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REPRESENTATION THEOREMS FOR FUZZY ORDERS AND QUASI-METRICS

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Abstract. Let L be a complete residuated lattice. Then we show that any L-preorder can be represented both by an implication-based graded inclusion as defined [1] and by a similarity-based graded inclusion as defined in [2]. Also, in accordance with a duality between [0,1]-orders and quasi-metrics, we obtain two corresponding representation theorems for quasi-metrics.

Keywords: fuzzy order, graded inclusion, quasi-metrics, Hausdorff excess.

1. Introduction.

Let $(S,\leq)$ be a preorder and denote by $P(S)$ the class of all the subsets of S. Also, consider the map $h : S\to P(S)$ associating any element $x\in S$ with the subset $h(x) = \{y\in S : y \leq x\}$. Then h is a homomorphism from $(S,\leq)$ to $(P(S),\subseteq)$. If $\leq$ is an order, and $C = h(S)$, then h is an isomorphism between $(S,\leq)$ and $(C,\subseteq)$. This means that any order can be represented by the inclusion relation. In this note we extend such a representation theorem to L-preorders and L-orders where L is a complete residuated lattice. To do this, we extend the set theoretical notion of inclusion into two different definitions of graded inclusion. The first one is the implication-based graded inclusion $Incl : L^S\times L^S \to L$ in the class L^S of all the L-subsets of S proposed by W. Bandler, L. Kohout in [1]. The latter is the similarity-based graded inclusion $Incl' : P(S)\times P(S) \to P(S)$ in the class $P(S)$ of all the subsets of S as defined by L. Biacino and G. Gerla in [2]. Both the definitions are logical in nature since both are interpretations in a multivalued logic of the classical definition of inclusion. As a matter of fact, in this paper we use as an heuristic tool the multivalued logics based on the residuated lattice L (see for example [5]).

Finally, in account of a natural duality between fuzzy orders and quasi-metrics, we prove two corresponding representation theorems for quasi-metric spaces related with the difference-based quasi-metric spaces and the Hausdorff excess spaces, respectively (see also [9] and [10]).

2. Preliminaries.

A *residuated lattice* is a structure $(L,\vee, \wedge,*, \to,0,1)$ such that

1. $(L,\vee, \wedge, 0,1)$ is a complete lattice
2. $(L,*,1)$ is a commutative monoid
3. $*$ is isotone in both arguments
4. $\to$ is a residuation operation with respect to $*$, i.e.

$$a*x \leq b \iff x\leq a\to b.$$

We say that $(L,\vee, \wedge,*, \to,0,1)$ is *complete* (*linear*) provided that L is complete (linear). The operation $\to$ is called an *implication*. Also, we define an *equivalence* operation by setting $x\leftrightarrow y = (x\to y)*(y\to x)$. The following proposition lists the main properties of a complete residuated lattice (see [5]).

Proposition 2.1. *Let $(L, \vee, \wedge, *, \to, 0, 1)$ be a complete residuated lattice, x, y and z be elements in L and $(x_i)_{i\in I}$ a family of elements in L. Then the following holds true:*

(*i*) $x\to x = 1$,

(*ii*) $(x\to y)*(y\to z)\leq x\to z$,

(*iii*) $x\to y = 1$ and $y\to x = 1 \Rightarrow x = y$

(*iv*) $x\to y = 1 \iff x\leq y$

(*v*) $x\to y = Sup\{z \in L : x*z \leq y\}$,

(*vi*) $(z\to y)*z \leq y$

(*vii*) $Sup_{i\in I}(x* x_i) = x*(Sup_{i\in I} x_i)$,

(*viii*) $Sup_{i\in I}(x\to x_i) \leq x\to(Sup_{i\in I} x_i)$,

(*ix*) $Sup_{i\in I}(x_i\to x) \leq (Inf_{i\in I} x_i)\to x$,

(*x*) $Inf_{i\in I}(x* x_i) \geq x*(Inf_{i\in I} x_i)$,

(*xi*) $Inf_{i\in I}(x\to x_i) = x\to(Inf_{i\in I} x_i)$,

(*xii*) $Inf_{i\in I}(x_i\to x) = (Sup_{i\in I} x_i)\to x$.

Moreover,

(*xiii*) $x\leftrightarrow x = 1$,

(*xiv*) $x\leftrightarrow y = 1 \iff x = y$

(*xv*) $(x\leftrightarrow y)*(y\leftrightarrow z)\leq x\leftrightarrow z$

(*xvi*) $x\leftrightarrow y = y\leftrightarrow x$.

In particular, we are interested to the residuated lattice in which L coincides with [0,1] and $*$ is a *continuous triangular norm*, i.e. an order-preserving continuous commutative monoid (see [7]). Precisely, we are interested to the following class of triangular norms.

Definition 2.2. A continuous triangular norm $*$ is called *Archimedean* if, for any $x, y \in [0,1]$, $y \neq 0$, an integer n exists such that $x^{(n)} < y$ where $x^{(n)}$ is defined by the equations $x^{(1)} = x$ and $x^{(n+1)} = x{*}x^{(n)}$.

In order to characterize the Archimedean triangular norms, consider the extended interval $[0,\infty]$ and assume that $x+\infty = \infty+x = \infty$ and that $x \leq \infty$ for any $x \in [0,\infty]$. Then we say that a map $f : [0,1]\to[0,\infty]$ is an *additive generator* provided that f is a continuous strictly decreasing function such that $f(1) = 0$. Also, the *pseudoinverse* $f^{[-1]} : [0,\infty] \to [0, 1]$ of f is defined by setting:

$$f^{[-1]}(y) = \begin{cases} f^{-1}(y) & \text{if } y \in f([0, 1]), \\ 0 & \text{otherwise.} \end{cases}$$

Trivially, $f^{[-1]}$ is order-reversing, $f^{[-1]}(0) = 1$ and $f^{[-1]}(\infty) = 0$. Moreover, for any $x\in S$, $f^{[-1]}(f(x)) = x$ and

$$f(f^{[-1]}(x)) = \begin{cases} x & \text{if } x \in f([0, 1]), \\ f(0) & \text{otherwise.} \end{cases}$$

Proposition 2.3. *An operation* $* : [0,1]\times[0,1] \to [0,1]$ *is a continuous Archimedean triangular norm if and only if an additive generator* $f : [0, 1] \to [0, \infty]$ *exists such that*

$$x{*}y = f^{[-1]}(f(x) + f(y)) \tag{2.1}$$

for all x, y in [0, 1]. *In such a case* $x\to y = f^{[-1]}(f(y) - f(x))$ (see [5], [7]).

As an example, assume that $f(x) = -log(x)$ (where, as usual, we set $-log(0) = \infty$). Then, $f^{[-1]}(y) = e^{-y}$ (where, as usual, we set $e^{-\infty} = 0$). Consequently, since

$$x * y = e^{-(-log(x)\ -\ log(y))} = e^{log(x\cdot y)} = x\cdot y,$$

the resulting triangular norm is the usual product in [0,1]. Assume that $f(x) = 1-x$. Then, since $f([0,1]) = [0,1]$, $f^{[-1]}(x) = f(x)$ if $x \in [0,1]$ and $f^{[-1]}(x) = 0$ otherwise. Consequently, in the case $x+y-1 \in [0,1]$ we obtain

$$x * y = 1-(1-x+1-y) = 1-1+x-1+y = x+y-1,$$

while, if $x+y-1 \notin [0,1]$, $x{*}y = 0$. So, in such a case $*$ coincides with the famous Lukasiewicz triangular norm. The minimum is an example of continuous triangular norm which is not Archimedean.

