THE CATEGORY OF FUZZY SUBSETS AND POINT-FREE ULTRAMETRIC SPACES

Giangiacomo Gerla
DMI
University of Salerno
Via S. Allende 84081
Baronissi (SA), Italy
ggerla@unisa.it

Cristina Coppola
DMA
University Federico II,
Via Cintia, 80126 Napoli,
Italy,
cristina.coppola@dma.unina.it

Tiziana Pacelli
DMI
University of Salerno
Via S. Allende 84081
Baronissi (SA), Italy
tpacelli@unisa.it

Abstract. Some attempts to establish a link between point-free geometry and the categorical approach to fuzzy set theory is exposed. In fact, it is possible to find functors between the category of fuzzy set as defined by Höhle in [4] and a category whose objects are the pointless ultrametric spaces.

Keywords: Point-free geometry, fuzzy set, similarity, category, ultrametrics.

1. Introduction.

The aim of point-free geometry is to give an axiomatic basis to geometry in which the notion of *point* is not assumes as a primitive. The first example in such a direction was furnished by Whitehead's researches [6, 7] where the primitives are the *regions* and the *inclusion* relation between regions. Later Whitehead proposed the topological notion of *connection* instead of the inclusion [8]. In [2] Gerla proposed a system of axioms in which *regions, inclusion, distance* and *diameter* are assumed as primitives.

In this note we expose some attempts to establish a link between point-free geometry and the categorical approach to fuzzy set theory. In fact, it is possible to find functors between the category of fuzzy set defined by Höhle in [4], and a category whose objects are the pointless ultrametric spaces. More precisely, Section 2 is devoted to give some preliminary notions. In Section 3, starting from the definition of *pm-spaces* given by Gerla in [2], we introduce the *pu-spaces* . In Section 4 we define the *psu-spaces* and we verify the relations between these structures and the previous ones. In Section 5 we consider a semi-similarity and we examine the relations existing with the *pu-spaces* and the *psu-spaces*. Besides we give a characterization of semi-similarities by the notion of *semi-equivalence* and we show an example of semi-similarity. Finally, in Section 6 we describe the category of fuzzy set defined by Höhle in [4] and the category of pointless ultrametric spaces and we examine a functor between these categories.

2. Preliminaries.

We introduce some basic notions such as *continuous t-norm* and *fuzzy relation.*

Definition 2.1. A *continuous triangular norm (t-norm)***,** is a continuous binary operation $*$ on [0, 1] such that, for all x, y, $x_1, x_2, y_1, y_2 \in [0, 1]$

- $*$ is commutative,
- $*$ is associative,
- $*$ is isotone in both arguments, i.e.,
  $x_1 \leq x_2 \Rightarrow x_1 * y \leq x_2 * y,$
  $y_1 \leq y_2 \Rightarrow x * y_1 \leq x * y_2,$
- $1 * x = x = x * 1$ and $0 * x = 0 = x * 0.$

The most important continuous t-norms are *minimum, product* and *Lukasiewicz conjunction* $a * b = max(0, a+b-1)$.

Definition 2.2. Let $*$ be a continuous t–norm . The *residuation is the operation* $\rightarrow_*$ defined by

$$a \rightarrow_* b = Sup\{x \,/\, a * x \leq b\}$$

It is immediate that

$$a * x \leq b \Leftrightarrow x \leq a \rightarrow_* b.$$

In this paper we confine ourselves to the *minimum t-norm.* In such a case the residuation operation is the *Gödel implication* $\rightarrow_G$:

$$a \rightarrow_G b = \begin{cases} 1 & \text{if } a \le b \\ b & \text{if } b < a. \end{cases} \quad (2.1)$$

Let S be a nonempty set. We call *fuzzy subset of S* any map $s: S \rightarrow [0,1]$; the value $s(x)$ represents the membership degree of x to s. A *fuzzy relation* is a fuzzy subset of $S \times S$, i.e., a map $r: S \times S \rightarrow [0, 1]$.

Definition 2.3. Let $ord: S \times S \rightarrow [0,1]$ be a fuzzy-relation on S and consider the following properties:

(i) $ord(x,x)=1$ (reflexivity)

(ii) $ord(x,y) \wedge ord(y,z) \le ord(x,z)$ (transitivity)

(iii) $ord(x,y)=ord(y,x)=1 \Rightarrow x=y$ (antisymmetry)

(iv) $ord(x,y)=ord(y,x)$ (symmetry)

for every $x, y, z \in S$.

