Proposition 3.10.1 in terms of fuzzy semantics as follows.

Proposition 3.10.2 A fuzzy semantics is logically compact iff, for every initial valuation V,

$$V \text{ satisfiable} \Leftrightarrow \text{every finite } V_f\ll V \text{ is satisfiable.} \tag{3.41}$$

This proposition suggests to call *compact* a fuzzy semantics M such that, for every fuzzy set of formulas V,

$$V \text{ satisfiable} \Leftrightarrow \text{every finite fuzzy subset of } V \text{ is satisfiable.}$$

Proposition 3.10.3 Every logically compact fuzzy semantics is compact while compact fuzzy semantics exist that are not logically compact.

Proof. The first part of the proposition is obvious. The fuzzy class $M=\{A\in F(F) \mid A(\alpha)<1 \text{ for every } \alpha\in F\}$ is an example of compact semantics that is not logically compact. Indeed, $F=F^1$ is limit of the class of satisfiable fuzzy subsets F^λ, $\lambda\neq 1$.

A simple compacity criterion for a fuzzy semantics is obtained by the notion of ultraproduct, (see, e.g., [Chang; Keisler 1966]). Recall that a class F of subsets of a set I is a *filter* provided that

$$X\in F \text{ and } Y\supseteq X \Rightarrow Y\in F \quad \text{and} \quad X\in F \text{ and } Y\in F \Rightarrow X\cap Y\in F$$

and that F is *prime* if, for every $X\in P(\mathrm{I})$, either $X\in F$ or $X^c\in F$. In this case, either F is generated by one element of I (i.e., is *principal*) or F contains the filter of co-finite subsets. One proves that a class C of subsets can be extended to a prime filter iff it satisfies the finite intersection property, i.e., the intersection of every finite family of elements of C is nonempty. Moreover, given a family $(\lambda_i)_{i\in I}$ of real numbers and a filter F, we write $\lim_F \lambda_i=\lambda$ provided that

$$\forall\varepsilon>0\ \exists X\in F\ \ \forall i\in X\ \ |\lambda-\lambda_i|\leq\varepsilon.$$

Equivalently, we can write

$$\text{for every neighborhood } (a,b) \text{ of } \lambda,\ \{i\in I \mid \lambda_i\in(a,b)\}\in F.$$

Such a notion of convergence satisfies the same properties of the classical one but, in addition, if F is prime, for every bounded family $(\lambda_i)_{i\in I}$, $\lim_F \lambda_i$ always exists. Also, assume that I is the set N of natural numbers, that U is a prime filter on N and that U is not principal. Then,

$$\lim_{n\to\infty} \lambda_n=\lambda \Rightarrow \lim_U \lambda_n=\lambda.$$

Given a sequence $(A_n)_{n\in N}$ of fuzzy subsets of a set U and a prime filter U on N, we call an *ultraproduct modulo* U of $(A_n)_{n\in N}$ the fuzzy subset $A=\lim_U A_n$ defined by

$$A(x)=\lim_U A_n(x) \qquad x\in U.$$

Theorem 3.10.4 Let M be a fuzzy semantics closed with respect to the ultraproducts and V an initial valuation. Then,
i) the ultraproduct of a family of models of V is a model of V
ii) for every formula α a model M of V exists such that

$$C(V)(\alpha)=M(\alpha)$$

iii) M is logically compact.

Proof. i) An immediate consequence of the definition of limit with respect to a filter.
ii) Let V be an initial valuation and α a formula. Then, since $C(V)(\alpha)=Inf\{M(\alpha) \mid M\in M,\ M\supseteq V\}$, a sequence $(M_n)_{n\in N}$ of models of V exists such that $M_n(\alpha)$ is decreasing and $C(V)(\alpha)=\lim_{n\to\infty}M_n(\alpha)$. Let U be a non principal ultrafilter on N and M the ultraproduct of $(M_n)_{n\in N}$ modulo U. Then, M is a model of V such that

$$M(\alpha)=\lim_U M_n(\alpha)=\lim_{n\to\infty}M_n(\alpha)=C(V)(\alpha).$$

iii) In order to apply Proposition 3.10.2, suppose that V_f is satisfiable for every V_f finite such that $V_f\ll V$. At first we prove that every finite fuzzy subset A of V is satisfiable. Indeed, it is easy to find an increasing sequence V_n of finite fuzzy subsets such that $A(x)=\lim_{n\to\infty}V_n(x)$ for every x and $V_n\ll A$. Since we have also that $V_n\ll V$, by hypothesis a sequence of models M_n exists such that $M_n\supseteq V_n$. Let U be a non-principal prime filter and let M be the ultraproduct of $(M_n)_{n\in N}$ modulo U. Then, since

$$M(\alpha)=\lim_U M_n(\alpha)\geq\lim_U V_n(\alpha)=\lim_{n\to\infty}V_n(\alpha)=A(\alpha),$$

we have that M is a model of A.

Denote by I the class of finite subsets of F and let $i\in I$. Then, since the restriction of V to i is satisfiable, an element M_i of M exists such that $M_i(x)\geq V(x)$ for every $x\in i$. We find a model M of V as a suitable ultraproduct of the obtained family $(M_i)_{i\in I}$. To this purpose, we have to find an ultrafilter U such that for every $x\in F$ the set $B(x)=\{i\in I\ /\ M_i(x)\geq V(x)\}\in U$. In turn, this is possible provided that the class $\{B(x) \mid x\in F\}$ of subsets of I satisfies the finite intersection property. Now, let $x_1,\ldots,x_n$ be formulas and

$i=\{x_1,...,x_n\}$. Then $M_i(x_j)\geq V(x_j)$ for j=1,...,n and therefore i belongs to $B(x_1)\cap\ldots\cap B(x_n)$. This concludes the proof.

Proposition 3.10.4 suggests a general method to obtain logically compact fuzzy semantics. In the following a *closed k-ary relation* is a closed subset R of R^k. The equality and the order relation are examples of closed binary relations. As it is usual, if $x_1,...,x_k$ are real numbers, we write $R(x_1,...,x_k)$ to denote that $(x_1,...,x_k)\in R$.

Proposition 3.10.5 Denote by $\mathcal{M}$ the class of fuzzy subsets M of F satisfying a set of conditions like

$$R(M(p_0(x_1,...,x_h)),...,M(p_k(x_1,...,x_h))) \qquad (3.42)$$

where

- $p_0..., p_k$ are partial operations on F defined in a domain $D\subseteq F^h$;
- $R\subseteq R^{k+1}$ is a closed relation.

Then $\mathcal{M}$ is closed with respect to the ultraproducts. Hence, if $F^1\notin\mathcal{M}$, $\mathcal{M}$ is a logically compact fuzzy semantics.

Proof. At first observe that, since R is closed, if $(\lambda_i^0)_{i\in I},...,(\lambda_i^k)_{i\in I}$ are families of real numbers such that $R(\lambda_i^0,...,\lambda_i^k)$ for every $i\in I$, then
$R(\lim_U \lambda_i^0,...,\lim_U\lambda_i^k)$. Indeed, set $\lambda^0=\lim_U\lambda_i^0,...,\lambda_k=\lim_U\lambda_i^k$ and assume that $(\lambda_0,...,\lambda_k)$ is not in R. Then, since R is closed, k+1 intervals $I_0,...,I_k$ exist such that $\lambda_0\in I_0,...,\lambda_k\in I_k$ and $I_0\times...\times I_k$ is disjoint from R. As a consequence, the sets

$$X_0=\{i\in I \mid \lambda_i^0\in I_0\}, \ldots ,X_k=\{i\in I \mid \lambda_i^k\in I_k\}$$

belong to U. Since U is a filter $X_0\cap...\cap X_k$ is nonempty, so, if j is any element of this intersection, we have $(\lambda_j^0,...,\lambda_j^k)\in I_0\times...\times I_k$. Thus, $(\lambda_j^0,...,\lambda_j^k)\notin R$ and this contradicts the hypothesis.

Now, let $(M_i)_{i\in I}$ be a family of elements of $\mathcal{M}$, U an ultrafilter on I and M the ultraproduct of $(M_i)_{i\in I}$ modulo U. Then, since for every $i\in I$

$$R(M_i(p_0(x_1,...,x_h)),...,M_i(p_k(x_1,...,x_h)),$$

in view of the property we have just proved,

$$R(\lim_U(M_i(p_0(x_1,...,x_h)),...,\lim_U(M_i(p_k(x_1,...,x_h))).$$

Thus, $M\in\mathcal{M}$.

As an example, consider the class $\mathcal{M}$ of the truth-functional valuations in a multivalued logic and assume that the interpretations of the logical connectives are all continuous maps. Then $\mathcal{M}$ is the class of fuzzy subsets M satisfying conditions like

$$M(h(\alpha_1,...,\alpha_k))=h'(M(\alpha_1),...,M(\alpha_k))$$

where h is a logical connective and h' is a continuous map interpreting h. We can apply Proposition 3.10.5 by setting p_i equal to the i-projection for i=1,...,k and

$$R =\{(\lambda_0,...,\lambda_k) \mid \lambda_0=h'(\lambda_1,...,\lambda_k)\} \quad ; \quad p_0(\alpha_1,...,\alpha_k)=h(\alpha_1,...,\alpha_k).$$

Thus $\mathcal{M}$ is closed with respect to the ultraproducts and therefore logically compact. The same holds for the class of finitely additive probabilities, the class of necessities, the class of the super additive measures and so on (see, e.g., [Dubois, Lang, Prade 1994], [Gerla 1997]).