Definition 2.4. Let S be a nonempty set. We call *L-subset of S* any map $s : S\to L$ and we denote by L^S the class of all the L-subsets of S. An *L-relation* is an L-subset of $S\times S$, i.e., a map $r : S\times S\to L$ (see [11] and [12]).

The intended meaning of an L-subset s is that, given any x in S, the value $s(x)$ is the membership degree of x to s. We denote by L^S the class of all the L-subsets of S. Such a class is a complete lattice, namely the direct power of L with index set S. We denote by $\subseteq$ the resulting order in L^S and we call it *Zadeh inclusion*. Then we set $s_1\subseteq s_2$ every time $s_1(x)\leq s_2(x)$ for any x in S. We call *crisp* any L-subset s such that $s(x)\in\{0,1\}$ for any $x\in S$. We can identify any subset X of S with the crisp L-subset $c_X : S \to L$ defined by setting $c_X(x) = 1$ if $x \in X$ and $c_X(x) = 0$ otherwise. More precisely, the map $H : P(S) \to L^S$ associating any $X \in P(S)$ with the L-subset $H(s) = c_X$ is an injective lattice homomorphism from $P(S)$ to L^S. Given a fuzzy subset s and $\lambda \in L$, the *λ-cut* of s is the set $C(s,\lambda) = \{x \in S : s(x) \geq \lambda\}$. In the case $L = [0,1]$, we use expressions as *fuzzy subset* and *fuzzy relation* instead of L-set, and L-relation, respectively.

3. *L*-preorders.

Let S be a set and $ord : S \times S \rightarrow L$ be an L-relation on S. Also, consider the following properties

(i) $ord(x,x) = 1$, (reflexivity)
(ii) $ord(x,y)*ord(y,z) \leq ord(x,z)$, (transitivity)
(iii) $ord(x,y) = ord(y,x) = 1 \Rightarrow x = y$, (antisimmetry)
(iv) $ord(x,y) = ord(y,x)$, (simmetry)

where, $x, y, z \in S$. Then ord is called:

- an *L-preorder* if it satisfies (i) and (ii),
- an *L-order*, provided that it satisfies (i), (ii) and (iii),
- an *L-similarity* if it satisfies (i), (ii) and (iv),
- a *strict L-similarity* if it satisfies (i), (ii), (iii) and (iv).

Given an L-preorder ord, the cut $C(ord,1)$ is a preorder relation we denote by $\leq$. In other words, $\leq$ is defined by setting $x\leq y$ if and only if $ord(x,y) = 1$. Then an L-preorder is an *L-order* if and only if $\leq$ is an order relation. Also, if ord is a similarity, then $C(ord,1)$ is an equivalence relation. If ord is strict, then $C(ord,1)$ is the identity relation.

Definition 3.1. Let (S, r) and (S', r') be two L-relations. We say that a map $h : S\rightarrow S'$ is a *homomorphism* from (S, r) to (S', r') provided that

$$r(x,y) = r'(h(x), h(y)).$$

We say that h is an *isomorphism* if h is a one-one homomorphism.

Proposition 3.2. *If* (S,ord) *is an L-order, then any homomorphism defined in* (S,ord) *is injective.*

Proof. Indeed, from $h(x) = h(y)$ it follows that $ord(x,y) = ord'(h(x),h(y)) = 1$ and $ord(y,x) = ord'(h(y),h(x)) = 1$ and therefore that $x = y$. □

The proof of the following proposition is trivial.

Proposition 3.3. *Let* (S,ord) *be an L-preorder whose associated preorder is* $\leq$, *let* S' *be a nonempty set and* $h : S'\rightarrow S$ *be a map. Also, define ord' by setting*

$$ord'(x,y) = ord(h(x), h(y)).$$

Then (S',ord') *is an L-preorder whose associated preorder* $\leq'$ *satisfies*

$$x\leq' y \Leftrightarrow h(x) \leq h(y).$$

Moreove h is a homomorphism from (S',ord') *to* (S,ord) *and*

- (S,ord) *is an L-order and h is injective* $\Rightarrow$ (S',ord') *is an L-order;*
- (S,ord) *is an L-similarity* $\Rightarrow$ (S',ord') *is an L-similarity;*
- (S,ord) *is a strict L-similarity and h is injective* $\Rightarrow$ (S',ord') *is a strict L-similarity.*

In particular, by assuming that h is the identity map, we obtain the following:

Proposition 3.4. *Let* (S,ord) *be an L-preorder, S' be a nonempty subset of S and ord' the restriction of ord to S'. Then* (S',ord') *is an L-preorder. Moreover,*

- (S,ord) *is an L-order* $\Rightarrow$ (S',ord') *is an L-order;*
- (S,ord) *is an L-similarity* $\Rightarrow$ (S',ord') *is an L-similarity;*
- (S,ord) *is an strict L-similarity* $\Rightarrow$ (S',ord') *is a strict L-similarity.*

In the following proposition we show that any L-order in L induces an L-preorder in the class of all the L-subsets of S.

Proposition 3.5. *Let* $ord : L\times L\rightarrow L$ *be an L-order in L whose associated order is the natural one in L, and define* $Incl : L^S\times L^S\rightarrow L$ *by setting*

$$Incl(s_1,s_2) = Inf\{ord(s_1(x),s_2(x)) : x\in S\}. \tag{3.1}$$

Then Incl is an L-order in L^S *whose associated order is the Zadeh inclusion.*

Proof. Trivially, $Incl(s,s) = Inf\{ord(s(x), s(x)) : x\in S\} = 1$. Also, by () of Proposition 2.1,

$$\begin{aligned} Incl(s_1,s_2)*Incl(s_2,s_3) &= (Inf\{ord(s_1(x),s_2(x)) : x\in S\})*(Inf\{ord(s_2(x'),s_3(x')) : x'\in S\}) \\ &\le Inf_{x\in S}[ord(s_1(x),s_2(x))*(Inf\{ord(s_2(x'),s_3(x')) : x'\in S\})] \\ &\le Inf_{x\in S}\, Inf_{x'\in S}\,[ord(s_1(x),s_2(x))*ord(s_2(x'),s_3(x'))] \\ &\le Inf_{x\in S}\, ord(s_1(x),s_2(x))*ord(s_2(x),s_3(x) \\ &\le Inf\{(ord(s_1(x),s_3(x)) : x\in S\} = Incl(s_1,s_3). \end{aligned}$$

Finally, if $Incl(s_1,s_2) = 1$ then $ord(s_1(x),s_2(x)) = 1$ and therefore $s_1(x)\le s_1(x)$ for any $x \in S$. □

4. Representing *L*-preorders by implication based inclusions.

In this section we consider a particular class of L-orders in the class L^S of L-subsets of a given set.

