Available online at www.sciencedirect.com
ScienceDirect
Fuzzy Sets and Systems 159 (2008) 1178–1190
FUZZY sets and systems
www.elsevier.com/locate/fss

Convergence and fixed points by fuzzy orders

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Available online 8 December 2007

Abstract

A general approach to fixed point theory is proposed which is related to the notion of fuzzy ordering. This approach extends both the fixed point theorems in metric spaces and the ones in ordered sets.

Keywords: Fixed points; Metric spaces; Ordered set; Fuzzy orders

1. Introduction

Fixed point theory for operators in a lattice is a basic tool in formal logic and, in particular, in logic programming. Indeed, given the power set $P(B_P)$ of the Herbrand base B_P of a given program P and the single-step operator $T_P : P(B_P) \to P(B_P)$, associated with P, the fixed points of T_P are the Herbrand models for P [13]. Now, to obtain the Herbrand models, fixed point theorems for ordered sets, such as Tarski theorem, are used. Unfortunately, such a theorem does not apply when the single-step operator is not monotone, for instance, in programs with negation. In such a case, it is possible to find in literature metric approaches (see [9,8,15,17,18]), so fixed point theorems in metric spaces result useful. This paper is an attempt to face these questions by unifying fixed point theory in ordered sets and fixed point theory in metric spaces. This is done by the notion of fuzzy order in account of the fact that it allows us to extend simultaneously both the metric notions and the ones of ordered set theory.

2. Preliminaries

Let S be a set. We call fuzzy subset of S any function $s : S \to [0, 1]$. Given two fuzzy subsets s_1 and s_2, we say $s_1 \subseteq s_2$ provided that $s_1(x) \leqslant s_2(x)$, for every $x \in S$.

Definition 2.1. A *triangular norm* (briefly t-norm) [11] is an associative and commutative operation $\otimes$ on [0, 1] such that $\otimes$ is isotone in both the arguments and it verifies the boundary conditions

$$1 \otimes x = x = x \otimes 1 \quad \text{and} \quad 0 \otimes x = 0 = x \otimes 0.$$

An example of t-norm is the minimum operation $\wedge$ in [0, 1]. We call it *Gödel* t-norm. An interesting class of t-norms is the one of the *Archimedean* t-norms satisfying the condition $x \otimes x < x$ for every x different from 0 and 1. In fact, these t-norms can be obtained by the notion of additive generator [14].

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0165-0114/$ - see front matter
doi:10.1016/j.fss.2007.12.006

Definition 2.2. Let $f : [0, 1] \to [0, \infty]$ be a continuous, strictly decreasing function such that $f(1) = 0$ and define in $[0, f(0)]$ the *truncated sum* $\oplus$ by setting $x \oplus y = (x + y) \wedge f(0)$. Also, set

$$x \otimes y = f^{-1}(f(x) \oplus f(y)), \tag{1}$$

then, we say that f is the additive generator of $\otimes$.

It is well known that the operation $\otimes$, defined by (1), is a continuous Archimedean t-norm, and that every continuous Archimedean t-norm can be obtained by a suitable additive generator.

Given a nonempty set S, a *binary fuzzy relation* on S is a map $\mathsf{ord} : S \times S \to [0, 1]$. Let $\otimes$ be a triangular norm and let $\mathsf{ord} : S \times S \to [0, 1]$ be a fuzzy relation on S. We are interested in the following properties (see, for example, [6,21]):

(1) $\mathsf{ord}(x, x) = 1$ (reflexivity),
(2) $\mathsf{ord}(x, y) = \mathsf{ord}(y, x)$ (symmetry),
(3) $\mathsf{ord}(x, y) \otimes \mathsf{ord}(y, z) \leqslant \mathsf{ord}(x, z)$ ($\otimes$-transitivity),
(4) $E_{\mathsf{ord}}(x, y) = 1 \Rightarrow x = y$ (antisymmetry),

where $E_{\mathsf{ord}}(x, y)$ denotes $\mathsf{ord}(x, y) \otimes \mathsf{ord}(y, x)$ and $x, y, z \in S$.