3.11 AN EXTENSION PRINCIPLE FOR DEDUCTION OPERATORS

Several notions in crisp mathematics are translated into the corresponding notions in fuzzy mathematics in a uniform way by the famous Zadeh's extension principle. So, it is very natural to put the following question:

Given a (crisp) closure operator, does there exist a canonical way to extend it in a fuzzy closure operator ?

The answer is yes and is suggested by the identification of the fuzzy subsets with the continuous chains (see [Ramik 1983], [Gerla 1994], [Biacino; Gerla 1996], [Gerla; Scarpati [1997]). Indeed, given a closure operator J in U and a fuzzy subset $A\in\mathcal{F}(U)$, it is natural to proceed in the following way

- identify A with the continuous chain $(A_\lambda)_{\lambda\in[0,1]}$
-apply J to each element of this chain by obtaining the chain $(J(A_\lambda))_{\lambda\in[0,1]}$
- consider the continuous chain $(J(A_\lambda)')_{\lambda\in[0,1]}$ generated by such a family
- assume as image of A the fuzzy subset $J^*(A)$ corresponding to such a chain.

The following diagram can picture such a procedure:

$$\begin{array}{ccccc} A & \rightarrow & (A_\lambda)_{\lambda\in[0,1]} & \rightarrow & (J(A_\lambda))_{\lambda\in[0,1]} \\ \downarrow & & & & \downarrow \\ J^*(A) & & \leftarrow & & (J(A_\lambda)')_{\lambda\in[0,1]} \end{array}$$

Since, by Proposition 3.7.8, $(J(A_\lambda))_{\lambda\in[0,1]}$ and $(J(A_\lambda)')_{\lambda\in[0,1]}$ define the same fuzzy subset, we can give the following definition.

Definition 3.11.1 For every operator J, we define the *canonical extension* J^* of J as the fuzzy operator defined by setting, for every $A\in\mathcal{F}(U)$

$$J^*(A)=\bigcup\lambda\wedge J(A_\lambda) \tag{3.43}$$

or, equivalently,

$$J^*(A)(x)=Sup\{\lambda\in[0,1] \mid x\in J(A_\lambda)\}. \tag{3.44}$$

It is possible to reformulate such a definition in logical terms. Let (F,D) be a crisp deduction system and V a fuzzy set of formulas. Then we say that a formula α *is a consequence of V at degree* λ, in brief $V\vdash_\lambda\alpha$, provided that α is a consequence of V_λ, i.e., α can be proved by formulas that are true at least at degree λ.

Definition 3.11.2 Let (F,D) be a crisp deduction system. The *canonical extension of* (F,D) is the fuzzy deduction system (F,D^*) where D^* is defined by

$$D^*(V)(\alpha)=Sup\{\lambda\in[0,1] \mid V\vdash_\lambda\alpha\}. \tag{3.45}$$

The following lemma, whose proof is abvious, gives some information on the cuts of $J^*(A)$.

Lemma 3.11.3 Let J be an a-c-operator. Then for every fuzzy subset A and $\mu\in[0,1]$

$$J^*(A)_{>\lambda}=\bigcup_{x>\lambda}J(A_x)\subseteq J(A_{>\lambda})\subseteq J(A_\lambda)\subseteq\bigcap_{x<\lambda}J(A_x)=J^*(A)_\lambda.$$

Note that, $J^*(A)_{>\lambda}\neq J(A_{>\lambda})$ and $J^*(A)_\lambda\neq J(A_\lambda)$, in general. For example, if J is the usual topological closure operator in the interval $[0,1]$ and A is a continuous map, then $J^*(A)=A$, and $J^*(A)_{>\lambda}\neq J(A_{>\lambda})$ since $J^*(A)_{>\lambda}=A_{>\lambda}$ is an open set, while $J(A_{>\lambda})$ is closed.

Proposition 3.11.4 Let J be an order-preserving operator then

$$J^*(A)=\bigcap\lambda\vee J(A_\lambda) \tag{3.46}$$

$$J^*(A)=\bigcup\lambda\wedge J(A_{>\lambda}) \tag{3.47}$$

$$J^*(A)=\bigcap\lambda\vee J(A_{>\lambda}) \tag{3.48}$$

or, equivalently,

$$J^*(A)(x)=Inf\{\lambda\in[0,1]\mid x\notin J(A_\lambda)\} \tag{3.49}$$

$$J^*(A)(x)=Sup\{\lambda\in[0,1]\mid x\in J(A_{>\lambda})\} \tag{3.50}$$

$$J^*(A)(x)=Inf\{\lambda\in[0,1]\mid x\notin J(A_{>\lambda})\}. \tag{3.51}$$

Proof. (3.46) follows from Proposition 3.7.9. To prove (3.47) observe that, since J is order preserving, $J(A_{>\lambda}))\subseteq J(A_\lambda)$ and, hence, for every fuzzy subset A and $x\in U$

$$Sup\{\lambda\in[0,1]\mid x\in J(A_{>\lambda}))\}\leq Sup\{\lambda\in[0,1]\mid x\in J(A_\lambda)\}=J^*(A)(x).$$

Moreover, by Lemma 3.11.3 we have that $J^*(A)_{>\lambda}\subseteq J(A_{>\lambda})$ and

$$J^*(A)(x)=Sup\{\lambda\in[0,1]\mid x\in J^*(A)_{>\lambda}\}\leq Sup\{\lambda\in[0,1]\mid x\in J(A_{>\lambda})\}.$$

Finally, (3.48) follows from (3.47) and Proposition 3.7.9.

Theorem 3.11.5 Let J be an operator. Then J^* is an extension of J and the following equivalencies hold:

J^* almost closure operator $\Leftrightarrow$ J almost closure operator

J^* closure operator $\Leftrightarrow$ J closure operator.

Proof. It is immediate that J^* is an extension of J. Assume that J^* is an a-c-operator (a closure operator) then, since J is the restriction of J^* to the crisp subsets, J is an a-c-operator (a closure operator). Let J be an a-c-operator. Then it is immediate that J^* is order-preserving. In order to prove that $J^*(A)\supseteq A$, observe that, since $A_\lambda\subseteq J(A_\lambda)$,

$$A(x)=Sup\{\lambda\in[0,1]\mid x\in A_\lambda\}\leq Sup\{\lambda\in[0,1]\mid x\in J(A_\lambda)\}.$$

Assume that J is a closure operator. In order to prove that $J^*(J^*(A))=J^*(A)$, we prove at first that every cut $J^*(A)_\lambda$ is a fixed point for J. Indeed, observe that the intersection of a class of fixed points for J is a fixed point for J and that $J^*(A)_\lambda=\bigcap_{x<\lambda}J(A_x)$. Thus

$$J^*(J^*(A))(x)=Sup\{\lambda\in[0,1]\mid x\in J(J^*(A)_\lambda)\}=Sup\{\lambda\in[0,1]\mid x\in J^*(A)_\lambda\}=J^*(A)(x).$$

Now, we will propose a way to extend a closure system C into a fuzzy closure system in accordance with the concept of canonical extension of a closure operator. Indeed, we set

$$C^*=\{A\in F(U)\mid A_\lambda\in C \text{ for every } \lambda\neq 0\}, \tag{3.52}$$

and we say that C^* is *the canonical extension of* C. The proof of the following proposition is immediate.

Proposition 3.11.6 Let C^* be the canonical extension of the class C of subsets. Then C coincides with the class of crisp elements of C^* and therefore C^* is an extension of C. Further,

C^* is a fuzzy closure system $\Leftrightarrow$ C is a closure system.

The following proposition shows that the notions of canonical extension of a closure system and canonical extension of a closure operator are strictly related in accordance with the following diagrams

$$\begin{array}{ccc} C & \to & \boldsymbol{J}(\mathrm{C}) \\ \downarrow & & \downarrow \\ C^* & \leftarrow & \boldsymbol{J}(C)^*=\boldsymbol{J}(C^*) \end{array} \qquad \begin{array}{ccc} J & \to & \boldsymbol{C}(J) \\ \downarrow & & \downarrow \\ J^* & \leftarrow & \boldsymbol{C}(J)^*=\boldsymbol{C}(J^*). \end{array}$$

Proposition 3.11.7 Let C be a closure system and J a closure operator. Then

$$C^*=\boldsymbol{C}(\boldsymbol{J}(C)^*) \text{ and } \boldsymbol{C}(J)^*=\boldsymbol{C}(J^*). \tag{3.53}$$

Moreover,

$$J^*=\boldsymbol{J}(\boldsymbol{C}(J)^*) \text{ and } \boldsymbol{J}(C)^*=\boldsymbol{J}(C^*). \tag{3.54}$$

Proof. In order to prove the first equation in (3.53), observe that if A is a fixed point for $\boldsymbol{J}(C)^*$, then by Lemma 3.11.3

$$A_\lambda=(\boldsymbol{J}(C)^*)(A)_\lambda=\cap\{\boldsymbol{J}(C)(A_x) \mid x<\lambda\}\in C.$$

As a consequence, $A\in C^*$. Conversely, if $A\in C^*$, then, every cut A_λ belongs to C and hence is a fixed point for $\boldsymbol{J}(C)$. Thus

$$\boldsymbol{J}(C)^*(A)(x)=Sup\{\lambda\in[0,1] \mid x\in\boldsymbol{J}(C)(A_\lambda)\}=Sup\{\lambda\in[0,1] \mid x\in A_\lambda\}=A(x).$$

This proves that A is a fixed point for $\boldsymbol{J}(C)^*$. The remaining part of the proposition is immediate.