Proposition 4.1. *Let L be a complete residuated lattice and S a nonempty set. Then the L-relation in* L^S *defined by setting:*

$$Incl(s_1,s_2) = Inf\{s_1(x)\to s_2(x) : x\in S\} \tag{4.1}$$

is an L-order whose associated order is the Zadeh inclusion between L-subsets.

Proof. By (i), (ii), (iii) and (iv) of Proposition 2.1, the map $\to$ is an *L-order* in L whose associated order coincides with the natural order in L. Then, it is sufficient to apply Proposition 3.5. □

Definition 4.2. Let L be a complete residuated lattice and S a nonempty set. Then we call *implication-based inclusion* the L-relation *Incl* defined by (4.1) and *implication-based inclusion space* any structure (C, *Incl*) where C is a class of *L-subsets* of S (see also [1]).

Observe that this definition is logic in nature. Indeed, consider the first order multivalued logic based on the residuated lattice L. Then, since the universal quantifier is interpreted by the operator *Inf*, we can interpret the number $Incl(s_1,s_2)$ as the valuation of the claim "for every x, if x belongs to s_1 then x belongs to s_2".

Given an L-preorder (S,*ord*) and $z \in S$, we indicate by $h(z)$ the L-subset of elements of S defined by setting, for any $x\in S$,

$$h(z)(x) = ord(x,z). \tag{4.2}$$

We interpret $h(z)$ as the L-subset of elements which are less than or equal to z. This enables us to prove the first representation theorem.

Theorem 4.3. *Let* (S, *ord*) *be an L-preorder, and let* $h : S \to L^S$ *be defined by* (4.2). *Then h is a homomorphism from* (S, *ord*) *to* (L^S, *Incl*)*, i.e. for any z and t in S,*

$$ord(z,t) = Incl(h(z),h(t)). \tag{4.3}$$

Consequently, any L-order, is isomorphic to an implication-based inclusion space.

Proof. Since $ord(x,z)*ord(z,t) \le ord(x,t)$ and $\to$ is a residuation with respect to $*$, it is $ord(z,t) \le ord(x,z)\to ord(x,t)$ and therefore

$$Incl(h(z),h(t)) = Inf\{h(z)(x)\to h(t)(x) : x\in S\} = Inf\{ord(x,z)\to ord(x,t) : x\in S\} \ge ord(z,t).$$

Moreover,

$$\begin{aligned} Incl(h(z),h(t)) &= Inf\{ord(x,z)\to ord(x,t) : x\in S\} \le ord(z,z)\to ord(z,t) \\ &= 1\to ord(z,t) = ord(z,t), \end{aligned}$$

and this proves (4.3). Assume that (S, *ord*) is an L-order and let z and z' be elements in S. Then

$$\begin{aligned} h(z) = h(z') &\Rightarrow h(z)(z) = h(z')(z) \text{ and } h(z)(z') = h(z')(z') \\ &\Rightarrow ord(z,z) = ord(z,z') \text{ and } ord(z',z) = ord(z',z') \\ &\Rightarrow 1 = ord(z,z') \text{ and } ord(z',z) = 1 \Rightarrow z = z'. \end{aligned}$$

This proves that $h : S \to L^S$ is injective. Thus, it is sufficient to set $C = h(S)$. □

Such a representation theorem, together with Proposition 3.5, gives a general tool to define L-orders.

Theorem 4.4. *Any L-preorder* (S, *ord*) *can be obtained by considering a map* $h : S \to L^S$ *and by defining* $ord(x,y) = Incl(h(x),h(y))$. *Also, any L-order can be obtained by choosing h injective.*

Theorem 4.3 entails the following basic representation theorem given in [8].

Theorem 4.5. *An L-relation ord* : $S\times S\to L$ *is an L-preorder if and only if a family* $(s_i)_{i\in I}$ *of L-subsets of S exists such that*

$$ord(x,y) = Inf_{i\in I}\, s_i(x)\to s_i(y). \qquad (4.4)$$

Proof. It is matter of routine to prove that an L-relation defined by (4.4) is an L-*preorder.* Conversely, let *ord* be any L-preorder. Then by Theorem 4.3 we have that

$$ord(z,t) = Inf_{x\in S} h(z)(x)\to h(t)(x).$$

Set $I = S$ and, for any $i\in I$, let s_i be the L-subset defined by setting $s_i(z) = h(z)(i)$. Then $(s_i)_{i\in I}$ is a family of L-subsets such that $ord(z,t) = Inf_{i\text{-}S}\, s_i(z)\to s_i(t)$. □

If *ord* is interpreted as a graded preference relation, then the logical meaning of (4.4) is the following. Assume that $(s_i)_{i\in I}$ is the family of the (graded) properties we consider desirable. Then $ord(x,y)$ is the valuation of the claims *"every desirable property satisfied by x is satisfied by y"*.

Finally, taking in account of the cardinalities, we can prove the following proposition where, as usual, $card(X)$ denotes the cardinality of a set X (see also [6]).

Proposition 4.6. *Let* α *be a cardinal number, and S be a set such that* $card(S) = \alpha$. *Then any L-order* (S', ord) *such that* $card(S') \le \alpha$, *is isomorphic to a substructure of* $(L^S, Incl)$.

Proof. By Theorem 4.3 a mapping $h : S'\to L^{S'}$ exists such that, for any z and t in S,

$$ord(z,t) = Incl(h(z),h(t)). \qquad (4.5)$$

Since $card(S')\le\alpha$, an injective mapping $k : S' \to S$ exists from S' to S. Let $h' : S' \to L^S$ be the function associating any $z\in S'$ with the fuzzy subset $h'(z)$ of S defined by setting, for any $x\in S$,

$$h'(z)(x) = \begin{cases} h(z)(k^{-1}(x)) & \text{if } x\in k(S') \\ 0 & \text{otherwise.} \end{cases}$$

Then h' is injective and

$$Incl(h'(z),h'(t)) = Inf_{x\in S}\, h'(z)(x)\to h'(t)(x) = Inf_{x\in k(S')}\, h(z)(k^{-1}(x)) \to h(t)(k^{-1}(x))$$
$$= Inf_{y\in S'}\, h(z)(y) \to h(t)(y) = Incl(h(z),h(t)) = ord(z,t),$$

i.e., h' is a homomorphism from (S',ord) to $(L^S,Incl)$. □

5. Representing *L*-preorders by similarity-based inclusions.

Another interesting class of L-preorders is related with the notion of similarity as follows. Let *sim* be an L-similarity relation in a set S. Then the map $Sim : L^S \to L^S$ is defined by setting, for any $s\in L^S$ and $x \in S$,

$$Sim(s)(x) = Sup\{sim(x,x')*s(x') : x'\in S\}. \qquad (5.1)$$

We interpret $Sim(s)$ as the L-subset of elements similar with an element of s. If X is a subset of S, we identify X with c_X and therefore we set $Sim(X) = Sim(c_X)$. Obviously,

$$Sim(X)(x) = Sup_{x'\in X}\, sim(x,x').$$

In [4] one proves the following proposition:

Proposition 5.1. *Let sim be a* $*$-*similarity and define Sim* : $L^S \to L^S$ *by* (5.1). *Then Sim is a closure operator in* L^S, *i.e.*

(*i*) *Sim is order preserving,*

(*ii*) $Sim(s) \supseteq s$,

(*iii*) $Sim(Sim(s)) = Sim(s)$.