Definition 2.3. A fuzzy relation $\mathsf{ord} : S \times S \to [0, 1]$ on a nonempty set S is called:

- $\otimes$-*fuzzy preorder* if it satisfies (1) and (3),
- $\otimes$-*fuzzy order*, if it satisfies (1), (3) and (4),
- $\otimes$-*similarity*, if it satisfies (1)–(3),
- *strict* $\otimes$-*similarity*, if it satisfies (1)–(4).

Usually, we denote a similarity by e instead of ord. Notice that the antisymmetry property is sometimes expressed by the implication

$$E_{\mathsf{ord}}(x, y) \neq 0 \Rightarrow x = y$$

(see [1]). However, we prefer to express such a property by (4) in order to have a duality with functions which are metric in nature, as we are going to see in the next section.

The proof of the following proposition is immediate.

Proposition 2.4. *Given a* $\otimes$-*fuzzy preorder* $\mathsf{ord} : S \times S \to [0, 1]$, *the relation* E_{ord} *is a* $\otimes$-*similarity*. *If* ord *is a* $\otimes$-*fuzzy order*, *then* E_{ord} *is a strict* $\otimes$-*similarity*.

Observe that a $\otimes$-fuzzy preorder is a $\otimes$-fuzzy order according to the definition proposed in [1,12] with respect to E_{ord}. Observe also that if $\otimes$ is the Gödel t-norm and e is a similarity, then $E_e = e$.

Definition 2.5. Let ord be a fuzzy preorder and $0 \leqslant \varepsilon < 1$. Then we denote by $\leqslant_\varepsilon$ the *open* ε-*cut* $\{(x, y) \in S \times S | \mathsf{ord}(x, y) > \varepsilon\}$ of ord and by $\equiv_\varepsilon$ the *open* ε-*cut* $\{(x, y) \in S \times S | E_{\mathsf{ord}}(x, y) > \varepsilon\}$ of E_{ord}. Also, given $0 < \varepsilon \leqslant 1$, we denote by $\leqslant^\varepsilon$ the *closed* ε-*cut* $\{(x, y) \in S \times S | \mathsf{ord}(x, y) \geqslant \varepsilon\}$ of ord and by $\equiv^\varepsilon$ the *closed* ε-*cut* $\{(x, y) \in S \times S | E_{\mathsf{ord}}(x, y) \geqslant \varepsilon\}$ of E_{ord}.

In particular, we are interested to the relations $\leqslant^1$ and $\equiv^1$ we denote by $\leqslant$ and $\equiv$, respectively. Observe that $\leqslant$ is a preorder and $\equiv$ is an equivalence relation. Also, $\leqslant$ is an order relation if and only if $\equiv$ coincides with the identity $=$.

3. A basic duality

We will introduce some notions metrical in nature, since there is an easy understandable duality between the notions of "closeness" and the one of "distance". Indeed, in comparing some objects, it is possible to use either a measure of how they are "similar" or a measure of how they are "distant". Obviously, the smaller the distance is, the bigger the closeness is. Also, as we will see, there is a duality between the notion of quasi-metric and the one of fuzzy order.

Let S be a nonempty set and $d : S \times S \to [0, \infty)$ be a mapping and consider the following properties where $x, y, z \in S$:

(d1) $d(x, y) = 0 \Rightarrow x = y$,
(d′1) $d(x, x) = 0$ (reflexivity),
(d2) $d(x, y) = d(y, x)$ (symmetry),
(d3) $d(x, z) \leqslant d(x, y) + d(y, z)$ (triangular inequality),
(d′3) $d(x, z) \leqslant d(x, y) \vee d(y, z)$ (strong triangular inequality),
(d4) $d(x, y) = 0$ and $d(y, x) = 0 \Rightarrow x = y$.

Definition 3.1. A map $d : S \times S \to [0, \infty)$ on a nonempty set S is called

- *metric distance*, if d satisfies (d1), (d′1), (d2) and (d3);
- *pseudometric distance*, if d satisfies (d′1), (d2) and (d3);
- *quasi-metric distance*, if d satisfies (d1), (d′1), (d3) and (d4);
- *quasi-pseudometric distance*, if d satisfies (d′1), (d3);
- *semi-metric distance*, if d satisfies (d2), (d3) and (d4).

Likewise, if we have axiom (d′3) instead of (d3), then d is called

- *ultrametric distance*,
- *pseudoultrametric distance*,
- *quasi-ultrametric distance*,
- *quasi-ultrapseudometric distance*,
- *semi-ultrametric distance*,

respectively. Finally, if axiom (d4) is not required, then they are called *generalized distances* (metric, ultrametric, pseudometric, etc.).