Then *ord* is called:

- *fuzzy preorder* if it satisfies (i) and (ii),
- *fuzzy order* if it satisfies (i), (ii) and (iii),
- *fuzzy similarity* if it satisfies (i), (ii) and (iv),
- *strict fuzzy similarity* if it satisfies (i), (ii), (iii) and (iv).

3. Pointless ultrametric spaces.

In order to give a metric approach to point-free geometry, G. Gerla in [2] defines the notion of *pointless metric space*, briefly *pm-space*. A *pm-space* is a structure $(R, \le, d, |\ |)$, where

- $(R,\le)$ is an ordered set,
- $d : R \times R \rightarrow [0,¥)$ is an order-reversing map,
- $|\ | : R \rightarrow [0, ¥]$ is an order-preserving map

and, for every $x, y, z \in R$ the following axioms hold:

(a1). $d(x, x) = 0$

(a2). $d(x, y) = d(y, x)$

(a3). $d(x, y) \le d(x, z) + d(z, y) + |z|$.

The elements of R are called *regions*, the number $d(x, y)$ is called *distance* between regions and the number $|x|$ *diameter* of the region x.

In alternative, we can consider the inclusion relation as a derived notion. In fact, the following hold true:

Proposition 3.1. Let $(R, d, |\ |)$ be a structure satisfying a1), a2) and a3) and define $\le$ by setting

$x \le y \Leftrightarrow |x| \le |y|$ and $d(x, z) \ge d(y, z)$

for every $z \in R$.

Then $(R, d, |\ |)$ is a pm-space.

Example 3.1. A class of basic examples of *pm-spaces* are obtained by considering a metric space (M, d) and a class C of nonempty subsets of M. The distance and the diameter are defined by setting

$$d(X, Y) = inf\{d(x, y) \,/\, x \in X,\ y \in Y\} \quad (3.1)$$

$$|X| = sup\{d(x, y) \,/\, x, y \in X\}. \quad (3.2)$$

and the relation $\le$ is the usual inclusion.

In fact, we can prove immediately (a1) and (a2); instead, let X, Y and Z be subsets of M, $x \in X$, $y \in Y$, z and $z' \in Z$; then

$d(X, Y) \le d(x, y) \le d(x, z) + d(z, z') + d(z', y) \le d(x, z) + d(z', y) + |Z|$

and therefore we prove (a3):

$d(X, Y) \le d(X, Z) + d(Z, Y) + |Z|$.

Any *pm*-space of this type is called *canonical*.

In this paper we are interested to a particular class of *pm-spaces* which is related with the notion of ultrametric space. In accordance with Proposition 3.1, we do not assume the inclusion as a primitive.

Definition 3.1. A *pointless ultrametric space,* briefly *pu-space,* is a structure $\mathcal{R}=(R,d,|\ |)$ where $d : R \times R \rightarrow [0, 1]$ and $|\ | : R \rightarrow [0, 1]$ are functions satisfying the following axioms, for every $x, y, z \in R$:

(A1). $d(x,x)=0$ (reflexivity)

(A2). $d(x,y)=d(y,x)$ (symmetry)

(A3). $d(x, y) \le d(x, z) \vee d(z, y) \vee |z|$ (generalized triangle-inequality)

where $\vee$ is the maximum.

We say that the region x *overlaps* the region y if there exists $z \in R$ such that $z \leq x$ and $z \leq y$.

Proposition 3.2. *The relation* $\leq$ *is a preorder. Moreover, in any pu-space the following hold:*
(1) $x \leq y \Rightarrow d(x, y) = 0$,
(2) x *and* y *overlap* $\Rightarrow d(x, y) = 0$.