Proof. Both (i) and (ii) are trivial. In order to prove (*iii*), observe that by the transitivity of *sim*,

$$\begin{aligned} Sim(Sim(s))(x) &= Sup_{x'\in S}\, sim(x',x)*Sim(s)(x') \\ &= Sup_{x'\in S}\, (sim(x',x)*(Sup_{x''\in S}\, sim(x'',x')*s(x''))) \\ &= Sup_{x'\in S}\, Sup_{x''\in S}\, (sim(x',x)*sim(x'',x')*s(x'')) \\ &= Sup_{x'\in S}\, Sup_{x''\in S}\, sim(x'',x')*sim(x',x)*s(x'') \end{aligned}$$

$$\leq \; Sup_{x''\in S}\, sim(x'',x) * s(x'') = Sim(s)(x).$$

This proves that $Sim(Sim(s)) \subseteq Sim(s)$. Since by (*ii*) $Sim(Sim(s)) \supseteq Sim(s)$, we have that $Sim(Sim(s)) = Sim(s)$. □

Since $Sim : L^S \to L^S$ is a map from the set L^S into the L-order $(L^S, Incl)$, by Proposition 3.3 we can obtain a new L-preorder in L^S by setting, for any s_1 and s_2 in L^S,

$$Incl'(s_1, s_2) = Incl(Sim(s_1), Sim(s_2)). \qquad (5.2)$$

Definition 5.2. Let *sim* be an L-similarity, then we call *similarity-based inclusion* the L-relation $Incl' : L^S \times L^S \to L$ defined by (5.2).

From Proposition 3.3 we obtain the following proposition:

Proposition 5.3. *The similarity-based inclusion* $(L^S, Incl')$ *is an L-preorder whose induced preorder* $\leq$ *satisfies*

$$s_1 \leq s_2 \Leftrightarrow Sim(s_1) \subseteq Sim(s_1).$$

Also, Sim is a homomorphism from $(L^S, Incl')$ to $(L^S, Incl)$.

In [2] and [4] the similarity-based inclusions are used to define a similarity-based fuzzy logic. The logical meaning of (5.2) is immediate. Indeed, given two subset X and Y, in classical logic we can define the inclusion $X \subseteq Y$ by the claim "$\forall x \in S$ ($\exists x' \in X$ such that $x' = x \Rightarrow \exists y \in Y$ such that $x = y$)". Then it is very natural to define the graded inclusion by the same claim in a multivalued logic where = is interpreted by a similarity.

In the following we are interested to the substructures of $(L^S, Incl')$ whose elements are crisp subsets of S.

Definition 5.4. We call *similarity-based inclusion space* any structure $(C, Incl')$ where C is a class of subsets of S.

The following proposition shows how calcolate $Incl'$ on the crisp subsets.

Proposition 5.5. *Let X and Y be two subsets of . Then*

$$Incl'(X,Y) = Incl(X, Sim(Y)) = Inf_{x\in X}\, Sim(Y)(x) = Inf_{x\in X} Sup_{y\in Y}\, sim(x,y). \qquad (5.3)$$

Proof. We have that

$$Incl(X,Sim(Y)) = Inf_{x\in S}(c_X(x) \to Sim(Y)(x)) = Inf_{x\in X}(1 \to Sim(Y)(x)) = Inf_{x\in X}\, Sim(Y)(x),$$

and it is immediate that $Inf_{x\in X}\, Sim(Y)(x) = Inf_{x\in X} Sup_{y\in Y}\, sim(x,y)$. Also, trivially, $Incl(X,Sim(Y)) \geq Incl(Sim(X),Sim(Y))$. Moreover, since $sim(x',x) * sim(x,y) \leq sim(x',y)$, we have that $sim(x,y) \leq sim(x',x) \to sim(x',y)$. Then

$$Sim(Y)(x) = Sup_{y\in Y}\, sim(x,y) \leq Sup_{y\in Y}\, (sim(x',x) \to sim(x',y))$$
$$\leq sim(x',x) \to (Sup_{y\in Y}\, sim(x',y)) = sim(x',x) \to Sim(Y)(x')$$

As a consequence, by (*xii*) of Proposition 2.1, for any $x' \in S$,

$$Incl(X,Sim(Y)) = Inf_{x\in X} Sim(Y)(x) \leq Inf_{x\in X}\, (sim(x',x) \to Sim(Y)(x'))$$
$$= (Sup_{x\in X}\, sim(x',x)) \to Sim(Y)(x') = Sim(X)(x') \to Sim(Y)(x'),$$

and finally

$$Incl(X,Sim(Y)) \leq Inf_{x'\in S}(Sim(X)(x') \to Sim(Y)(x')) = Incl(Sim(X),Sim(Y)).$$

This proves (5.3). □

The logical meaning of (5.3) is evident. Indeed, in classical logic we can define the inclusion $X \subseteq Y$ by the claim "$\forall x \in X \exists y \in Y$ such that $x = y$". Then the graded inclusion $Incl'$ is defined by the same claim in a multivalued logic where = is interpreted by a similarity.

Now, we are ready to prove the second representation theorem. To this purpose, we call *L-singleton* or *L-point* any L-subset s such that $Supp(s) = \{x \in S : s(x) \neq 0\}$ is a singleton. Given $a \in S$ and $\lambda \in L$, $\lambda \neq 0$, we denote by s_a^{λ} the L-singleton such that $Supp(s_a^{\lambda}) = \{a\}$ and $s_a^{\lambda}(a) = \lambda$. We say the an L-point s_a^{λ}

belongs to s, in brief $s_a^{\lambda} \in s$, provided that $s_a^{\lambda} \subseteq s$, i.e. provided that $\lambda \leq s(a)$. Finally, if we denote by S' the set of L-points of S, then we define the map $Fp : L^S \to P(S')$ by setting

$$Fp(s) = \{s_a^{\lambda} : s_a^{\lambda} \in s\}.$$

If L is linearly ordered, then Fp is an injective lattice homomorphism (see, for example, [3]).