As an immediate consequence, we obtain the following theorem.

Theorem 3.11.8 Let T be the class of theories of (F,D). Then T^* is the class of theories of (F,D^*), i.e., for every fuzzy subset T of formulas,

$$T \text{ is a theory of } (F,D^*) \iff \text{every cut of } T \text{ is a theory of } (F,D).$$

The following proposition gives a very simple way to obtain the canonical extension of a closure system.

Proposition 3.11.9 Let C be a closure system and set

$$\mathrm{Q}(C)=\{\lambda\vee X \mid X\in C, \lambda\in[0,1]\}. \tag{3.55}$$

Then, we have

$$C^* =\mathrm{c}(\mathrm{Q}(C)), \tag{3.56}$$

i.e., C^* is the fuzzy closure system generated by $\mathrm{Q}(C)$.

Proof. Given X in C and $\lambda\in[0,1]$, for every $\mu\in[0,1]$, since $(\lambda\vee X)_\mu=U$ if $\mu\leq\lambda$ and $(\lambda\vee X)_\mu=X$ if $\mu>\lambda$, we have that $\lambda\vee X\in C^*$. So, $\mathrm{Q}(C)\subseteq C^*$ and therefore $\mathrm{c}(\mathrm{Q}(C))\subseteq C^*$. Let $A\in C^*$, then, since $A=\bigcap\lambda\vee A_\lambda$ and every A_λ belongs to C, we have that $A\in\mathrm{c}(\mathrm{Q}(C))$. Thus, $\mathrm{c}(\mathrm{Q}(C))\supseteq C^*$ and (3.56) is completely proven.

Let T be a theory of (F,D). Then the fuzzy subset $\lambda\vee\mathrm{T}$ is a theory of (F,D^*), we call a *λ-theory*. If T denotes the class of theories of D, then $\mathrm{Q}(T)$ is the class of λ-theories, obviously. As an immediate consequence of Propositions 3.11.8 and 3.11.9, we have the following

Theorem 3.11.10 Every theory in (F,D^*) is intersection of λ-theories.

3.12 EXTENDING COMPACT DEDUCTION SYSTEMS

The canonical extension (F,D^*) of a compact deduction system (F,D) can be obtained as follows.

Proposition 3.12.1 Let J be a compact a-c-operator. Then J^* is continuous and, $J^*(V)(x)=1$ if $x\in J(\emptyset)$ and

$$J^*(V)(x)=Sup\{V(x_1)\wedge...\wedge V(x_n) \mid x\in J(\{x_1,...,x_n\})\} \tag{3.57}$$

otherwise.

Proof. Let $(A_i)_{i\in I}$ be a directed family of fuzzy subsets and $A=\cup_{i\in I}A_i$. Then, since $J(A_\lambda)=J(\cup(A_i)_\lambda)= \cup J((A_i)_\lambda)$, we have that

$$\begin{aligned}J^*(A)(x)&=Sup\{\lambda\in[0,1] \mid x\in J((A_i)_\lambda) \text{ for a suitable } i\in I\}\\ &=Sup_{i\in I}\{Sup\{\lambda\in[0,1] \mid x\in J((A_i)_\lambda)\}\\ &=Sup_{i\in I}J^*(A_i)(x).\end{aligned}$$

This proves the continuity of J^*. In order to prove (3.57), let $x\in J(\emptyset)$. Then, since $J^*(V)\supseteq J^*(\emptyset)=J(\emptyset)$, we have that $J^*(V)(x)=1$. Suppose $x\notin J(\emptyset)$ and let $\lambda\in[0,1]$ such that $V_\lambda\neq\emptyset$, then, since J is compact,

$$\begin{aligned}J(V_\lambda) &= \cup\{J(\{x_1,...,x_n\}) \mid \{x_1,...,x_n\}\subseteq V_\lambda\}\\ &= \cup\{J(\{x_1,...,x_n\}) \mid V(x_1)\geq\lambda,...,V(x_n)\geq\lambda\}\\ &= \cup\{J(\{x_1,...,x_n\}) \mid V(x_1)\wedge...\wedge V(x_n)\geq\lambda\}.\end{aligned}$$

As a consequence,

$$\begin{aligned}J^*(V)(x)&=Sup\{\lambda\in[0,1] \mid x\in J(V_\lambda)\}\\ &=Sup\{\lambda\in[0,1] \mid \exists x_1,...,\exists x_n \text{ s.t. } V(x_1)\wedge...\wedge V(x_n)\geq\lambda \text{ and } x\in J(\{x_1,...,x_n\})\}\\ &=Sup\{V(x_1)\wedge...\wedge V(x_n) \mid x\in J(\{x_1,...,x_n\})\}.\end{aligned}$$

In terms of a deduction operator D we obtain immediately the following theorem in which we write $\alpha_1,...,\alpha_n \vdash \alpha$ instead of $\alpha\in D(\{\alpha_1,...,\alpha_n\})$.

Theorem 3.12.2 Let (F,D) be a crisp deduction system and assume that D is compact. Then $D^*(V)(\alpha)=1$ if α is a tautology and

$$D^*(V)(\alpha) = Sup\{V(\alpha_1)\wedge...\wedge V(\alpha_n) \mid \alpha_1,...,\alpha_n\vdash \alpha\} \tag{3.58}$$

otherwise.

In the case of compact operators we have that the canonical extension can be obtained as follows.

Proposition 3.12.3 Let $J:P(U)\to P(U)$ be a compact classical a-c-operator, then

$$J^*(A)(x)=Sup\{\lambda\in[0,1] \mid x\in J(A_{>\lambda}))\}. \tag{3.59}$$

Proof. It is sufficient to prove that

$$J^*(A)_{>\lambda}=J(A_{>\lambda}). \tag{3.60}$$

Indeed, since $(A_\mu)_{\mu>\lambda}$ is a directed family, by Lemma 3.11.3 we have that

$$J^*(A)_{>\lambda}=\cup_{x>\lambda}J(A_x)=J(\cup_{x>\lambda}A_x)=J(A_{>\lambda})).$$

Proposition 3.12.4 For every classical compact a-c-operator J we have that

$$c(J^*)=c(J)^*,$$

i.e.,

$$\cup_{n\in N}(J^*)^n=(\cup_{n\in N}J^n)^*. \tag{3.61}$$

Proof. Observe at first that, for every $n\in N$,

$$(J^*)^n(A)_{>\lambda}=J^n(A_{>\lambda}). \tag{3.62}$$

Indeed, in the case n=1 (3.62) coincides with (3.60). Assume (3.62) for the integer n, then

$$(J^*)^{n+1}(A)_{>\lambda}=J^*(J^{*n}(A))_{>\lambda}=J(J^{*n}(A)_{>\lambda}))=J(J^n(A_{>\lambda}))=J^{n+1}(A_{>\lambda}).$$

Also, we have that

$$c(J^*)(A)_{>\lambda}=c(J)(A_{>\lambda}). \tag{3.63}$$

Indeed,

$$c(J^*)(A)_{>\lambda}=(\cup_{n\in N}J^{*n}(A))_{>\lambda}=\cup_{n\in N}(J^{*n}(A)_{>\lambda}))=\cup_{n\in N}J^n(A_{>\lambda})=c(J)(A_{>\lambda}).$$

Finally, by (3.63)

$$c(J^*)(A)(x)=Sup\{\lambda\in[0,1] \mid x\in(c(J^*)(A))_{>\lambda}\}$$
$$=Sup\{\lambda\in[0,1] \mid x\in c(J)(A_{>\lambda})\}=(c(J))^*(A)(x).$$

The proof of the following proposition is immediate.