Proof. It is easy to prove that $\leq$ is a preorder. To prove (1), set $z = x$ in Definition (2.3); then by (A1) we have $d(x, y) \leq d(x, x) = 0$.
To prove (2), assume that $z \leq x$ and $z \leq y$; then by (1) and by Definition (2.3)
$$d(x, y) \leq d(x, z) = 0.$$

Proposition 3.3. *Any pu-space is a pm-space.*

Proof. We need only to observe that
$$d(x, y) \leq d(x, z) \vee d(z, y) \vee |z| \leq d(x, z) + d(z, y) + |z| .$$

Example 3.2.
We obtain canonical examples of *pu-spaces* as in Example 3.1. In fact, recalling that (M, d) is a quasi-ultrametric space if the following hold:
(A1*). $d(x, x) = 0$,
(A2*). $d(x, y) = d(y, x)$,
(A3*). $d(x, y) \leq d(x, z) \vee d(z, y)$
and defining the distance and the diameter as in (3.1) and (3.2) , it results
$$d(X, Y) \leq d(x, y) \leq d(x, z) \vee d(z, z') \vee d(z', y) \leq d(x, z) \vee d(z', y) \vee |Z| ,$$
and therefore $(R,d,|\,|)$ is a *pu*-space. Any *pu*-space of this type is called *canonical*.

4. Semi-ultrametric spaces.

We introduce a new class of structures which verifies symmetry and a triangular inequality, but not reflexivity.

Definition 4.1. A *semi-ultrametric space*, briefly *psu-space*, is a structure (R, d) where R is a set whose elements are called *regions* and $d : R \times R \rightarrow [0, 1]$ is a function, we call *semi-distance*, such that, for any $x, y, z \in R$:
(B1) $d(x, y) = d(y, x)$,
(B2) $d(x, y) \leq d(x, z) \vee d(z, y)$.

Given a semi-distance d, we set:
$$|x|_d = d(x, x) , \qquad (4.1)$$
Observe that by setting $z = x$ in (B2), we obtain that $d(x, x) \leq d(x, y) \vee d(y, z)$ and therefore by (B1) that $d(x, x) \leq d(x, y)$. Likewise we have that $d(y, y) \leq d(x, y)$ and therefore it results
$$d(x,y) \geq |x|_d \vee |y|_d . \qquad (4.2)$$
It is possible to associate any *pu*-space $(R,d,|\,|)$ with a *psu*-space (R, d_d) by setting
$$d_d(x, y) = d(x, y) \vee |x| \vee |y| , \quad (4.3)$$
for every $x, y \in R$.

Proposition 4.1. *Let* $(R,d,|\,|)$ *be a pu-space, then the structure* (R, d_d) *defined by* (4.3) *is a psu-space such that* $|\,|_d = |\,|$.

Proof. (B1) and the equality $|\,|_d = |\,|$ are trivial. Besides,
$$\begin{aligned} d_d(x, z) &= d(x, z) \vee |x| \vee |z| \\ &\leq d(x, y) \vee d(y, z) \vee |y| \vee |x| \vee |z| \\ &= (d(x,y) \vee |x| \vee |y|) \vee (d(y, z) \vee |y| \vee |z|) \\ &= d_d(x,y) \vee d_d(y, z). \end{aligned}$$

Conversely, we can associate any *psu-space* (R, d) with a *pu-space* (R,d_d) by setting

$$d_d(x,y) = \begin{cases} 0 & \text{if } d(x,y) = |x|_d \vee |y|_d \\ d(x,y) & \text{if } d(x,y) > |x|_d \vee |y|_d \end{cases} \qquad (4.4)$$
for every $x, y \in R$..

Proposition 4.2. *Let* (R, d) *be a psu-space, then the structure* $(R,d_d,|\,|_d)$ *defined by* (4.1) *and* (4.4) *is a pu-space* .

Proof. Axioms (A1), (A2) are immediate. To prove (A3) observe that if $d_d(x, y) = 0$ then the generalized triangle-inequality is trivially verified.
Assume that $d_d(x, y) = d(x, y)$ i.e. $d(x, y) > |x|_d \vee |y|_d$ [*].
We can consider four cases:
I) $d_d(x, z) = d(x, z)$ and $d_d(z, y) = d(z, y)$.
We have that

$d_d(x, y) = d(x, y) \leq d(x, z) \vee d(z, y) = d_d(x, z) \vee d_d(z, y) \leq d_d(x, z) \vee d_d(z, y) \vee |z|_d$.

II) $d_d(x, z) = 0$ and $d_d(z, y) = d(z, y)$.

It means that

$d(x, z) = |x|_d \vee |z|_d$ [**]

and $|z|_d \vee |y|_d < d(z, y)$.

By [**] it results

$d(x, y) \leq d(x, z) \vee d(z, y) = |x|_d \vee |z|_d \vee d(z, y)$, but by [*]

$d(x, y) \not\leq |x|_d$ and therefore

$d(x, y) \leq |z|_d \vee d(z, y)$, i.e.