Proposition 5.6. *Assume that L is linearly ordered and let S' be the set of L-singletons of S. Then a similarity-based inclusion Incl' in S' exists such that, Fp is an injective homomorphism from* $(L^S, Incl)$ *to* $(P(S'), Incl')$, *i.e.,*

$$Incl(s_1,s_2) = Incl'(Fp(s_1),Fp(s_2)). \tag{5.4}$$

Proof. Let id: $S \times S \to R^+$ be the characteristic function of the identity relation, i.e. the function defined by setting $id(x,y) = 1$ if $x = y$ and $id(x,y) = 0$ otherwise. Define the map $sim' : S' \times S' \to L$ by setting

$$sim'(s_a^{\lambda}, s_b^{\mu}) = id(a,b) * (\lambda \leftrightarrow \mu),$$

where $\leftrightarrow$ is the equivalence relation associated with $*$. Then it is easy to prove that (S',sim') is a strict similarity. Let s_2 be an L-subset of S and s_a^{λ} an L-singleton. Then, in the case $\lambda \leq s_2(a)$, it is $Sim(Fp(s_2))(s_a^{\lambda}) = 1$. Otherwise, since L is linearly ordered, $\lambda > s_2(a)$ and therefore

$$\begin{aligned} Sim(Fp(s_2))(s_a^{\lambda}) &= Sup\{sim'(s_a^{\lambda}, s_b^{\mu}) : s_b^{\mu} \in Fp(s_2)\} \\ &= Sup\{id(a,b) * (\lambda \leftrightarrow \mu) : s_b^{\mu} \in Fp(s_2)\} \\ &= Sup\{\lambda \leftrightarrow \mu : s_a^{\mu} \in Fp(s_2)\} \\ &= Sup\{\lambda \leftrightarrow \mu \ : \mu \leq s_2(a) < \lambda\} \\ &= Sup\{\lambda \to \mu \ : \mu \leq s_2(a) < \lambda\} = \lambda \to s_2(a). \end{aligned}$$

Consequently, in the case $s_1 \subseteq s_2$ it is immediate that $Incl(s_1,s_2) = Incl'(Fp(s_1),Fp(s_2)) = 1$. Otherwise,

$$\begin{aligned} Incl(s_1,s_2) &= Inf\{s_1(a) \to s_2(a) : \ s_2(a) < s_1(a)\} \\ &= Inf\{\lambda \to s_2(a) : s_2(a) < \lambda \leq s_1(a)\} \\ &= Inf\{Sim(Fp(s_2))(s_a^{\lambda}) : s_a^{\lambda} \in Fp(s_1)\} = Incl'(Fp(s_1),Fp(s_2)). \end{aligned}$$

□

Theorem 5.7. *Let L be linearly ordered and let* (S,ord) *be any L-preorder. Then a similarity-based inclusion Incl' exists and a homomorphism map h' from* (S,d) *to* $(P(S'),Incl')$. *Consequently, any L-order is isomorphic to a similarity-based inclusion space.*

Proof. Denote by h the homomorphism from (S,d) to $([0,1]^S,Incl)$ given by Theorem 4.3. Then, since

$$ord(z,t) = Incl(h(z),h(t)) = Incl'(Fp(h(z)),Fp(h(t))),$$

it is sufficient to consider the map $h' : S \to P(S')$ defined by setting,

$$h'(x) = Fp(h(x)).$$

The remaining part of the proposition is immediate. □

6. Quasi-metrics.

Given a nonempty set S and a map $d : S \times S \to [0,\infty]$, we consider the following axioms where x, y, z are elements in S :

(d1) $d(x, x) = 0$,

(d2) $d(x, y) + d(y, z) \geq d(x, z)$,

(d3) $d(x, y) = d(y, x) = 0 \Rightarrow x = y$,

(d4) $d(x,y) = d(y,x)$.

Then we say that:

- (S, d) is an *extended generalised quasi-metric space* if d satisfies (d1) and (d2),
- (S, d) is an *extended quasi-metric space* if d satisfies (d1), (d2) and (d3),
- (S, d) is an *extended generalized metric space* if d satisfies (d1), (d2) and (d4),
- (S, d) is an *extended metric space* if d satisfies (d1), (d2), (d3) and (d4).

In the case that $d(x,y) \neq \infty$ for any x and y, we omit the word "*extended*". Then, by referring to the usual definition of metric space, the word *"extended"* means the possibility that a distance is infinite, the word "*quasi*" refers to the lack of the symmetry axiom, the word "*generalised*" refers to the lack of Axiom (d3). Any extended generalized quasi-metric space (S,d) defines an order relation $\leq$ obtained by setting $x \leq y$ if and only if $d(x,y) = 0$. Then an extended generalized quasi-metric space is an extended quasi-metric space if and only if $\leq$ is an order relation. Moreover, if (S,d) is an extended generalized metric space then $\leq$ is an equivalence relation, if (S,d) is an extended metric space, then $\leq$ is the identity relation. The *diameter* $D(X)$ of a subset X of S is the number in $[0,\infty]$ defined by setting

$$D(X) = Sup\{d(x,y) : x,y \in X\}.$$

If $D(X) \neq \infty$, then we say that X is *bounded*. We say that the space (S,d) is *bounded* provided that S is bounded.

Definition 6.1. Let (S,d) and (S',d') be two extended generalized quasi-metric spaces. Then a map $k : S \to S'$ is called an *isometry* provided that

$$d(x,y) = d'(k(x),k(y)) \tag{6.1}$$

for any x, y in S. An *isomorphism* is an one-one isometry.

If (S,d) is an extended quasi-metric space, then any isometry k defined in (S,d) is injective. In fact,

$$k(x) = k(y) \Rightarrow d(x,y) = d'(k(x),k(y)) = 0 \text{ and } d(y,x) = d'(k(y),k(x)) = 0 \Rightarrow x = y.$$

Proposition 6.2. *Let (S,d) be an extended generalized quasi-metric spaces, S' be a nonempty subset of S and d' the restriction of d to S'. Then (S',d') is an extended, generalized quasi-metric space.*

Proof. Trivial. □

Given a quasi-metric in $[0,1]$, we can define a quasi-metric in the class $[0,1]^S$ of all fuzzy subsets of S in a natural way.

Proposition 6.3. *Let S be a nonempty set and $d : [0,1]\times[0,1] \to [0,\infty]$ an extended generalized quasi-metric in $[0,1]$ whose associated order is the usual one in $[0,1]$. Then the map $\delta : [0,1]^S \times [0,1]^S \to R^+$ defined by setting*

$$\delta(s_1,s_2) = Sup\{d(s_1(x),s_2(x)) : x \in S\}. \tag{6.2}$$

is an extended generalized quasi-metric whose associated order is the Zadeh inclusion. Consequently, if d is an extended quasi-metric, then δ is an extended quasi-metric.