Theorem 3.12.5 Let $(P(F),J,A)$ be a step-by-step deduction system and define H and $(P(F),D)$ as in Section 3.5, i.e., for $X\in P(F)$, $H(X)=J(X)\cup A\cup X$. Then the fuzzy deduction system $(F(F),D^*)$ can be obtained by setting

$$D^*=\cup_{n\in N}(H^*)^n. \tag{3.64}$$

3.13 SIMILARITY LOGIC

By following [Ying 1994] and [Biacino; Gerla 1998], we will consider logics in which the reasoning may be approximate by allowing the antecedent clause of a rule to match its premises only approximately. Such an idea was explored in the direction of a similarity-based Prolog in [Ferrante; Formato; Sessa 1998]. As an example we can have an inference like

1. x is a thriller book $\Rightarrow$ x is good for me +
2. b is a black book +
3. "black book" is similar to "thriller book" =

b is good for me

But the similarity among predicates is a fuzzy notion, in general, so the degree at which we can admit the conclusion "b is good for me" depends on the degree of similarity between "black book" and "thriller book". Notice that such an approach, syntactical in nature, is rather different from the one proposed in [Esteva; Garcia;

Godo; Rodriguez 1997] and [Dubois; Esteva; Garcia; Godo; Prade 1997], semantical in nature. Indeed in the latter the similarity is defined on the set of worlds and not in the set of predicates.

A fuzzy relation S:$F\times F\rightarrow[0,1]$ in a set F is called a *similarity* if, for every x, y, $z\in F$:

a) $S(x,x)=1$ (reflexivity)

b) $S(x,y)\geq S(x,z)\wedge S(z,y)$ (transitivity)

c) $S(x,y)=S(y,x)$ (symmetry),

Obviously, the crisp similarities coincide with the equivalence relations. The following proposition can be easily proved.

Proposition 3.13.1 Let S be a fuzzy similarity and define SIM:$\mathcal{F}(F)\rightarrow\mathcal{F}(F)$ by setting, for any $A\in\mathcal{F}(F)$ and $x\in F$,

$$SIM(A)(x)=Sup\{S(x',x)\wedge A(x') \mid x'\in F\}. \tag{3.65}$$

Then SIM is a continuous closure operator we call the *fuzzy closure operator associated with S*.

We can interpret $SIM(A)$ as the fuzzy subset of formulas similar to some formula in A. We say that a fuzzy subset A is *closed with respect to S* if A is a fixed point of SIM. It is immediate that a fuzzy subset A is closed with respect to S if and only if

$$A(x)\geq A(x')\wedge S(x',x) \tag{3.66}$$

for every x, x' in F. In particular, x, x' in F,

$$SIM(A)(x)\geq SIM(A)(x')\wedge S(x',x). \tag{3.67}$$

Let D be the deduction operator of a continuous fuzzy logic and consider the operator

$$K=D\circ SIM. \tag{3.68}$$

Given an initial valuation V, $K(V)$ is the fuzzy subset of formulas that can be deduced, by D, from formulas that are either in V or are similar to formulas in V. We have that K is a continuous almost closure operator but K is not a closure operator, in general. Therefore it is rather natural to define a similarity logic as a step-by-step logic.

Definition 3.13.4 An *abstract similarity logic* is a fuzzy logic $(\mathcal{F}(F),D_S)$ whose deduction operator D_S is the fuzzy closure operator generated by the operator K defined in (3.68) where D is the deduction operator of a continuous fuzzy logic and SIM the closure operator associated with a similarity S.

Since K is continuous, we have that, for every initial valuation V,

$$D_S(V)=\bigcup K^n(V), \tag{3.69}$$

i.e.,

$$V\subseteq SIM(V) \subseteq D(SIM(V)) \subseteq SIM(D(SIM(V))) \subseteq \ldots\rightarrow D_S(V).$$

The proof of the following theorem is matter of routine.

Theorem 3.13.5 $D_S(V)$ is the least fuzzy subset of formulas containing V closed with respect to the similarity relation S and to the deduction operator D.

3.14 YING'S SIMILARITY LOGIC

In [Ying 1994] the starting point is the classical propositional calculus (F, $\Rightarrow$, f) where F is the set of formulas, $\Rightarrow$ is the implication and f is a constant to denote the false. Also only *natural* similarities are considered, i.e., similarities S:$F\times F\to[0,1]$ obtained from a similarity S':$VAR\to[0,1]$ on the set VAR of propositional variables by

(1) $S(x,y)=S'(x,y)$ for every $x, y\in VAR$,
(2) $S(f,x)=S(x,f)=1$ if $x=f$ and $S(f,x)=S(x,f)=0$ if $x\neq f$,
(3) $S(x\Rightarrow y, x'\Rightarrow y') = S(x,x')\wedge S(y,y')$,
(4) $S(x,y)=0$ otherwise.

S is extended to a fuzzy relation between sets of formulas by setting, for X and Y subsets of F,

$$S''(X,Y)=Inf_{y\in Y}Sup_{x\in X}S(x,y) \tag{3.70}$$

The number $S''(X,Y)$ is a multivalued valuation of the claim that every element in Y is similar to a suitable element in X. If S is the identity relation then S'' is the (characteristic function of the) inclusion relation. This enables to define a consequence relation Con:$\mathcal{P}(F)\times F\to[0,1]$ by setting

$$Con(X,\alpha)=Sup\{S''(X\cup Tau,Y) \mid Y\vdash\alpha\} \tag{3.71}$$

where $\vdash$ is the deduction relation in the classical propositional calculus and Tau is the related set of tautologies. In [Biacino; Gerla 1998] these definitions are extended by considering any crisp deduction system ($\mathcal{P}(F)$,D) and any similarity S on the set of formulas F. Then the relation S'':$\mathcal{F}(F)\times\mathcal{P}(F)\to[0,1]$ is defined by setting

$$S''(V,Y)=Inf_{y\in Y}Sup_{x\in F}S(x,y)\wedge V(x), \tag{3.72}$$

i.e.,

$$S''(V,Y)=Inf_{y\in Y}SIM(V)(y). \tag{3.73}$$

The number $S''(V,Y)$ gives the extent at which each formula of Y is similar to a formula of V.

Definition 3.14.1 The fuzzy consequence relation Con:$\mathcal{F}(F)\times F\to[0,1]$ associated with a compact deduction system ($\mathcal{P}(F)$,D) and a similarity S is defined by

$$Con(V,\alpha)=Sup\{S''(V\cup Tau,Y) \mid Y\subseteq F,\ Y\vdash\alpha\} \tag{3.74}$$

where Tau is the set of tautologies of D, V an initial valuation, α a formula and $\vdash$ the deduction relation associated with D.

Obviously, due to the compactness of the relation $\vdash$, the set Y in (3.74) can be assumed to be finite. The meaning of $Con(V,\alpha)$ is immediate, it represents the degree at which we can prove α by using formulas that are similar to formulas in V or to tautologies. Note that if S is the identity relation, then

$$Con(V,\alpha)=D^*(V)(\alpha) \tag{3.75}$$

where D^* is the canonical extension of D. Consequently, Definition 3.14.1 extends the notion of canonical extension.

Theorem 3.14.2 Define the operator H by setting, for every initial valuation V,

$$H(V)=SIM(V\cup Tau). \tag{3.76}$$

Then H is a continuous closure operator such that

$$Con(V, .\ )=(D^*\circ H)(V). \tag{3.77}$$

Proof. We have that $S''(V\cup A,Y)=Inf_{y\in Y}H(V)(y)$ and, hence,

$$Con(V,\alpha)=Sup\{H(V)(y_1)\wedge...\wedge H(V)(y_n) \mid y_1,...,y_n \vdash \alpha\}=D^*(H(V))(\alpha).$$

We conclude this section by noticing that, by confining ourselves to the similarity relations considered by Ying, we can prove that the composition $D^*\circ H$ is a closure operator (see [Biacino; Gerla; Ying 1998]).

Theorem 3.14.3 Under Ying's hypothesis $D^*\circ H$ is a continuous closure operator. Consequently, Ying's logic is an abstract similarity logic.

3.15 STRATIFIED FUZZY LOGIC

The formula for the canonical extension of a closure operator enables us to apply a crisp deduction apparatus to fuzzy information, i.e., information "stratified" at several levels of validity. It is also possible to have crisp information and "stratified" deduction apparatus, i.e., different deductive tools each with a related degree of validity. We can represent such a state of affairs by assuming that, for every $\lambda\in[0,1]$, a crisp deduction operator D_λ is defined and that, given a set X of formulas and a formula α, $\alpha\in D_\lambda(X)$ means that α is a consequence of X (at least) at degree λ. More generally, it is possible that the available information and the deduction apparatus are both stratified. In this case, if V is the initial fuzzy information, it is rather natural to claim that α is a consequence of V at least at degree λ everywhere $\alpha\in D_\lambda(V_\lambda)$. Since we must consider the better lower bound for the truth degree of α we are able to obtain, it is natural to consider the number

$$D(V)(\alpha)=Sup\{\lambda\in[0,1] \mid \alpha\in D_\lambda(V_\lambda)\}$$

as the better lower constraint for this truth degree. This suggests the following generalization of the formula for the canonical extension of a classical closure operator.

Definition 3.15.1 Let $(J_\lambda)_{\lambda\in[0,1]}$ be a family of operators in a set U and let J be the fuzzy operator defined by setting, for every $A\in\mathcal{F}(U)$ and $x\in U$,

$$J(A)(x)=Sup\{\lambda\in[0,1] \mid x\in J_\lambda(A_\lambda)\}. \tag{3.78}$$

Then we say that J is the *fuzzy operator associated with* $(J_\lambda)_{\lambda\in[0,1]}$.