$d_d(x, y) \leq d_d(x, z)\, d_d(z, y) \vee |z|_d$.

III) $d_d(x, z) = d(x, z)$ and $d_d(z, y) = 0$. It is analogue to II).

IV) $d_d(x, z) = 0$ and $d_d(z, y) = 0$, i.e.

$d(x, z) = |x|_d \vee |z|_d$ [**]

and

$d(z, y) = |z|_d \vee |y|_d$ [***] .

By [**] and [***] it results

$d_d(x, y) = d(x, y) \leq d(x, z) \vee d(z, y) = |x|_d \vee |z|_d \vee |y|_d$,

but by [*]

$d(x, y) \not\leq |x|_d$ and $d(x, y) \not\leq |y|_d$; therefore $d_d(x, y) \leq |z|_d$, i.e.

$$d_d(x, y) \leq d_d(x, z) \vee d_d(z,y) \vee |z|_d.$$

5. Semi-similarities

In order to give a general approach to fuzzy set theory based on the notion of category, now we consider a fuzzy relation E on R that represents the dual concept of semi-distance. According to the terminology of M. Fourman and D.S. Scott [1], this relation is a [0, 1]-*valued equality* and the pair (R, E) is a [0, 1]-*valued set*. In the next section, we shall see that (R, E) are objects of a fuzzy category described in Höhle [4].

Definition 5.1. A *semi-similarity* is a fuzzy relation E on R such that the following

(e1) $E(x, y) = E(y, x)$ (symmetry)

(e2) $E(x, z) \wedge E(z, y) \leq E(x, y)$ ($\wedge$-transitivity)

hold for every $x, y, z \in R$. A *similarity* is a semi-similarity such that

(e3) $E(x,x) = 1$.

$E(x, y)$ is regarded as truth-value of a statement like $x =_R y$. Observe that by setting $x = y$ in (e2) we obtain $E(x, z) \wedge E(z, x) \leq E(x, x)$ and therefore that $E(x, z) \leq E(x, x)$. This entails also that

$$E(x, z) \leq E(x, x) \wedge E(z, z).$$

We can associate any *psu*-space (R, d) with a semi-similarity E_d by setting

$$E_d(x, y) = 1\text{-}d(x, y) \quad (5.1)$$

for every $x, y \in R$.

Proposition 5.1. *Let d be a semi-distance, then the fuzzy-relation E_d defined by* (5.1) *is a semi-similarity.*

Proof. Condition (e1) is immediate. To prove (e2) observe that

$E_d(x, y) \wedge E_d(y, z) = (1\text{-}\, d(x, y)) \wedge (1\text{-}\, d(y, z))$
$= 1\text{-}(d(x, y) \vee d(y, z)) \leq 1\text{-}\, d(x, z)$
$= E_d(x, z)$.

Conversely, we can associate any semi-similarity E with a *psu-space* (R, d_E) by setting

$$d_E(x,y) = 1\text{-}E(x,y) \quad (5.2)$$

for every $x, y \in R$.

Proposition 5.2. *Let E be a semi-similarity, then the structure* (R, d_E), *defined by* (5.2), *is a psu-space.*

Proof. Axiom (B1) is immediate. To prove (B2) it is sufficient to observe that

$d(x, y) = 1\text{-}\, E(x, y) \leq (1\text{-}E(x, z) \vee 1\text{-}E(z, y))$
$= d(x, z) \vee d(z, y)$.

In accordance with Proposition 4.2 and Proposition 5.2, we can associate directly any semi-similarity with a *pu*-space. Indeed the following proposition holds.

Proposition 5.3. *Let E be a semi-similarity, define* $|\ |_E : R \to [0, 1]$ *by setting*

$$|x|_E = 1 - E(x, x) \quad (5.3)$$

and $d_E : R\times R \to [0, 1]$ *by*

$$d_E(x,y) = \begin{cases} 0 & \text{if } E(x,y) = E(x,x) \wedge E(y,y) \\ 1 - E(x,y) & \text{if } E(x,y) < E(x,x) \wedge E(y,y) \end{cases} \quad (5.4)$$

for every $x, y \in R$. *Then* $\mathcal{R}_E = (R, d_E, |\ |_E)$ *is a pu-space.*

In accordance with Proposition 4.1 and Proposition 5.1, we can associate any *pu*-space with a semi-similarity.