Proof. Let d be an extended generalized quasi-metric. Then, trivially, $\delta(s,s) = 0$. Moreover,

$$\begin{aligned}\delta(s_1,s_2)+\delta(s_2,s_3) &= Sup\{d(s_1(x),s_2(x)) : x \in S\} + Sup\{d(s_2(x),s_3(x)) : x \in S\} \\ &\geq Sup\{d(s_1(x),s_2(x)) + d(s_2(x),s_3(x)) : x \in S\} \\ &\geq Sup\{d(s_1(x),s_3(x)) : x \in S\} = \delta(s_1,s_3).\end{aligned}$$

Finally,

$$\delta(s_1,s_2) = 0 \Leftrightarrow d(s_1(x),s_2(x)) = 0 \text{ for any } x \in S \Leftrightarrow s_1(x) \leq s_2(x) \text{ for any } x \in S \Leftrightarrow s_1 \subseteq s_2.$$ □

7. A connection between fuzzy preorders and quasimetrics.

In the following, if the residuated complete lattice L is defined by a triangular norm in $[0,1]$, then we write $*$*-fuzzy preorder*, $*$*-fuzzy order*, $*$*-similarity* instead of *L-preorder*, *L-order*, *L-similarity,* respectively. In this section, in accordance with [8], we emphasize that an interesting connection among $*$-fuzzy preorders and extended generalized quasi-metrics exists. To this purpose, given an additive generator $f : [0,1] \to [0,\infty]$, we associate any extended generalised quasi-metric $d : S\times S \to [0,\infty]$, with the fuzzy relation $o_f(d) : S\times S \to [0,1]$ defined by setting

$$o_f(d)(x, y) = f^{[-1]}(d(x,y)). \tag{7.1}$$

Theorem 7.1. *Let $f : [0,1] \to [0,\infty]$ be an additive generator of a triangular norm $*$. Then, for any extended generalised quasi-metric (S,d), the fuzzy relation $o_f(d)$ defined by (7.1) is a $*$-fuzzy preorder. Moreover,*

- *the preorder associated with $o_f(d)$ coincides with the preorder defined by d ;*
- *any isometry from (S,d) to (S',d'), is a homomorphism from $(S,o_f(d))$ to $(S',o_f(d'))$.*

Proof. Since $o_f(d)(x,x) = f^{[-1]}(d(x,x)) = f^{[-1]}(0) = 1$, $o_f(d)$ is reflexive. To prove the $*$-transitivity, i.e. that

$$o_f(d)(x, y)*o_f(d)(y, z) \leq o_f(d)(x, z), \tag{7.2}$$

observe that in the case $d(x,y)\notin f([0,1])$ we have that $o_f(d)(x, y) = f^{[-1]}(d(x,y)) = 0$ and in the case $d(y,z)\notin f([0,1])$, we have that $o_f(d)(y, z) = f^{[-1]}(d(y,z)) = 0$. In both the cases, (7.2) is trivial. Assume that both $d(x,y)$ and $d(y,z)$ are in $f([0,1])$, then, by (2.1),

$$\begin{aligned} o_f(d)(x, y) * o_f(d)(y, z) &= f^{[-1]}(d(x,y)) * f^{[-1]}(d(y,z)) \\ &= f^{[-1]}(f(f^{[-1]}(d(x,y))) + f(f^{[-1]}(d(y,z)))) \\ &= f^{[-1]}(d(x,y) + d(y,z)) \leq f^{[-1]}(d(x, z)) = o_f(d)(x, z). \end{aligned}$$

Assume that $d(x,y) = 0$, then $o_f(d(x,y)) = f^{[-1]}(d(x,y)) = f^{[-1]}(0) = 1$. Assume that $o_f(d(x,y)) = 1$ and therefore that $f(o_f(d(x,y)) = 0$. Then, $d(x,y) = f(f^{-1}(d(x,y)) = f(f^{[-1]}(d(x,y)) = f(o_f(d(x,y)) = 0$.

The remaining part of the theorem is trivial. □

The proof of the following proposition is immediate.

Proposition 7.2. *Let* $d : S\times S \to [0,\infty]$ *be a map, then:*

i) d is an extended quasi-metric $\Leftrightarrow$ $o_f(d)$ *is a* $*$*-fuzzy order;*

ii) d is an extended generalized metric $\Rightarrow$ $o_f(d)$ *is a* $*$*-similarity;*

iii) d is an extended metric $\Rightarrow$ $o_f(d)$ *is a strict* $*$*-similarity.*

As an example, let d be the usual distance in an Euclidean space and $f(x) = 1\text{-}x$. Then $o_f(d)(x,y) = 1\text{-}d(x,y)$ if $d(x,y) \leq 1$ and $o_f(d)(x,y) = 0$ otherwise and $o_f(d)$ is a $*$-fuzzy order where $*$ is the Lukasiewicz norm. Assume that $f(x) = -log(x)$ and therefore set $o_f(d)(x,y) = e^{-d(x,y)}$. Then we obtain a $*$-fuzzy order where $*$ is the usual product. As a matter of fact, in both the examples $o_f(d)$ is a strict $*$-similarity.

Conversely, we will associate any fuzzy order with a metric. To this aim, given an additive generator $f : [0,1] \to [0,\infty]$, we associate any $*$-fuzzy preorder $ord : S\times S \to [0,1]$, with the map $d_f(ord) : S\times S \to [0,\infty]$ defined by setting

$$d_f(ord)(x, y) = f(ord(x,y)). \tag{7.3}$$

Theorem 7.3. *Let* $f : [0,1] \to [0,\infty]$ *be an additive generator and* $*$ *the related triangular norm. Then, for any* $*$*-fuzzy preorder ord* : $S\times S \to [0,1]$, $d_f(ord)$ *is an extended generalised quasi-metric. Moreover,*

- the preorder associated with $d_f(ord)$ *coincides with the order defined by ord ;*

- any L-homomorphism from (S,ord) *to* (S',ord')*, is an isometry from* $(S,d_f(ord))$ *to* $(S',d_f(ord'))$.

Proof. For any $x\in S$, $d_f(ord)(x,x) = f(ord(x,x)) = f(1) = 0$. To prove the triangle property, i.e. that $f(ord(x,y)) + f(ord(y,z)) \geq f(ord(x,z))$, observe that since f is order-reversing and ord is $*$-transitive, $f(ord(x, y)*ord(y, z)) \geq f(ord(x, z))$ and therefore

$$f(f^{[-1]}(f(ord(x, y)) + f(ord(y, z)))) \geq f(ord(x, z)).$$

Now, if $f(ord(x, y)) + f(ord(y, z)) \in f([0,1])$, we obtain that

$$f(ord(x, y)) + f(ord(y, z)) \geq f(ord(x, z)),$$

Otherwise,

$$f(ord(x, y)) + f(ord(y, z)) \geq f(0) \geq f(ord(x, z)).$$

Assume that $ord(x,y) = 1$, then $d_f(ord)(x,y) = f(ord(x,y)) = f(1) = 0$. Assume that $d_f(ord)(x,y) = 0$ and therefore that $f(ord(x,y)) = 0$. Then, $ord(x,y) = f^{[-1]}(f(d(x,y)) = f^{[-1]}(d_f(ord)(x,y)) = f^{[-1]}(0) = 1$.
The remaining part of the theorem is trivial. □

The proof of the following proposition is trivial.

Proposition 7.4. *Let ord* : $S\times S\to[0,1]$ *be a fuzzy preorder. Then*

i) ord is a $*$*-fuzzy order* $\Rightarrow$ $d_f(ord)$ *is an extended quasi-metric;*

ii) ord is a $*$*-similarity* $\Rightarrow$ $d_f(ord)$ *is an extended generalized metric;*

iii) ord is a strict $*$*-similarity* $\Rightarrow$ $d_f(ord)$ *is an extended metric.*

The established connection between the extended generalized quasi-metrics and the $*$-fuzzy orders is not completely satisfactory, in a sense. Indeed in the next proposition we observe that while $o_f(d_f(ord)) = ord$, it is $d_f(o_f(d)) \neq d$, in general. Indeed, we have the following proposition.