Let us observe that (d′3) entails (d3). So, any ultrametric distance is a metric distance. In the case that the map d takes values in the closed interval $[0, \infty]$ the distances are called *extended*.

The following propositions, whose proofs are immediate, extend to fuzzy orders and quasi-(ultra)metric distances a connection between similarities and metrics exposed, for example, in [20,5]. We start by analyzing the case involving the Gödel t-norm.

Proposition 3.2. *Let $\otimes$ be the Gödel* t-norm, *let $d : S \times S \to [0, 1]$ be a map, and let us set*

$$\mathsf{ord}(x, y) = 1 - d(x, y).$$

Then

(i) ord *is a $\otimes$-similarity if and only if d is a pseudoultrametric*;
(ii) ord *is a $\otimes$-fuzzy preorder if and only if d is a generalized quasi-ultrametric*;
(iii) ord *is a $\otimes$-fuzzy order if and only if d is a quasi-ultrametric*.

If we consider t-norms different from the Gödel one it is possible to obtain a result which is analogous to Proposition 3.2.

Proposition 3.3. *Let $\otimes$ be a continuous Archimedean* t-norm *and $f : [0, 1] \to [0, \infty]$ be an additive generator of $\otimes$. Moreover, let $d : S \times S \to [0, 1]$ be a map and let us define the fuzzy relation $\mathsf{ord}_f(d) : S \times S \to [0, 1]$ by setting*

$$\mathsf{ord}_f(d)(x, y) = f^{-1}(d(x, y) \wedge f(0)).$$

Then

(i) *d is an extended (generalized) pseudometric* $\Rightarrow \mathsf{ord}_f(d)$ is a $\otimes$-*similarity*;
(ii) *d is an extended quasi-metric* $\Rightarrow \mathsf{ord}_f(d)$ *is a $\otimes$-fuzzy order*;
(iii) *d is an extended generalized quasi-metric* $\Rightarrow \mathsf{ord}_f(d)$ *is a $\otimes$-fuzzy preorder*;
(iv) *d is an extended metric* $\Rightarrow \mathsf{ord}_f(d)$ *is a strict $\otimes$-similarity*.

Conversely, the following proposition shows how it is possible to associate some fuzzy relations with extended distances.

Proposition 3.4. *Let $f : [0, 1] \to [0, \infty]$ be an additive generator and $\otimes$ be the related* t-norm. *Let* $\mathsf{ord} : S \times S \to [0, 1]$ *be a map and let us consider the function $d_f(\mathsf{ord}) : S \times S \to [0, \infty]$ defined by setting*

$$d_f(\mathsf{ord})(x, y) = f(\mathsf{ord}(x, y)).$$

Then

(i′) ord *is a $\otimes$-similarity* $\Rightarrow d_f(\mathsf{ord})$ *is an extended generalized pseudometric*;
(ii′) ord *is a $\otimes$-fuzzy order* $\Rightarrow d_f(\mathsf{ord})$ *is an extended quasi-pseudometric*;
(iii′) ord *is a $\otimes$-fuzzy preorder* $\Rightarrow d_f(\mathsf{ord})$ *is an extended generalized quasi-pseudometric*;
(iv′) ord *is a strict $\otimes$-similarity* $\Rightarrow d_f(\mathsf{ord})$ *is an extended metric*.

Examples. Let d be the usual distance in an Euclidean space and let $f(x) = 1 - x$. Then if we set $\mathsf{ord}_f(d)(x, y) = 1 - d(x, y)$ if $d(x, y) \leqslant 1$, and $\mathsf{ord}_f(d)(x, y) = 0$ otherwise, $\mathsf{ord}_f(d)$ is a $\otimes$-fuzzy order, where $\otimes$ is the Łukasiewicz t-norm. As another example, let us assume that $f(x) = -\log(x)$. Therefore, we set $\mathsf{ord}_f(d)(x, y) = \mathrm{e}^{-d(x,y)}$ and we obtain a $\otimes$-fuzzy order, where $\otimes$ is the product t-norm.