Proposition 5.4. Let $(R,d,|\,|)$ be a pu-space and define $E_{d,|\,|}:R\times R \to [0, 1]$ by setting

$$E_{d,|\,|}(x,y) = 1 - (d(x,y)\vee|x|\vee|y|) \qquad (5.5)$$

Then $(R, E_{d,|\,|})$ is a semisimilarity.

5.1. Semi-equivalences.

Let S be a set, R be a relation on S and $D_R = \{x\in S \,/\, \exists\, y: (x, y)\in R\}$ the domain of R .

Definition 5.1.1. Let S be a set. A relation R on S is called *semi-equivalence* provided that is symmetric and transitive.

Observe that if R is a semi-equivalence relation it results $(x, y) \in R \Rightarrow (x,x) \in R$ for every $x, y \in S$, i.e. R is reflexive in its domain D_R . Equivalently, if x is not related to itself, it cannot be related to any element. Therefore, every semi-equivalence relation R on S is an equivalence relation on its domain D_R and viceversa.

The notion of semi-equivalence is related to the notion of semi-similarity, as we can see in the following propositions.

Proposition 5.1.1. *Assume that E is a semi-similarity and let* $R_l = C(E, l) = \{(x, y)/ E(x, y) \geq l\}$ *be a* l*-cut of E, where* $\lambda \in [0,1]$ *and x, y are regions. Then* R_l *is a semi-equivalence.*

Conversely, let $(R_l)_{l\in[0,1]}$ be an order-reversing family of semi-equivalence relations, i.e., if $l = \mu$ then $R_\mu \quad R_l$ and let

$$E(x, y) = Sup\{ l \,/ (x, y) \in R_l \} \qquad (5.1.1)$$

(where $Sup(f) = 0$).

Proposition 5.1.2. *Let* $(R_l)_{l\in[0,1]}$ *be an order-reversing family of semi-equivalence relations, then the fuzzy relation E defined by* (5.1.1) *is a semi-similarity*

.

Proof. Condition (e1) is immediate by symmetry of R_l. To prove (e2), let us consider

$E(x, z) = Sup\{ l \,/(x, z) \in R_l \} = m$
$E(z, y) = Sup\{ l \,/(z, y) \in R_l \} = x$
$E(x, y) = Sup\{ l \,/(x, y) \in R_l \} = h$.

Suppose $m \leq x$ (likewise $x \leq m$). Since $(R_l)_{l\in[0,1]}$ is an order-reversing family of relations, it results $R_x \subseteq R_m$..Therefore we have (x, z) ,$(z, y) \in R_m$..and then, by transitivity, $(x, y) \in R_m$. But $h = Sup\{ l \,/(x, y) \in R_l \}$, then $h \geq m$ and, since $m \wedge x = m$, the condition $(e2)$

$$E(x, z) \wedge E(z, y) \leq E(x, y)$$

is verified.

5.2. Example of semi-similarity.

Let X and Y be two nonempty sets and denote by $F(X, Y)$ the class of partial functions from X to Y. If $f \in F(X, Y)$ we denote by D_f the domain of f. We consider a function *rel:* $X\to [0,1]$ which gives the *"degree of relevance"* of any element $x \in X$.

Definition 5.2.1. Let S be a subset of X. and *rel:* $X\to[0,1]$. The *degree of relevance of S* , is

$$Rel(S)= Sup\{rel(x) \,/\, x \in S\}.$$

Let f, g be elements of $F(X, Y)$ and consider the *equalizer* of f and g , defined by

$$eq(f, g) = \{x \in D_f \cap D_g : f(x)=g(x)\}.$$

We set

$$E(f, g) = 1- Sup\,\{rel(x) \,/\, x\notin eq(f, g)\} = Inf\,\{1-rel(x) \,/\, x\notin eq(f, g)\}. \qquad (5.2.1)$$

Observe that $x\notin eq(f, g)$ means that x belongs to the set $C_{fg}\cup F\cup G$, where

$C_{fg} = \{x\in X \,/\, x \in D_f \cap D_g \text{ and } f(x) \neq g(x)\}$,
$F = \{x\in X \,/\, x\notin D_f\}$,
$G = \{x\in X \,/\, x\notin D_g\}$. (5.2.2)

In other words, elements not belonging to $eq(f, g)$ are the elements on which f and g *"contrast"*. Then, in a sense, $E(f, g)$ measures the *similarity* between f and g, because it gives the degree of *"irrelevance"*, by $1-rel(x)$, of the elements above.