Proposition 7.5. *Let f be an additive generator. Then, for any fuzzy preorder ord,*

$$o_f(d_f(ord)) = ord .$$

Moreover, for any extended generalized quasi-metri $d : S\times S \to [0,\infty]$,

$$d_f(o_f(d)) = d\wedge f(0).$$

Proof. Observe that $f(f^{[-1]}(d(x,y))) = d(x,y)$ if $d(x,y)\leq f(0)$ and $f(f^{[-1]}(d(x,y))) = f(0)$ otherwise. □

Then, given an additive generator f, the resulting connection among $*$-fuzzy preorders and extended generalized quasi-metrics works well only for the extended generalized quasi-metrics (S, d) such that $D(S)\leq f(0)$. As an example, if f coincides with $-log$, then since $f(0) = \infty$ it is all O.K. Instead, if $f(x) = 1-x$, and d is the usual Euclidean distance, then $d_f(o_f(d))(x,y) = d(x,y)$ if $d(x,y) \leq 1$ and $d_f(o_f(d))(x,y) = 1$ otherwise.

8. Representation theorems for quasi-metric spaces.

The connection between quasi-metric spaces and fuzzy orders enables us to translate the representation theorems for fuzzy preorders into corresponding representation theorems for extended generalized quasi-metrics. Firstly, given an additive generator $f : [0,1]\to[0,\infty]$, we define the map $\delta^f : [0,1]^S\times[0,1]^S\to[0,\infty]$ by setting

$$\delta^f(s_1,s_2) = Sup\{f(s_2(x))-f(s_1(x)) : s_1(x)\geq s_2(x)\}. \tag{8.1}$$

Proposition 8.1. *Let* $f : [0,1]\to[0,\infty]$ *be an additive generator,* $\to$ *the associated implication and Incl fuzzy inclusion in* $[0,1]^S$ *based on* $\to$. *Then*

$$\delta^f = d_f(Incl) \tag{8.2}$$

and therefore $([0,1]^S,\delta^f)$ *is an extended quasimetric space. Moreover,*

$$o_f(\delta^f) = Incl. \tag{8.3}$$

Proof. Observe that, in the case $s_1(x)\geq s_2(x)$, since $f(s_1(x))\leq f(s_2(x))$, we have that

$$f(s_2(x)\to s_1(x)) = f(f^{[-1]}(f(s_1(x))-f(s_2(x))) = f(s_1(x))-f(s_2(x))$$

while, in the case $s_1(x)<s_2(x)$, $f(s_2(x)\to s_1(x)) = f(1) = 0$. Then,

$$\begin{aligned} d_f(Incl)(s_1,s_2) &= f(Incl(s_1,s_2)) = f(Inf\{s_1(x)\to s_2(x) : x\in S\}) \\ &= Sup\{f(s_1(x)\to s_2(x)) : x\in S\} \\ &= Sup\{f(s_2(x))-f(s_1(x)) : s_1(x)\geq s_2(x)\}. \end{aligned}$$

□

Definition 8.2. *Let* $f : [0,1]\to[0,\infty]$ *be an additive generator*. Then we call *difference-based space* a structure (C, δ^f) such that C is a class of fuzzy subsets of a given set and δ^f is defined by (8.1).

Then, by Proposition 8.1, the difference-based spaces are the spaces corresponding to the implication-based inclusions. Now we are able to prove the first representation theorem for quasi-metric spaces.

Theorem 8.3. *Let* (S,d) *be an extended generalised quasi-metric, and* $f : [0,1] \to [0,\infty]$ *an additive generator such that* $D(S)\leq f(0)$. *Also, let* $k : S \to [0,1]^S$ *be defined by setting*

$$k(z)(x) = f^{[-1]}(d(x,z)). \tag{8.4}$$

Then k is an isometry from (S,d) *to* $([0,1]^S,\delta^f)$. *Consequently, any extended quasi-metric is isomorphic to a difference-based space.*

Proof. By Theorem 7.1, $o_f(d)$ is a $*$-fuzzy preorder and therefore by Theorem 4.3 the map h defined in (4.2) is a homomorphism from $(S,o_f(d))$ to $([0,1]^S,Incl)$. Trivially, h coincides with k. Moreover, by Theorem 7.3, h is also an isometry from the space $(S,d) = (S,d_f(o_f(d))$ to $([0,1]^S,d_f(Incl))$ $= ([0,1]^S,\delta^f)$. □

As an example, assume that the additive generator is the function $-log$. Then any extended quasi-metric space (S,d) is isomorphic with the difference-based space (C,d') such that C is equal to the class $\{k(z) : z\in S\}$ of fuzzy subsets of S, where $k(z) = e^{-d(x,z)}$ for any $z\in S$, and d' be defined by setting $d'(s_1,s_2) = 0$ if $s_1\subseteq s_2$ and

$$d'(s_1,s_2) = Sup\{log(s_1(x))-log(s_2(x)) : s_1(x)\geq s_2(x)\}$$

otherwise.

A second representation theorem can be derived from the similarity-based representation theorem for $*$-fuzzy preorders. To this aim, we define the following class of extended quasi-metrics. Let (S,d) a generalized extended metric space. Then we set, for any $x \in S$ and $X\in P(S)$,

$$d(x,X) = Inf\{d(x,x') : x'\in X\}. \tag{8.5}$$

Also, we define $\delta_H : P(S)\times P(S)\to[0,1]$ by setting

$$\delta_H(X,Y) = Sup\{d(x,Y) : x\in X\}. \tag{8.6}$$

We call δ_H the *Hausdorff excess.* The word *"Hausdorff"* is justified by the fact that the usual Hausdorff distance in a metric space is defined by setting $\delta(X,Y) = \delta_H(X,Y)\vee\delta_H(Y,X)$ for any pair of nonempty closed bounded subsets X and Y. Observe that $d(x,\varnothing) = Inf(\varnothing) = \infty$ for any $x\in S$ and that $\delta_H(\varnothing,Y) = Sup(\varnothing) = 0$ for any $Y\in P(S)$. In the following proposition we set $cl(X) = \{x\in S : d(x,X) = 0\}$.

Proposition 8.4. *Let (S,d) be an extended generalized metric space and f be an additive generator. Then*

$$o_f(\delta_H) = Incl' \tag{8.7}$$

where Incl' is the similarity-based inclusion associated with the similarity $o_f(d)$. If $D(S)\leq f(0)$,

$$\delta_H = d_f(Incl'). \tag{8.8}$$

As a consequence, $(P(S),\delta_H)$ is an extended, generalized quasi-metric space whose associate preorder $\leq$ is such that

$$X\leq Y \iff X\subseteq cl(Y).$$

Proof. We have that

$$o_f(\delta_H)(X,Y) = f^{[-1]}(\delta_H(X,Y)) = f^{[-1]}(Sup_{x\in X}Inf_{y\in Y}d(x,y))$$
$$= Inf_{x\in X}Sup_{y\in Y}f^{[-1]}(d(x,y)) = Inf_{x\in X}Sup_{y\in Y}o_f(d)(x,y) = Incl'(X,Y).$$

The remaining part of the proposition is trivial. □

Definition 8.5. *Let d* : $S\times S\to[0,\infty]$ be a generalized extended metric space and C a class of subsets of S. Then we say that (C,δ_H) is a *Hausdorff excess space* (see [9] and [10]).