Vice versa, considering the Łukasiewicz t-norm and its additive generator $f(x) = 1 - x$, we obtain a distance by setting $d_f(\mathsf{ord})(x, y) = 1 - \mathsf{ord}(x, y)$. If $\otimes$ is the product t-norm and $f(x) = -\log(x)$ its additive generator, we obtain a distance by $d_f(\mathsf{ord})(x, y) = -\log(\mathsf{ord}(x, y))$.

4. Convergence by fuzzy orders

Every fuzzy order is associated with some notions topological in nature.

Let us remark that we will denote by $\mathbb{N}_0$ the set of natural numbers including 0 and by $\mathbb{N}$ the set of natural numbers without 0.

Recall that a *sequence* of elements of a set S is a map $f : \mathbb{N} \to S$, we denote by $(x_n)_{n\in\mathbb{N}}$, where $x_n = f(n)$. We denote by $(x_n)_{n \geqslant \underline{n}}$ the sequence of $y_i = x_{\underline{n}+i-1}$.

Definition 4.1. A sequence $(x_n)_{n\in\mathbb{N}}$ of elements of S is said to be *forward Cauchy* in (S, ord) if, for every $0 \leqslant \varepsilon < 1$, there exists a natural number n_0 such that $m \geqslant n \geqslant n_0$ entails $x_n \leqslant_\varepsilon x_m$.

Equivalently, $(x_n)_{n\in\mathbb{N}}$ is forward Cauchy provided that, for every $0 \leqslant \varepsilon < 1$ there is n_0 such that $m \geqslant n \geqslant n_0$ entails $\mathsf{ord}(x_n, x_m) > \varepsilon$. The following proposition shows that the notion of forward Cauchy sequence extends both the one of order-preserving sequence in an ordered set and the one of Cauchy sequence in a metric space.

Proposition 4.2. *Let* ord *be the characteristic function of a crisp partial order $\leqslant$. Then a sequence $(x_n)_{n\in\mathbb{N}}$ is forward Cauchy if and only if there is a natural number $\underline{n}$ such that $(x_n)_{n \geqslant \underline{n}}$ is increasing.*

Let e be the strict similarity defined by a metric d and an additive generator f. Then a sequence is forward Cauchy in (S, e) if and only if it is a Cauchy sequence in the metric space (S, d).

Proof. The first part of the proposition is obvious. Let $e(x, y) = f^{-1}(d(x, y) \wedge f(0))$ and assume that $(x_n)_{n\in\mathbb{N}}$ is forward Cauchy. Then for every $0 \leqslant \varepsilon < 1$ there is n_0 such that $m \geqslant n \geqslant n_0$ entails $f^{-1}(d(x_n, x_m) \wedge f(0)) > \varepsilon$. Let δ be such that $0 < \delta < f(0)$ and set $\varepsilon = f^{-1}(\delta)$. Then, since $f^{-1}(d(x_n, x_m) \wedge f(0)) \geqslant \varepsilon = f^{-1}(\delta)$ if and only if $d(x_n, x_m) \leqslant \delta$, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the metric space (S, d). In a similar way we can prove that every Cauchy sequence in (S, d) is also a forward Cauchy sequence in (S, e). □

Definition 4.3. We say that $l \in S$ is a *limit* of a sequence $(x_n)_{n\in\mathbb{N}}$ in (S, ord) if, for every $x \in S$, we have that the sequence $\mathsf{ord}(x_n, x)_{n\in\mathbb{N}}$ converges and

$$\lim_{n\to\infty} \mathsf{ord}(x_n, x) = \mathsf{ord}(l, x).$$

Proposition 4.4. *Let e be the strict similarity corresponding to a metric d, then the convergence, defined in Definition* 4.3, *coincides with the usual one in the metric space* (S, d).

It is possible that a sequence has more than one limit. Nevertheless the following proposition holds.