We are interested in families of operators with some natural properties. Namely, we say that a family $(J_\lambda)_{\lambda\in[0,1]}$ of operators is a *chain* provided that $(J_\lambda(X))_{\lambda\in[0,1]}$ is a chain for every subset X, i.e.,

(i) J_0 is the map constantly equal to U ;

(ii) $(J_\lambda)_{\lambda\in[0,1]}$ is order-reversing

We say that $(J_\lambda)_{\lambda\in[0,1]}$ is a *continuous chain* provided that $(J_\lambda(X))_{\lambda\in[0,1]}$ is a continuous chain for every subset X, i.e.,

(j) J_0 is the map constantly equal to U ;

(jj) $J_\lambda(X)=\bigcap_{x<\lambda}J_x(X)$ for every subset X and for every $\lambda\in[0,1]$.

In terms of a stratified deduction apparatus, these conditions look to be rather natural. Indeed, (j) means that, given any set X of formulas, every formula can be considered as a consequence of X (at least) at degree zero. The inclusion

$J_\lambda(X)\subseteq\bigcap_{x<\lambda}J_x(X)$, i.e., the order-reversing condition, means that if α is a consequence of X (at least) at degree λ, then α is a consequence of X (at least) at degree x for every $x<\lambda$. The continuity condition $J_\lambda(X)\supseteq\bigcap_{x<\lambda}J_x(X)$ claims that if α is a consequence of X (at least) at degree x for every $x<\lambda$, then α is a consequence of X (at least) at degree λ, too.

It is obvious that if $J_\lambda=H$ for every $\lambda\in[0,1]$, then the operator defined by (3.78) is the canonical extension of H. Obviously, we are interested to families of closure operators.

Proposition 3.15.2 Let $(J_\lambda)_{\lambda\in[0,1]}$ be a family of closure operators and let J be the associated operator. Then J is a fuzzy a-c-operator (but J is not a closure operator, in general). If $(J_\lambda)_{\lambda\in[0,1]}$ is a chain, then J is a fuzzy closure operator.

Observe that by Proposition 3.7.9 we have that, if $(J_\lambda)_{\lambda\in[0,1]}$ is a chain of closure operators and J the associated operator, then, for every $\lambda\in[0,1]$

$$J(A)_{>\lambda} = \bigcup_{x>\lambda}J_x(A_x) \subseteq J_\lambda(A_{>\lambda})) \subseteq \bigcap_{x<\lambda}J_x(A_x) = J(A)_\lambda. \qquad (3.79)$$

As an application of Proposition 3.7.8 it is possible to show that every fuzzy closure operator obtained by a chain of closure operators can be obtained by a continuous chain of closure operators, too.

Proposition 3.15.3 Let $(J_\lambda)_{\lambda\in[0,1]}$ be any chain of closure operators and set, for every $\lambda\in[0,1]$ and $X\subseteq F$

$$J_\lambda'(X) = \bigcap_{x<\lambda}J_x(X).$$

Then $(J_\lambda')_{\lambda\in[0,1]}$ is a continuous chain of closure operators whose associated fuzzy closure operator coincides with the one associated with $(J_\lambda)_{\lambda\in[0,1]}$.

Definition 3.15.4 Let $(J_\lambda)_{\lambda\in[0,1]}$ be a family of closure operators and let J be the associated fuzzy a-c-operator. We define the *fuzzy closure operator associated with* $(J_\lambda)_{\lambda\in[0,1]}$ as the closure operator c(J) generated by J. In this case we say that c(J) is *stratified*. If $(J_\lambda)_{\lambda\in[0,1]}$ is a chain then we say that c(J)=J is *well stratified*.

We are not able either to prove or disprove that a stratified closure operator that is not well stratified exists. Now, we define a notion of stratified closure system that is well related with the one of stratified closure operator.

Definition 3.15.5 Let U be a set and $(C_\lambda)_{\lambda\in[0,1]}$ a family of classes of subsets of U. Then the fuzzy system

$$C=\{A\in\mathcal{F}(U) \mid A_\lambda\in C_\lambda \text{ for every } \lambda\neq 0\} \qquad (3.80)$$

is said to be *the fuzzy system associated with* $(C_\lambda)_{\lambda\in[0,1]}$.

We have the following obvious proposition.

Proposition 3.15.6 Let $(C_\lambda)_{\lambda\in[0,1]}$ be a family of closure systems. Then the fuzzy system C associated with $(C_\lambda)_{\lambda\in[0,1]}$ is a fuzzy closure system.

Proof. Let $(A_i)_{i\in I}$ be a family of elements of C. Then, since, for every $\lambda\in[0,1]$, $(A_i)_\lambda\in C_\lambda$ and $(\bigcap_{i\in I}A_i)_\lambda = \bigcap_{i\in I}(A_i)_\lambda\in C_\lambda$, we have that $\bigcap_{i\in I}A_i\in C$.

Obviously, (3.80) generalizes the formula for the canonical extension of a classical closure system. In order to give a notion of continuous chain for closure systems that is well related to the one of continuous chain for closure operators, we say that a family of closure systems $(C_\lambda)_{\lambda\in[0,1]}$ is a *chain* if

(i) $C_0=\{U\}$; (ii) $(C_\lambda)_{\lambda\in[0,1]}$ is order-preserving.

Such a family is called a *continuous chain* if, for every $\lambda\in[0,1]$

(j) $C_0=\{U\}$; (jj) $C_\lambda=Sup\{C_x \mid x<\lambda\}$,

Here the operator *Sup* is the join in the lattice of closure systems and hence (jj) means that C_λ is the closure system generated by $\bigcup_{x<\lambda}C_x$.

Definition 3.15.7 Let C be a fuzzy closure system associated with a family $(C_\lambda)_{\lambda\in[0,1]}$ of closure systems, then C is said to be *stratified*. If $(C_\lambda)_{\lambda\in[0,1]}$ is a chain, then C is said to be *well stratified*.

Any family $(J_\lambda)_{\lambda\in[0,1]}$ of closure operators defines a corresponding family $(\boldsymbol{C}(J_\lambda))_{\lambda\in[0,1]}$ of closure systems and any family $(C_\lambda)_{\lambda\in[0,1]}$ of closure systems defines a corresponding family $(\boldsymbol{J}(C_\lambda))_{\lambda\in[0,1]}$ of closure operators. We have the following equivalencies whose proof is matter of routine.

Proposition 3.15.8 Let $(J_\lambda)_{\lambda\in[0,1]}$ be a family of closure operators. Then

$$(J_\lambda)_{\lambda\in[0,1]} \text{ is a chain} \Leftrightarrow (\boldsymbol{C}(J_\lambda))_{\lambda\in[0,1]} \text{ is a chain}$$
$$(J_\lambda)_{\lambda\in[0,1]} \text{ is a continuous chain} \Leftrightarrow (\boldsymbol{C}(J_\lambda))_{\lambda\in[0,1]} \text{ is a continuous chain.}$$

The following proposition shows that the associated fuzzy closure operators and the associated fuzzy closure systems are related in a natural way.

Proposition 3.15.9 Let $(J_\lambda)_{\lambda\in[0,1]}$ be a family of closure operators and J the associated fuzzy closure operator. Besides, let $(\boldsymbol{C}(J_\lambda))_{\lambda\in[0,1]}$ be the corresponding family of closure systems and C the associated fuzzy closure system. Then, J=$\boldsymbol{J}(C)$, that is,

$$\begin{array}{ccc} (J_\lambda)_{\lambda\in[0,1]} & \rightarrow & (\boldsymbol{C}(J_\lambda))_{\lambda\in[0,1]} \\ \downarrow & & \downarrow \\ \boldsymbol{J}(C) & \leftarrow & C \end{array}$$

Proof. In order to prove that J=$\boldsymbol{J}(C)$ we prove that $\boldsymbol{C}(J)=C$. Let A be an element of C, then every cut A_λ belongs to $\boldsymbol{C}(J_\lambda)$ and A_λ is a fixed point for J_λ. Then

$$J(A)(x)=Sup\{\lambda\in[0,1] \mid x\in J_\lambda(A_\lambda)\}=Sup\{\lambda\in[0,1] \mid x\in A_\lambda\}=A(x)$$

and this proves that $A\in\boldsymbol{C}(J)$. Conversely, if $J(A)=A$, then, for every $x\in U$, $A(x)=J(\mathrm{A})(x)=\mathrm{Sup}\{\lambda\in[0,1] \mid x\in J_\lambda(A_\lambda)\}$. In other words, $x\in J_\lambda(A_\lambda)$ implies $\lambda\leq A(x)$, i.e., $x\in A_\lambda$. Then, since $J_\lambda(A_\lambda)$ is contained in A_λ, A_λ is a fixed point for J_λ. Thus $A\in C$.

In a similar way one proves the following proposition.