The proof of the following proposition is trivial.

Proposition 8.6. *Let C be a class of closed subsets of (S,d), then (C,δ_H) is an extended quasi-metric space. Let M be a class of nonempty bounded closed subsets of (S,d). Then (M,δ_H) is a quasi-metric space. In both the cases, the associated order is the inclusion relation.*

The following proposition shows that any similarity-based inclusion is associated with an Hausdorff excess.

Proposition 8.7. *Let f be an additive generator, sim* : $S \times S \to [0,1]$ *be a similarity and Incl' be the related similarity-based inclusion in $P(S)$. Then,*

$$d_f(Incl') = \delta_H$$

where δ_H is the Hausdorff excess associated with the extended generalized metric $d_f(sim)$.

Proof. Indeed,

$$d_f(Incl')(X,Y) = f(Incl'(X,Y)) = f(Inf_{x\in X}Sup_{y\in Y}sim(x,y))$$
$$= Sup_{x\in X}Inf_{y\in Y}f(sim(x,y))) = Sup_{x\in X}Inf_{y\in Y}\,d_f(sim)(x,y) = \delta_H(X,Y).$$

□

We are ready to prove the second representation theorem for quasi-metric spaces.

Theorem 8.8. *Let* $f : [0,1] \to [0,\infty]$ *be an additive generator and* (S,d) *be a generalized extended quasi-metric space such that* $D(S) \leq f(0)$. *Then a metric space* (S',d') *and a map* $h' : S \to P(S')$ *exist such that* h' *is an isometry from* (S,d) *to* $(P(S'),\delta_H)$. *Consequently, any quasi-metric space is isomorphic to a Hausdorff excess space.*

Proof. Let $f : [0,1] \to [0,\infty]$ be an additive generator such that $D(S) \leq f(0)$ and consider the $*$-fuzzy order $o_f(d)$ defined by (7.1). Then, by Theorem 5.7, a similarity-based inclusion space ($P(S')$, $Incl'$) and a homomorphism h' from $(S,o_f(d))$ to ($P(S')$, $Incl'$) exist. Also, by Theorem 7.3, h' is an isometry from from $(S,d_f(o_f(d))) = (S,d)$ to $(P(S'), d_f(Incl')) = (P(S'), \delta_H)$. □

9. Category theory

The language of category theory enables us to express all the observations in this paper in a more elegant and natural way. Indeed, we consider the *category of the extended generalized quasi-metrics*, i.e. the category *EGQM* whose objects are the extended generalized quasi-metrics and whose maps are the isometries. Also, given an Archimedean triangular norm $*$, we consider the *category of the* $*$*-fuzzy preorders*, i.e. the category $*$*-FP* whose objects are the $*$-fuzzy preorders and whose maps are the homomorphisms as defined in Definition 3.1.

Theorem 9.1. *Let* $f : [0,1] \to [0,\infty]$ *be an additive generator of the triangular norm* $*$ *and let* $\underline{d}_f$ *be the correspondence defined by setting*

- $\underline{d}_f((S,ord)) = (S, d_f(ord))$
- $\underline{d}_f(h) = h$ *for any homomorphism* h *from* (S,ord) *to* (S',ord')

where (S,ord) and (S',ord') *are* $*$*-fuzzy preorders. Then* $\underline{d}_f$ *is a functor from* $*$*-FP to EGQM. Moreover,* $\underline{d}_f$ *associates*

- *any implication-based inclusion space with a difference-space*
- *any similarity-based inclusion space with a Hausdorff excess.*

Proof. By Theorem 7.3, Proposition 8.1 and Proposition 8.7. □

Theorem 9.2. *Let* $f : [0,1] \to [0,\infty]$ *be an additive generator of the triangular norm* $*$ *and let* $\underline{o}_f$ *be the correspondence defined by setting*

- $\underline{o}_f((S,d)) = (S,o_f(d))$
- $\underline{o}_f(k) = k$ *for any isometry* k *from* (S,d) *to* (S',d')

where (S,d) *and* (S',d') *are extended generalized quasi-metrics. Then* $\underline{o}_f$ *is a functor from EGQM to* $*$*-FP. Moreover,* $\underline{o}_f$ *associates*

- *any difference-space with a implication-based inclusion space*
- *any Hausdorff excess with a similarity-based inclusion space.*

Proof. By Theorem 7.1, Proposition 8.1 and Proposition 8.4. □

Theorem 9.3. *Let* $f : [0,1] \to [0,\infty]$ *be an additive generator and* $*$ *the corresponding triangular norm. Then the category* $*$*-FP is isomorphic to the subcategory of the extended quasi-metric spaces whose diameter is less than or equal to* $f(0)$. *Consequently, if* $f(0) = \infty$, *then* $*$*-FP and EGQM are isomorphic.*

Proof. Let (S,d) be an extended generalized quasi-metric whose diameter is less than $f(0)$. Then, by Proposition 7.5,

$$d_f(o_f(d)) = d \wedge f(0) = d.$$

Let (S,ord) be a $*$-fuzzy preorder, then

$$o_f(d_f(ord)) = ord.$$

Thus, both $\underline{d}_f \circ \underline{o}_f$ and $\underline{o}_f \circ \underline{d}_f$ coincide with the identity functor and this proves that $*$*-FP* and *EGQM* are isomorphic. □

References

1. W. Bandler, L. Kohout, Fuzzy power sets and fuzzy implication operators, *Fuzzy Sets and Systems*, 4 (1980) 13-30.

2. L. Biacino, G. Gerla, Logics with Approximate Premises, *Int. J. of Int. Systems*, 13 (1998) 1-10.
3. G. Gerla, On the concept of fuzzy point, *Fuzzy Sets and Systems*, 18 (1986) 1-14.
4. G. Gerla, Fuzzy logic: *Mathematical Tools for Approximate Reasoning*, Kluwer Academic Press (2000).
5. V. Novak, I. Perfilieva, J. Mockor, *Mathematical Principles of Fuzzy Logic*, Kluwer Academic Publishers, London 1999.
6. S. V. Ovchinnikov, Representations of transitive relations, in H. J. Skala, S. Termini, E. Trillas, (Eds), *Aspect of Vagueness*, D. Reidel Publ. Co., Dordrecht, 1984, 105-118.
7. B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, North-Holland, New York 1983.
8. L. Valverde, On the Structure of F-Indistinguishability Operators. *Fuzzy Sets and Systems*, 17 (1985) 313-328.
9. P. Vitolo, A representation theorem for quasi-metric spaces, *Topology and its Applications* 65 (1995) 101-104.
10. P. Vitolo, The Representation of Weighted Quasi-Metric Spaces, *Rend. Ist. Mat. Univ. Trieste*, 21 (1999) 95-100.
11. L.A. Zadeh, Fuzzy Sets, *Information and Control*, 8 (1965) 338-353.
12. L.A. Zadeh, Similarity relations and fuzzy orderings, *Inform. Sci.*,3 (1971) 177-200.