Proposition 4.5. *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence in* (S, ord) *and assume that l is a limit for it. Then* l' *is a limit of* $(x_n)_{n\in\mathbb{N}}$ *if and only if* l' *is similar with l, i.e.* $l' \equiv l$ (*see Definition* 2.5). *As a consequence if* ord *is a* $\otimes$*-fuzzy order, a sequence admits at most a limit.*

Proof. Let us assume that l is a limit of $(x_n)_{n\in\mathbb{N}}$. Then in the case that also l' is a limit, we have that, $\mathsf{ord}(l, x) = \lim_{n\to\infty} \mathsf{ord}(x_n, x)$ and $\mathsf{ord}(l', x) = \lim_{n\to\infty} \mathsf{ord}(x_n, x)$ for every $x \in S$. In particular, by setting $x = l'$,

$$1 = \mathsf{ord}(l, l) = \lim_{n\to\infty} \mathsf{ord}(x_n, l) = \mathsf{ord}(l', l'),$$

by setting $x = l'$,

$1 = \mathsf{ord}(l', l') = \lim_{n\to\infty} \mathsf{ord}(x_n, l') = \mathsf{ord}(l, l')$. Then $1 = \mathsf{ord}(l', l) = \mathsf{ord}(l, l')$ and l' is similar with l'. Conversely, assume l' is an element such that $l' \equiv l$. Then it is immediate that $\mathsf{ord}(l', x) = \mathsf{ord}(l, x) = \lim_{n\to\infty} \mathsf{ord}(x_n, x)$ for every $x \in S$. □

In accordance with Proposition 4.5 we will write $\lim_{n\to\infty}(x_n) \equiv l$ to denote that l is a limit of $(x_n)_{n\in\mathbb{N}}$.

The proof of the following proposition is immediate.

Proposition 4.6. *Assume that* ord *is the characteristic function of a partial order* $\leqslant$ *on S, then l is a limit of* $(x_n)_{n\in\mathbb{N}}$ *if and only if*

$$\forall x \in S(l \leqslant x \Leftrightarrow \exists m \forall n \geqslant m, x_n \leqslant x).$$

As a consequence, if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and $\underline{n}$ is a natural number such that $(x_n)_{n\geqslant\underline{n}}$ is order-preserving, then $\lim_{n\to\infty}(x_n) \equiv l$ if and only if $l = \sup\{x_n | n \geqslant \underline{n}\}$.

Definition 4.7. A structure (S, ord) is called *complete* if every forward Cauchy sequence converges to a limit.

If ord is a crisp order, the completeness coincides with the existence of the least upper bound for any increasing sequence. If e is the strict similarity corresponding to a metric d, then the completeness coincides with the usual completeness in (S, d).

Definition 4.8. Let ord be a $\otimes$-fuzzy preorder on S. Then a map f is called *continuous* if for every convergent sequence $(x_n)_{n\in\mathbb{N}}$ in S,

$$\lim_{n\to\infty} x_n \equiv l \Rightarrow \lim_{n\to\infty} f(x_n) \equiv f(l).$$

Obviously, when ord is a partial order, f is continuous if and only if it preserves upper bounds of chains. If ord is the similarity associated with a distance d, then such a notion of continuity coincides with the continuity in the metric space (S, d).

5. Fuzzy inclusions based on a possibility

Given a nonempty set S, we call *generalized $\otimes$-fuzzy inclusion* a $\otimes$-fuzzy preorder $Incl : P(S) \times P(S) \rightarrow [0, 1]$ such that

(1) $Incl$ extends the classical inclusion relation, i.e. $X \subseteq Y \Rightarrow Incl(X, Y) = 1$;
(2) $X_1 \subseteq X_2 \Rightarrow Incl(X_1, Y) \geqslant Incl(X_2, Y)$;
(3) $Y_1 \subseteq Y_2 \Rightarrow Incl(X, Y_1) \leqslant Incl(X, Y_2)$.

The value $Incl(X, Y)$ is called the *degree of inclusion* of X in Y. We say that $Incl$ is a *$\otimes$-fuzzy inclusion* if $Incl$ is a $\otimes$-fuzzy order.