Proposition 5.2.1. *Let C be a nonempty class of partial functions. Then the relation E on C, defined by* (5.2.1) *is a semi-similarity.*

Proof. (e1) is immediate. To prove (e2) , observe that for every $f, g, h \in C$, the set C_{fg}, defined in (5.2.2), is contained in $C_{fh} \cup C_{hg} \cup H$, where

$C_{fh} = \{x \in X \,/\, x \in D_f \cap D_h \text{ and } f(x) \neq h(x)\}$,

$C_{hg} = \{x \in X \,/\, x \in D_h \cap D_g \text{ and } h(x) \neq g(x)\}$,

$H = \{x \in X \,/\, x \notin D_h\}$.

So,

$\{ril(x) \,/\, x \in C_{fg}\} \subseteq \{ril(x) \,/\, x \in C_{fh} \cup C_{hg} \cup H\}$

and

$Sup\{ril(x) \,/\, x \in C_{fg}\} \leq Sup\{ril(x) \,/\, x \in C_{fh} \cup C_{hg} \cup H\}$ i.e. $Ril(C_{fg}) \leq Ril(C_{fh} \cup C_{hg} \cup H)$ from which

$1\text{-}Ril(C_{fg}) \geq 1\text{-}Ril(C_{fh} \cup C_{hg} \cup H)$.

Then,

$(1\text{-}Ril(C_{fg})) \wedge (1\text{-}Ril(F \cup G)) \geq (1\text{-}Ril(C_{fh} \cup C_{hg} \cup H)) \wedge (1\text{-}Ril(F \cup G))$, i.e.

$1\text{-}(Ril(C_{fg}) \vee Ril(F \cup G)) \geq 1\text{-}(Ril(C_{fh} \cup C_{hg} \cup H) \vee Ril(F \cup G))$,

that is equivalent to

$1\text{-}Ril(C_{fg} \cup F \cup G) \geq$

$1\text{-}(Ril(C_{fh} \cup C_{hg} \cup H \cup F \cup G) =$

$1\text{-}(Ril(C_{fh} \cup F \cup H) \vee Ril(C_{hg} \cup H \cup G)) =$

$(1\text{-}Ril(C_{fh} \cup F \cup H)) \wedge (1\text{-}Ril(C_{hg} \cup H \cup G))$, i.e.

$E(f, g) \geq E(f, h) \wedge E(h, g)$.

6. The categories of the semi-similarities and of the *pu*-spaces

In order to organize the class of semi-similarities into a category, we refer to the categories of *M*-SET* as described by Hohle in [4]. Namely, while Hohle defines this category for any *GL-monoid*, we are interested only with the particular *GL*-monoid in [0,1] defined by the t-norm $\wedge$. In such a case we have the following simplified definition.

Definition 6.1. The *category of the semi-similarities* is the category ***SS*** such that:

- *the objects* are the semi-similarities;
- *a morphism* from (R', E') to (R', E') is a map $f : R \to R'$ satisfying the axioms

(M1) $E'(f(x), f(x)) \leq E(x, x)$

(M2) $E(x, y) \leq E'(f(x), f(y))$

for every $x, y \in X$.

Observe that from M2 we have that $E(x, x) \leq E'(f(x), f(x))$ and therefore, by M1,

$$E(x, x) = E'(f(x), f(x))$$

The second category we consider is defined by the class of *pu*-spaces.

Definition 6.2. The category ***PU*** *of the pu-spaces* is the category such that

- the *objects* are the *pu*-spaces;
- a *morphism* from $(R, d, |\,|)$ to $(R', d', |\,|')$ is a map $f: R \to R'$ such that

(1) $d(x, y) \geq d'(f(x), f(y))$

(2) $|x| \geq |f(x)|'$

In both the categories the *composition* is the usual composition of maps and the *identities* are the identical maps. Proposition 5.3 enables us to associate any semi-similarity (R,E) with a *pu*-space $(R, d_E, |\,|_E)$. This suggests the definition of a suitable functor.

Proposition 6.1. We define a functor *F from* ***SS*** to ***PU*** by setting

- $F((R, E)) = (R, d_E, |\,|_E)$
- $F(f) = f$.