Proposition 3.15.10 Let $(C_\lambda)_{\lambda\in[0,1]}$ be any family of closure systems and C the associated fuzzy closure system. Besides, let $(\boldsymbol{J}(C_\lambda))_{\lambda\in[0,1]}$ be the corresponding family of closure operators and J the associated fuzzy closure operator. Then, C=$\boldsymbol{C}(J)$, that is,

$$\begin{array}{ccc} (C_\lambda)_{\lambda\in[0,1]} & \rightarrow & (\boldsymbol{J}(C_\lambda))_{\lambda\in[0,1]} \\ \downarrow & & \downarrow \\ C=\boldsymbol{C}(J) & \leftarrow & J \end{array}$$

Corollary 3.15.11 If J is a closure operator, then

$$J \text{ stratified} \Leftrightarrow \boldsymbol{C}(J) \text{ stratified}$$
$$J \text{ well stratified} \Leftrightarrow \boldsymbol{C}(J) \text{ well stratified.}$$

If C is a fuzzy closure system, then

$$C \text{ stratified} \Leftrightarrow \boldsymbol{J}(C) \text{ stratified.}$$
$$C \text{ well stratified} \Leftrightarrow \boldsymbol{J}(C) \text{ well stratified.}$$

We can rewrite all the definitions and results in this section in terms of deduction systems.

Definition 3.15.12 Let $((F,D_\lambda))_{\lambda\in[0,1]}$ be a family of crisp deduction systems and D the closure operator associated with $(D_\lambda)_{\lambda\in[0,1]}$. Then (F,D) is called, *the fuzzy deduction system associated with* $((F,D_\lambda))_{\lambda\in[0,1]}$. In this case, we say that (F,D) is *stratified* and, if $((F,D_\lambda))_{\lambda\in[0,1]}$ is a chain, we say that (F,D) is *well stratified*.

We can reformulate formula (3.78) in logical terms. Let $((F,D_\lambda))_{\lambda\in[0,1]}$ be a family of deduction systems. We say that a formula α *is a consequence of an initial valuation* V *at degree* λ, in brief $V \vdash_\lambda \alpha$, provided α is a consequence of V_λ by (F,D_λ). Then the formula for the deduction system associated with $((F,D_\lambda))_{\lambda\in[0,1]}$ becomes

$$D(V)(\alpha)=Sup\{\lambda\in[0,1] \mid V \vdash_\lambda \alpha\}. \qquad (3.81)$$

for every initial valuation V and $\alpha\in F$.

We conclude this section by emphasizing the following immediate characterization of the theories of a stratified fuzzy deduction system.

Theorem 3.15.13 Let (F,D) be a fuzzy deduction system. Then (F,D) is stratified iff the class of its theories is a stratified closure system. Also, if (F,D) is associated with the family $((F,D_\lambda))_{\lambda\in[0,1]}$ of deduction systems, then

$$T \text{ is a theory of } (F,D) \Leftrightarrow \text{ every cut } T_\lambda \text{ of } T \text{ is a theory of } (F,D_\lambda).$$

3.16 GRADED CONSEQUENCE RELATIONS

The notion of a stratified deduction system will now be applied to the concept of graded consequence relation proposed in [Chakraborty 1988] and in [Chakraborty 1995]. The idea is to extend the concept of consequence relation $\vdash$ that plays a central role in any crisp logic.

Definition 3.16.1 We call *conclusion relation* any relation $\vdash$ from $P(F)$ to F. For $X\in P(F)$ and $\alpha\in F$, we write $X\vdash\alpha$ to denote that $(X,\alpha)\in\vdash$. We say that a conclusion relation $\vdash$ is a *consequence relation* if, for every X, Y, Z in $P(F)$ and $\alpha\in F$,

(i) $X\vdash\alpha$ whenever $\alpha\in X$

(ii) $X\vdash\alpha \Rightarrow X\cup Y\vdash\alpha$

(iii) $X\vdash\beta$ for every $\beta\in Z$ and $X\cup Z\vdash\alpha \Rightarrow X\vdash\alpha$.

If $\vdash$ is a consequence relation and $X\vdash\alpha$, then we say that α *is a consequence of* X. The meaning of the above conditions is immediate. Condition (i) says that every formula in X is a consequence of X, condition (ii) that the logic under consideration is monotone, (iii) that if each formula in Z follows from X and we are able to prove α from $X\cup Z$, then we may prove α directly from X. There is a strict connection between the operators and the conclusion relations.

Definition 3.16.2 Given an operator J we call *conclusion relation associated with* J the relation $\vdash_J$ defined by

$$X\vdash_J\alpha \Leftrightarrow \alpha\in J(X). \tag{3.82}$$

Given a conclusion relation $\vdash$ we call *operator associated with* $\vdash$ the operator $J_\vdash$ defined by

$$J_\vdash(X)=\{\alpha\in F \mid X\vdash\alpha\}. \tag{3.83}$$

The following proposition, whose proof is matter of routine, shows that (3.82) and (3.83) define a bijective correspondence between the class of the operators and the class of the conclusion relations.

Proposition 3.16.3 Let J be an operator and $\vdash$ a conclusion relation. Then

- the operator associated with $\vdash_J$ coincides with J
- the relation associated with $J_\vdash$ coincides with $\vdash$.

Definitions (3.82) and (3.83) establish also a one-to-one correspondence between crisp consequence relations and closure operators.

Proposition 3.16.4 Let $\vdash$ be a conclusion relation. Then

$\vdash$ is a consequence relation $\Leftrightarrow$ $J_\vdash$ is a closure operator.

Let J be an operator. Then

J is a closure operator $\Leftrightarrow$ $\vdash_J$ is a consequence relation.

Proof. At first we prove that if $\vdash$ is a consequence relation, then $J_\vdash$ is a closure operator. Indeed, from (i) it follows that $J_\vdash(X)\supseteq X$ and from (ii) that $X\supseteq Y$ implies $J_\vdash(X)\supseteq J_\vdash(Y)$. In order to prove that $J_\vdash(J_\vdash(X))=J_\vdash(X)$, observe that, since $X\vdash\beta$ for every $\beta\in J_\vdash(X)$, by (iii),

$$J_\vdash(X)\vdash\alpha \Rightarrow X\vdash\alpha.$$

Thus, $J_\vdash(J_\vdash(X)) = \{\alpha\in F \mid J_\vdash(X)\vdash\alpha\} \subseteq J_\vdash(X)$ and therefore $J_\vdash(J_\vdash(X))=J_\vdash(X)$. Now, we prove that if J is a closure operator, then $\vdash_J$ is a consequence relation. Indeed, (i) and (ii) are immediate. In order to prove (iii) suppose $X\vdash_J\beta$ for every $\beta\in Z$ and $X\cup Z\vdash_J\alpha$, i.e., $Z \subseteq J(X)$ and $\alpha \in J(X \cup Z)$. Then, since $X \cup Z \subseteq J(X)$, $\alpha \in J(X \cup Z) \subseteq J(J(X)) = J(X)$ and this proves that $X \vdash_J \alpha$. Thus $\vdash_J$ is a consequence relation.

Assume that $\vdash_J$ is a consequence relation. Then, since by Proposition 3.16.3 the operator associated with $\vdash_J$ is J, by the first implication we have just proven J is a closure operator. Finally, assume that $J_\vdash$ is a closure operator. Then, since we have just proven that the relation associated with $J_\vdash$ is a consequence operator, by Proposition 3.16.3 $\vdash$ is a consequence relation.

The following corollary shows that the theory of consequence relations coincides with the theory of the closure operators.

Corollary 3.16.5 A conclusion relation $\vdash$ is a consequence relation iff a deduction system (F,D) exists such that

$$X \vdash \alpha \iff \alpha \in D(X). \tag{3.84}$$

In order to extend the just given notions, we consider fuzzy binary relations from $P(F)$ to F. We call a *graded conclusion relation* any fuzzy relation $g: P(F) \times F \to [0,1]$. By following [Chakraborty 1988], we write $g(X \vdash \alpha)$ instead of $g(X,\alpha)$.

Definition 3.16.6 We say that a graded conclusion relation g is a *graded consequence relation* if, for every $X, Y, Z \in P(F)$ and $\alpha \in F$

(i) $g(X \vdash \alpha) = 1$ for every $\alpha \in X$

(ii) $g(X \cup Y \vdash \alpha) \geq g(X \vdash \alpha)$

(iii) $g(X \vdash \alpha) \geq (Inf\{g(X \vdash z) \mid z \in Z\}) \wedge g(X \cup Z \vdash \alpha)$.