Proposition 5.1. *Let $\mu : P(S) \rightarrow [0, 1]$ be a possibility measure, i.e. a map such that $\mu(X \cup Y) = \mu(X) \vee \mu(Y)$* and *$\mu(\emptyset) = 0$, and set*

$$Incl(X, Y) = 1 - \mu(X - Y).$$

Then Incl is a generalized fuzzy inclusion with respect to the minimum t-norm. *Moreover, if $\mu(X) \neq 0$, for any $X \neq \emptyset$, then Incl is a fuzzy inclusion.*

Proof. First let us observe that if $X \subseteq Y$, then $X - Y = \emptyset$, and therefore $Incl(X, Y) = 1$. This proves that $Incl$ is an extension of $\emptyset$. Reflexivity follows trivially by the definition. To prove that $Incl(X, Z) \geqslant Incl(X, Y) \wedge Incl(Y, Z)$, let us observe that

$$X - Z \subseteq ((X - Y) \cup (Y - Z)). \tag{2}$$

In fact, let $x \in X - Z$. If $x \in Y$, then we have that $x \in Y - Z$, otherwise, if $x \notin Y$, we have that $x \in X - Y$. Therefore, thanks to (2) we can write

$$\mu(X - Z) \leqslant \mu((X - Y) \cup (Y - Z)) = \mu(X - Y) \vee \mu(Y - Z)$$

and then

$$1 - \mu(X - Z) \geqslant 1 - (\mu((X - Y) \cup (Y - Z))) = (1 - \mu(X - Y)) \wedge (1 - \mu(Y - Z)).$$

So the $\wedge$-transitivity is satisfied and $Incl$ is a $\wedge$-fuzzy preorder.

Moreover, let us assume that $\mu(X) \neq 0$ for every $X \neq \emptyset$. Then $\mu(X) = 0$ entails that $X = \emptyset$. Thus, from $\mu(X-Y) = 0$ it follows that $X - Y = \emptyset$, and therefore $X \subseteq Y$. Similarly, from $\mu(Y - X) = 0$ it follows that $Y - X = \emptyset$ and $Y \subseteq X$. So, in such a case $Incl$ results a fuzzy inclusion. $\square$

Proposition 5.2. *The fuzzy inclusion associated with a possibility measure satisfies the following properties*:

(i) $Incl(X, Y) = Incl(X - Y, \emptyset) = Incl(S, Y \cup -X)$;
(ii) $Incl(X_1 \cup X_2, Y) = Incl(X_1, Y) \wedge Incl(X_2, Y)$;

Conversely, assume that Incl is a $\otimes$-fuzzy inclusion satisfying (i) *and* (ii). *Then Incl is the fuzzy inclusion associated with a suitable possibility measure.*

Proof. To prove (i), it is sufficient to observe that $Incl(X, Y) = 1-\mu(X-Y) = 1-\mu((X-Y)-\emptyset) = Incl(X-Y, \emptyset) = 1 - \mu(S - (Y \cup -X)) = Incl(S, Y \cup -X)$. To prove (ii), observe that

$$\begin{aligned} Incl(X_1 \cup X_2, Y) &= 1 - \mu(X_1 \cup X_2 - Y) = 1 - \mu((X_1 - Y) \cup ((X_2 - Y)) \\ &= 1 - (\mu(X_1 - Y) \vee \mu(X_2 - Y)) \\ &= (1 - \mu(X_1 - Y)) \wedge (1 - \mu(X_2 - Y)) \\ &= Incl(X_1, Y) \wedge Incl(X_2, Y). \end{aligned}$$

Conversely, assume (i) and (ii) and set $\mu(X) = 1 - Incl(X, \emptyset)$. Then $\mu(\emptyset) = 1 - Incl(\emptyset, \emptyset) = 0$ and, by (ii),

$$\begin{aligned}\mu(X \cup Y) &= 1 - Incl(X \cup Y, \emptyset) = 1 - (Incl(X, \emptyset) \wedge Incl(Y, \emptyset)) \\ &= (1 - Incl(X, \emptyset)) \vee (1 - Incl(Y, \emptyset)) \\ &= \mu(X) \vee \mu(Y).\end{aligned}$$

This proves that μ is a possibility measure. Also, by (i),

$$Incl(X, Y) = Incl(X - Y, \emptyset) = 1 - \mu(X - Y). \quad \square$$

In the following proposition, where we denote by $-X$ the complement of X in S, we list some basic properties of *Incl*.