Proof. We have only to prove that if f is a morphism from (R,E) to (R', E'), then f is a morphism from $(R, d_E, |\,|_E)$ to $(R', d_{E'}, |\,|_{E'})$. Indeed, it is immediate that

$$|f(x)|_{E'} = 1\text{-}E(f(x), f(x)) = 1\text{-}E(x,x) = |x|_E.$$

To prove that

$$d_E(x,y) \geq d_{E'}(f(x), f(y)) \qquad (6.1)$$

it is not restrictive to assume that $d_{E'}(f(x), f(y)) \neq 0$ and therefore that

$$E'(f(x), f(y)) < E'(f(x), f(x)) \wedge E'(f(y), f(y)).$$

and $d_{E'}(f(x), f(y)) = 1\text{-}E'(f(x), f(y))$. In such a case, since

$$\begin{aligned} E(x,y) &\leq E'(f(x), f(y)) \\ &< E'(f(x), f(x)) \wedge E'(f(y), f(y)) \\ &= E(x,x) \wedge E(y,y), \end{aligned}$$

we have that $d_E(x,y) = 1\text{-}E(x,y)$. So, (6.1) is a trivial consequence of M2.

Observe that in proving that F is a functor we obtain that $|f(x)|_{E'} = |x|_E$ .On the other hand, it is easy to find a morphism h in ***PU*** such that $|f(x)|_{E'} < |x|_E$ for a suitable region. As an example we can consider the morphism induced by a contraction in the canonical *pu*-space associated with a Euclidean space. Then the proposed functor is not surjective.

Proposition 5.4. Let $(R,d,|\,|)$ be a pu-space and define $E_{d,|\,|}:R\times R \to [0, 1]$ by setting

$$E_{d,|\,|}(x,y) = 1 - (d(x,y)\vee|x|\vee|y|) \qquad (5.5)$$

Then $(R, E_{d,|\,|})$ is a semi-similarity.

Proposition 5.4 suggest a definition of a functor from the category of the *pu*-spaces into the category of the semi-similarities.

Proposition 6.2. We define a functor F' *from* ***PU*** to ***SS*** by setting

- $F'((R, d, |\,|)) = (R, E_{d,|\,|})$
- $F(f)=f$.

Proof. Let $(R, d, |\,|)$ and $(R', d, |\,|')$ be two *pu*-spaces and denote by $(R, E_{d,|\,|})$ and $(R', E_{d',|\,|'})$ the associated semi-similarities. Then, for any morphism f from $(R, d, |\,|)$ to $(R', d, |\,|')$, we have that

$$E'(f(x),f(x)) = 1-|f(x)|' \geq 1-|x| = E(x,x).$$

Moreover,

$$\begin{aligned} E(x,y) &= 1-(d(x,y)\vee|x|\vee|y|) \\ &\leq 1-(d'(f(x),f(y))\vee|f(x)|'\vee|f(y)|') \\ &= E'(f(x),f(y)). \end{aligned}$$

Conclusions and future works.

This paper is a first attempt to establish a link between point-free geometry and fuzzy set theory. In spite of some promising results, the proposed functor is not yet satisfactory. Indeed, a complete equivalence between the categories we are interested is not yet obtained. Another open question is to give a geometric interpretation of the objects of the category of the fuzzy sets as suggested by the obtained results. Future works will be addressed to this aims.

References

[1] Fourman M., Scott D.S., *Sheaves and logic*, in “Applications of Sheaves”, Lecture Notes in Mathematics 753, Springer-Verlag (1979), 302-401.

[2] Gerla G., *Pointless Metric Space*, Journal of Symbolic Logic 55, num.1, (1990), 207-219.

[3] Gerla G., *Pointless Geometries*, in “Handbook of Incidence Geometries”, F. Buekenhouted (1994) Elsevier Science, 1015-1031.

[4] Höhle U., *Presheaves over GL-monoids.* In *Nonclassical Logics and Their Applications to Fuzzy Subsets* (U. Höhle, E. P. Klement, eds), Kluwer, 1995.

[5] Novak V., Perfilieva I., Mockor J., *Mathematical Principles of Fuzzy Logic*, Kluwer Academic Publishers, London, 1999.

[6] Whitehead A. N., *An inquiry concerning the Principles of Natural Knowledge*, Univ. Press, Cambridge, 1919.

[7] Whitehead A. N., *The Concept of Nature*, Univ. Press, Cambridge, 1920.

[8] Whitehead A. N., *Process and Reality*, McMillian, N.Y., 1929.