The question arises wether an analogous of Corollary 3.16.5 holds for the graded consequence relations or not. Now, let J be a fuzzy closure operator and define a graded conclusion relation g by setting

$$g(X \vdash \alpha) = J(X)(\alpha) \tag{3.85}$$

for $X \subseteq F$ and $\alpha \in F$. Then g satisfies (i) and (ii) but not (iii) and hence is not a graded consequence relation, in general. The following example is due to M.K. Chakraborty. Let $F = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and define A_1, A_2 by setting

$$A_1(\alpha_1) = A_1(\alpha_3) = 1,\ A_1(\alpha_2) = 0.7,\ A_1(\alpha_4) = 0.8,\ A_2(\alpha_1) = A_2(\alpha_3) = A_2(\alpha_4) = 1,\ A_2(\alpha_2) = 0.9.$$

Then, the class $C = \{A_1, A_2\}$ defines the fuzzy closure operator $J = \boldsymbol{J}(C)$ where, for every fuzzy subset A and $\alpha \in F$, $J(A)(\alpha) = Inf\{A_i(\alpha) \mid A_i \supseteq A\}$. Take $X = \{\alpha_1, \alpha_3\}$ and $Z = \{\alpha_4\}$, then a simple calculation gives

$$J(X)(\alpha_2) = 0.7\ ,\ J(X)(\alpha_4) = 0.8\ ,\ J(X \cup Z)(\alpha_2) = 0.9\ .$$

So, we have that

$$g(X \vdash \alpha_2) = 0.7,\ Inf\{g(X \vdash z) \mid z \in Z\} = 0.8 \text{ and } g(X \cup Z \vdash \alpha_2) = 0.9.$$

Hence,

$$g(X \vdash \alpha_2) < (Inf\{g(X \vdash z) \mid z \in Z\}) \wedge g(X \cup Z \vdash \alpha_2),$$

and this proves that (iii) is not satisfied.

The following theorem extends Corollary 3.16.5 in terms of stratified deduction systems (see [Gerla 1997]).

Theorem 3.16.7 A graded conclusion relation $g: P(F) \times F \to [0,1]$ is a consequence relation iff a well stratified deduction system (F,D) exists such that

$$g(X \vdash \alpha)=D(X)(\alpha) \tag{3.86}$$
for every X subset of F and $\alpha \in F$.

Proof. Let g be a graded consequence relation and, for every $\lambda \in [0,1]$, let D_λ be the operator defined by setting
$$D_\lambda(X)=\{\alpha \in F \mid g(X \vdash \alpha) \geq \lambda\}. \tag{3.87}$$
Then $(D_\lambda)_{\lambda \in [0,1]}$ is a continuous chain of closure operators. Indeed, it is immediate that each D_λ satisfies the inclusion and monotony properties. Let $\alpha \in D_\lambda(D_\lambda(X))$, i.e., $g(D_\lambda(X) \vdash \alpha) \geq \lambda$. For every $z \in D_\lambda(X)$ we have $g(X \vdash z) \geq \lambda$, and therefore, by condition (iii),
$$g(X \vdash \alpha) \geq Inf\{g(X \vdash z), z \in D_\lambda(X)\} \wedge g(D_\lambda(X) \vdash \alpha\} \geq \lambda.$$
Thus, $\alpha \in D_\lambda(X)$. In order to prove that $(D_\lambda)_{\lambda \in [0,1]}$ is continuous, let X be a set of formulas. Then it is immediate that $D_0(X)=F$ and, if $\mu \in [0,1]$, then
$$\alpha \in D_\mu(X) \Leftrightarrow g(X \vdash \alpha) \geq \mu \Leftrightarrow g(X \vdash \alpha) \geq \lambda \text{ for every } \lambda < \mu \Leftrightarrow \alpha \in \bigcap_{\lambda<\mu} D_\lambda(X).$$
Let D be the fuzzy closure operator associated with $(D_\lambda)_{\lambda \in [0,1]}$. It is immediate that
$$g(X \vdash \alpha)=Sup\{\lambda \mid g(X \vdash \alpha) \geq \lambda\}=Sup\{\lambda \mid \alpha \in D_\lambda(X)\}=D(X)(\alpha).$$
Conversely, let (F,D) be the fuzzy deduction system associated with a given continuous chain $((F,D_\lambda))_{\lambda \in [0,1]}$ of deduction systems and define g by (3.86). It is immediate that g satisfies (i) and (ii). In order to prove (iii) we have to prove that
$$Inf_{z \in Z} Sup\{\lambda \mid z \in D_\lambda(X)\} \wedge Sup\{\lambda \mid \alpha \in D_\lambda(X \cup Z)\} \leq Sup\{\lambda \mid \alpha \in D_\lambda(X)\}. \tag{3.88}$$
Let $\nu = Inf_{z \in Z} Sup\{\lambda \mid z \in D_\lambda(X)\}$. Then for every $z \in Z$, $Sup\{\lambda \mid z \in D_\lambda(X)\} \geq \nu$ whence, since $(D_\lambda)_{\lambda \in [0,1]}$ is a continuous chain, we have that, for every $\lambda \leq \nu$, $Z \subseteq D_\lambda(X)$ and therefore $D_\lambda(X \cup Z)=D_\lambda(X)$. Then, if $Sup\{\lambda \mid \alpha \in D_\lambda(X \cup Z)\} < \nu$, we have that
$$Sup\{\lambda \mid \alpha \in D_\lambda(X \cup Z)\} \leq Sup\{\lambda \mid \alpha \in D_\lambda(X)\}$$
and (3.88) holds. In the case $Sup\{\lambda \mid \alpha \in D_\lambda(X \cup Z)\} \geq \nu$, it is $Sup\{\lambda \mid \alpha \in D_\lambda(X)\} \geq \nu$. Indeed, otherwise we would have $\alpha \notin D_\nu(X)$ but $\alpha \in D_\nu(X \cup Z)$, while $D_\nu(X \cup Z)=D_\nu(X)$. Thus (3.88) holds again.

We conclude this section by considering the compactness property for a graded consequence relation. In the following, if X is a set we denote by $P_f(X)$ the class of finite subsets of X. A conclusion relation $\vdash$ is *compact*, provided that
$$X \vdash \alpha \Leftrightarrow \text{there exists } X_f \in P_f(X) \text{ such that } X_f \vdash \alpha.$$
It is immediate that
$$\vdash \text{ is compact} \Leftrightarrow J_\vdash \text{ is compact}$$
and that if J is an operator,
$$J \text{ is compact} \Leftrightarrow \vdash_J \text{ is compact.}$$
A graded conclusion relation g is said to be *compact* if
$$g(X \vdash \alpha)=Sup\{g(X_f \vdash \alpha) \mid X_f \in P_f(X)\}. \tag{3.89}$$
Finally, g is called *strongly compact* if
$$g(X \vdash \alpha)=Max\{g(X_f \vdash \alpha) \mid X_f \in P_f(X)\}. \tag{3.90}$$

Theorem 3.16.8 A graded conclusion g is a compact graded consequence iff there exists a chain $(K_\lambda)_{\lambda \in [0,1]}$ of compact closure operators such that
$$g(X \vdash \alpha)=Sup\{\lambda \in [0,1] \mid \alpha \in K_\lambda(X)\}. \tag{3.91}$$

Proof. Let $(K_\lambda)_{\lambda\in[0,1]}$ be a chain of compact closure operators such that (3.91) holds then, by Theorem 3.16.7, g is a graded consequence relation. To prove that g is compact, observe that

$$\{\lambda\in[0,1] \mid \alpha\in K_\lambda(X)\}$$
$$= \{\lambda\in[0,1] \mid \alpha\in\bigcup\{ K_\lambda(X_f) \mid X_f\in P_f(X)\}$$
$$= \bigcup\{\lambda\in[0,1] \mid \alpha\in K_\lambda(X_f), X_f\in P_f(X)\}.$$

Then,

$$g(X \vdash \alpha)=Sup\{\lambda\in[0,1] \mid \alpha\in K_\lambda(X)\}$$
$$= Sup\{\lambda\in[0,1] \mid X_f\in P_f(X), \alpha\in K_\lambda(X_f)\}$$
$$= Sup\{g(X_f\vdash \alpha) \mid X_f\in P_f(X)\}.$$

Conversely, suppose that g is a compact graded consequence relation and set, for every $\lambda<1$ and $X\subseteq F$,

$$K_\lambda(X)=\{\alpha\in F \mid g(X\vdash \alpha)>\lambda\}. \tag{3.92}$$

Then it is immediate that K_λ is compact and that satisfies inclusion and monotony properties. To prove idempotence, i.e., $K_\lambda(K_\lambda(X))\subseteq K_\lambda(X)$, observe that

$$\alpha\in K_\lambda(K_\lambda(X)) \Leftrightarrow g(K_\lambda(X) \vdash \alpha)>\lambda$$
$$\Leftrightarrow \text{a finite subset } Z_f \text{ of } K_\lambda(X) \text{ exists s.t. } g(Z_f\vdash \alpha)>\lambda.$$

Since, $g(X \vdash z)>\lambda$ for every $z\in Z_f$ and $g(X\cup Z_f\vdash \alpha)\geq g(Z_f\vdash \alpha)>\lambda$, we have that

$$g(X\vdash\alpha) \geq (Inf\{g(X\vdash z), z\in Z_f\})\wedge g(X\cup Z_f\vdash \alpha)>\lambda,$$

and this proves that $\alpha\in K_\lambda(X)$.

Set K_1 be equal to the identity map, then the family $(K_\lambda)_{\lambda\in[0,1]}$ is a chain of compact closure operators such that (3.91) holds.

The chain in the previous proposition is not continuous, in general. Indeed the following proposition holds:

Proposition 3.16.9 Let g be a graded conclusion. Then g is a strongly compact consequence relation iff there exists a continuous chain $(K_\lambda)_{\lambda\in[0,1]}$ of compact closure operators such that (3.91) holds.