Proposition 5.3. *The $\otimes$-fuzzy inclusion Incl associated with a possibility measure satisfies the following properties*:

(i) $Incl(X, Y) = Incl(-Y, -X)$;
(ii) $Incl(X, Y) = Incl(X - Y, Y)$;
(iii) $Incl(X, Y) = Incl(X, X \cap Y)$;
(iv) $Incl(X, Y) = Incl(X \cap Z, Y) \wedge Incl(X - Z, Y)$;
(v) $(Incl(X, Y_1 \cap Y_2) = Incl(X, Y_1) \wedge Incl(X, Y_2)$;
(vi) $Incl(X, Y_1 \cup Y_2) = Incl(X - Y_1, Y_2)$;
(vii) $Incl(X_1 \cap X_2, Y) = Incl(X_1, Y \cup -X_2)$;
(viii) $Incl(X, Y) \leqslant Incl(X \cap Z, Y \cap Z)$;
(ix) $Incl(X, Y) \leqslant Incl(X \cup Z, Y \cup Z)$.

Proof. Proving the properties is quite trivial. Let us prove, for example, (i), (iv) and (viii). For (i) we have $Incl(X, Y) = 1 - \mu(X - Y) = 1 - \mu(-Y - (-X)) = Incl(-Y, -X)$; for (iv) we have $Incl(X, Y) = 1 - \mu(X - Y) = 1 - (\mu((X \cap Z - Y) \cup ((X - Z) - Y))) = 1 - (\mu((X \cap Z - Y) \vee \mu((X - Z) - Y))) = (1 - \mu(X \cap Z - Y)) \wedge (1 - \mu((X - Z) - Y)) = Incl(X \cap Z, Y) \wedge Incl(X - Z, Y)$; to prove (viii) observe that, since $(X \cap Z) - (Y \cap Z) \subseteq (X - Y)$, it is $\mu(X - Y) \geqslant \mu((X \cap Z) - (Y \cap Z))$, so $1 - \mu(X - Y) \leqslant 1 - (\mu((X \cap Z) - (Y \cap Z)))$; analogously for (ix). $\square$

As an example, given a set S (think, for example, to the set of facts in a program), let us consider a fuzzy set $rl : S \to [0, 1]$, we interpret as the *fuzzy subset of the relevant elements*. Then, it is possible to define the function $\mu : P(S) \to [0, 1]$ by setting

$$\mu(X) = \sup\{rl(x) | x \in X\}.$$

The number $\mu(X)$ is the truth degree of the claim "there is a relevant element in X". In accordance, if $irl(x) = 1 - rl(x)$, a $\otimes$-fuzzy inclusion *Incl* on $P(S)$ is defined by setting

$$Incl(X, Y) = 1 - \mu(X - Y) = \inf\{irl(x) | x \in X - Y\}. \tag{3}$$

We interpret $Incl(X, Y)$ as the truth degree of the claim "all the elements belonging to X and not to Y are non-relevant", or, in other words, "all the relevant elements of X are in Y". Let us investigate the meaning of being forward Cauchy with respect to the fuzzy relation *Incl*.

Proposition 5.4. *Set $R_\varepsilon = \{x \in S | rl(x) > \varepsilon\}$. Then*

$$X \leqslant_\varepsilon Y \Leftrightarrow X \cap R_{1-\varepsilon} \subseteq Y.$$

In accordance, a sequence $(X_n)_{n \in \mathbb{N}}$ of subsets of S is forward Cauchy if, for every $0 \leqslant \delta < 1$, there exists a natural number n_0 such that $X_n \cap R_\delta \subseteq X_m$ whenever $m \geqslant n \geqslant n_0$.

Proof. We have

$$\begin{aligned}Incl(X, Y) \geqslant \varepsilon &\Leftrightarrow \inf\{irl(x) | x \in X - Y\} \geqslant \varepsilon \\ &\Leftrightarrow \inf\{1 - rl(x) | x \in X - Y\} \geqslant \varepsilon\end{aligned}$$

$$\begin{aligned}
&\Leftrightarrow \forall x \in X - Y, 1 - rl(x) \geqslant \varepsilon \\
&\Leftrightarrow X - Y \subseteq \{x | 1 - rl(x) \geqslant \varepsilon\} \\
&\Leftrightarrow (X - Y) \cap \{x | rl(x) > 1 - \varepsilon\} = \emptyset \\
&\Leftrightarrow X \cap R_{1-\varepsilon} \subseteq Y. \qquad \square
\end{aligned}$$

In other words a sequence is forward Cauchy if for every δ, it is definitely increasing with respect to the δ-relevant elements. In particular, we have the following proposition.