Proof. Assume that g is representable by a continuous chain of compact closure operators $(K_\lambda)_{\lambda\in[0,1]}$ and let $g(X \vdash \alpha)=\mu$, i.e., $\mu=\sup\{\lambda\in[0,1] \mid \alpha\in K_\lambda(X)\}$. Then, since $\alpha\in K_\lambda(X)$ for every $\lambda<\mu$ and $K_\mu(X)=\bigcap_{\lambda<\mu}K_\lambda(X)$, we have that $\alpha\in K_\mu(X)$. By the compactness of K_μ this happens whenever we have that a finite part X_f of X exists such that $\alpha\in K_\mu(X_f)$. So, $g(X_f \vdash \alpha)\geq\mu$ and therefore $g(X \vdash \alpha)=g(X_f \vdash \alpha)$. This proves that g is strongly compact.

Conversely, suppose that g is a strongly compact graded consequence and let $(D_\lambda)_{\lambda\in[0,1]}$ be defined by (3.87). In proving Theorem 3.16.7 we have early observed that $(D_\lambda)_{\lambda\in[0,1]}$ is a continuous chain of closure operators. To prove that each D_λ is compact, observe that

$$\alpha\in D_\lambda(X) \Leftrightarrow g(X_f\vdash \alpha)\geq\lambda \text{ for a suitable } X_f \text{ finite part of } X$$
$$\Leftrightarrow \alpha\in\bigcup\{D_\lambda(X_f) \mid X_f \text{ finite part of } X\}.$$

We conclude by noticing that Ying's similarity logic (see Section 3.14) defines a compact graded consequence.

Proposition 3.16.10. Let $Cons:P(F)\times F\rightarrow[0,1]$ be the graded conclusion defined in Ying's similarity logic. Then *Cons* is a compact graded consequence.

Proof. By Theorem 3.14.2, $Cons(X,\alpha)=D^*(H(X))(\alpha)$ where D^* is the canonical extension of the classical deduction operator and H is defined by (3.76). By Theorem 3.14.3, $D^*\circ H$ is a closure operator. So, we have only to prove that a chain of compact closure operators can represent $D^*\circ H$. Now, at first observe that H is associated with a suitable chain of compact closure operators. Indeed, set, for any $\lambda\in[0,1]$,

$$H_\lambda(X)=\{x\in F \mid S(x,x')\geq\lambda \text{ for a suitable } x'\in X\cup Tau\}.$$

Then we obtain a chain $(H_\lambda)_{\lambda\in[0,1]}$ of (compact) closure operators. Moreover,

$$\begin{aligned}H(V)(x)&=SIM(V\cup Tau)(x)=Sup\{S(x,x')\wedge(V\cup Tau)(x') \mid x'\in F\}\\ &=Sup\{\lambda\in[0,1] \mid \lambda\leq S(x,x') \text{ and } \lambda\leq(V\cup Tau)(x') \text{ for a suitable } x'\in F\}\\ &=Sup\{\lambda\in[0,1] \mid x\in H_\lambda((V\cup Tau)_\lambda)\}=Sup\{\lambda\in[0,1] \mid x\in H_\lambda(V_\lambda)\},\end{aligned}$$

where the last equality is justified by the fact that

$$H_\lambda((V\cup Tau)_\lambda) = H_\lambda(V_\lambda\cup Tau)=H_\lambda(V_\lambda).$$

Since H is associated with $(H_\lambda)_{\lambda\in[0,1]}$, by (3.79) we have that $H(A)_{>\lambda}\subseteq H_\lambda(A_\lambda)\subseteq H(A)_\lambda$ and therefore, by the monotony of D,

$$D(H(A)_{>\lambda})\subseteq D(H_\lambda(A_\lambda)))\subseteq D(H(s)_\lambda).$$

Due to the compactness of D and Proposition 3.12.3, we have that

$$D^*(H(A))(x) \leq Sup\{\lambda\in[0,1] \mid x\in D(H_\lambda(A_\lambda))\} \leq D^*(H(A))(x)$$

and therefore that $D^*\circ H$ is associated with the chain $(D\circ H_\lambda)_{\lambda\in[0,1]}$ of compact operators. Finally, to prove that each $D\circ H_\lambda$ is a closure operator, denote by H_λ' the operator associated with the crisp similarity S_λ by (3.76). Then, by Theorem 3.14.3, $D^*\circ H_\lambda'$ is a closure operator. Thus, since H_λ coincides with H_λ' and D^* with D on the class of crisp subsets, $D\circ H_\lambda$ is a closure operator.

References

Biacino L. (1993). Generated envelopes, *Journal of Math. Anal. Appl.*, 172, 179-190.

Biacino L. and Gerla G. (1984). Closure systems and L-subalgebras, *Information Sciences*, 32, 181-195.

Biacino L. and Gerla G. (1992). Generated necessities and possibilities, *International Journal of Intelligent Systems*, 7, 445-454.

Biacino L. and Gerla G. (1996). An extension principle for closure operators, *J. of Math. Anal. Appl.*, 74, 432-440.

Biacino L. and Gerla G. (1998). Logics with approximate premises, *International J. of Intell. Syst.*, 13, 1-10.

Biacino L. and Gerla G. and Ying M.S. (1998). Approximate reasoning based on similarity, *submitted*.

Brown D.J. and Suszko R. (1973). Abstract Logics, *Dissertationes Mathematicae*, 102, 9-42.

Chang C.C. and Keisler H. J. (1966). *Continuous Model Theory*, Princeton, Princeton University Press.

Chakraborty M.K. (1988). Use of fuzzy set theory in introducing graded consequence in multiple valued logic, in *Fuzzy Logic in Knowledge-Based Systems, Decision and Control*, M.M. Gupta and T.Yamakawa (eds), North-Holland, Amsterdam, 247-257.

Chakraborty M.K. (1995). Graded consequence: further studies, *Journal of Applied non-classical Logic*, 5 227-237.

Conrad F. (1980). Fuzzy topological concepts, *J. Math. Anal. Appl.*, 74, 432-440.

Dubois D. and Lang J. and Prade H. (1994). *Possibilistic logic*, In: D.Gabbay, C.Hogger and J.Robinson (eds.) Handbook of Logic in Artificial Intelligence and Logic Programming, vol. 3. Clarendon Press.

Dubois D. and Prade H. (1987). Necessity measures and the resolution principle, *IEEE Trans. Systems Man and Cybernetics*, 17, 474-478.

Dubois D. and Prade H. (1988). *Possibility Theory*, Plenum Press, New York and London.

Dubois D., Esteva F., Garcia P., Godo L and Prade H. (1997). A Logical Approach to Interpolation Based on Similarity Relations, *Int. J. Approximate Reasoning*, 17, 1-36.

Esteva F., Garcia P., Godo L. and Rodriguez R. (1997). A Modal Account of Similarity-Based Reasoning, *Int. J. Approximate Reasoning*, 16, 235-260.

Ferrante F., Gerla G. and Sessa I. M. (1998). Similarity-based Unification, submitted.

Gerla G. (1994). An extension principle for fuzzy logics, *Mathematical Logic Quarterly*, 40, 357-80.

G. Gerla (1998). Closure operators, fuzzy logic and constraints, to appear on the book *Fuzzy Sets, Logics, and Knowledge-Based Reasoning.*

Gerla G. (1996). Closure operators, fuzzy logic and constraints, to appear in *Fuzzy Sets, Logics and Reasoning about knowledge*.

Gerla G. (1997). Graded consequence relations and closure operators, *Journal of Applied non-classical Logic*, 6, 369-379.

Gerla, G.(1997), Probability-like functionals and fuzzy logic, *J. of Math. Anal. Appl.*, 216, 438-465.

Gerla G. and Scarpati L. (1998). Extending the extension principles, *Inf. Sciences*, 106, 49-69.

Goguen J.A. (1968/69). The logic of inexact concepts, *Synthese*, 19, 325-373.

Murali V. (1991). Lattice of fuzzy subalgebras and closure systems in I^X, *Fuzzy Sets and Systems*, 41, 101-111.

Negoita C.V. and Ralescu D.A. (1975). Representation theorems for fuzzy concepts, *Kybernetes*, 4, 169-174.

Pavelka J. (1979). On fuzzy logic I: Many-valued rules of inference, *Zeitschr. f. math. Logik und Grundlagen d. Math.*, Bd.25, 45-52.

Ramik J. (1983). Extension principle and fuzzy-mathematical programming, *Kybernetika*, 19, 513-25.

Tarski A. (1956). *Logic, semantics and metamathematics*, Clarendon Press, Oxford.

Ward M. (1942). The closure operators of a lattice, *Annals of Mathematics*, 43, 191-196.

Ying M.S. (1994). A logic for approximate reasoning, *The Journal of Symbolic Logic*, 59, 830-837

Zadeh L.A. (1965). Fuzzy Sets, *Information and Control*, 12, 338-353.
Zadeh, L.A. (1975). Fuzzy logic and approximate reasoning, *Synthese*, 30, 407-428.