Proposition 5.5. *Assume that there exists* $\varepsilon < 1$ *such that for every* $x \in S$, $rl(x) > \varepsilon$. *Then a sequence* $(X_n)_{n\in\mathbb{N}}$ *of subsets of S is forward Cauchy if and only if it is definitely increasing with respect to the classical inclusion. In such a case, there is* n_0 *such that* $\lim_{n\to\infty} X_n = \bigcup_{n\geqslant n_0} X_n$.

Proof. Since $(X_n)_{n\in\mathbb{N}}$ is definitely increasing, it is $Incl(X_n, X_{n+1}) = 1$, from a suitable n_0 on and so the first part of the proposition is proved. Besides, a limit L is such that $Incl(L, X) = \lim_{n\to\infty} Incl(X_n, X)$, for every $X \in P(S)$. Reminding that $Incl(X_n, X) \geqslant Incl(X_{n+1}, X)$, we have, for every X, $\lim_{n\to\infty} Incl(X_n, X) = \inf_{n\geqslant n_0} Incl(X_n, X) = \inf_{n\geqslant n_0}\{irl(x)|x \in X_n - X\} = \inf\{irl(x)|x \in \bigcup_{n\geqslant n_0} X_n\}$. So we have that $\lim_{n\to\infty}(X_n) = \bigcup_{n\geqslant n_0} X_n$.

The second part of the proposition is trivially proved. $\square$

It is also interesting to examine the quasi-metric associated with a fuzzy relevance-based fuzzy inclusion.

Proposition 5.6. *Let Incl be a fuzzy inclusion based on a relevance function* $rl : S \to [0, 1]$ *and define* $d : P(S) \times P(S) \to [0, 1]$ *by setting*

$$d(X, Y) = \inf\{\varepsilon \in [0, 1] | X \cap R_\varepsilon \subseteq Y\}. \tag{4}$$

Then d is an ultra-quasimetric such that

$$Incl(X, Y) = 1 - d(X, Y).$$

Proof. Obviously, if $\lambda \geqslant \varepsilon$, then $X \cap R_\varepsilon \subseteq Y$, entails $X \cap R_\mu \subseteq Y$. This means that $\{\varepsilon \in [0, 1] | X \cap R_\varepsilon \subseteq Y\}$ is an interval and,

$$\begin{aligned}
d(X, Y) &= \inf\{\varepsilon \in [0, 1] | X \cap R_\varepsilon \subseteq Y\} \\
&= \sup\{\varepsilon \in [0, 1] | X \cap R_\varepsilon \text{ is not contained in Y}\} \\
&= \sup\{\varepsilon \in [0, 1] | x \in X \text{ exists such that } x \in R_\varepsilon \text{ and } x \notin Y\} \\
&= \sup\{\varepsilon \in [0, 1] | x \in X - Y \text{ exists such that } rl(x) > \varepsilon\} \\
&= \sup\{rl(x) | x \in X - Y\} = \mu(X - Y).
\end{aligned}$$

So, $Incl(X, Y) = 1 - d(X, Y) = 1 - \mu(X - Y)$. $\square$

6. An example from logic programming

We denote by P a program in a first-order language and by B_P the Herbrand basis associated with it. We call *Herbrand interpretation* any subset of B_P, i.e. any set of facts. Equivalently, since we identify a subset with the related characteristic function, an Herbrand interpretation is an element in $\{0, 1\}^{B_P}$, i.e. any mapping $v : B_P \to \{0, 1\}$. If P^* denotes the set of all the ground instances of the clauses in P, then the single-step operator $T_P : \{0, 1\}^{B_P} \to \{0, 1\}^{B_P}$ is defined for every $v \in \{0, 1\}^{B_P}$ in two steps:

(1) we extend v to the negations of the ground atoms by the equation

$$v(\neg X) = 1 - v(X),$$

(2) we set:
$T_P(v)(A) = 1$ if there is $A \leftarrow B_1, \ldots, B_n$ in P^* such that $v(B_1) = 1, \ldots, v(B_n) = 1$; $T_P(v)(A) = 0$ otherwise.

Now, if P is definite, T_P is a monotone operator in the complete lattice $\{0,1\}^{B_P}$ and therefore the Tarski theorem applies. For a general logic program P (that is a program with possible negation) T_P cannot be monotone. In such a case, techniques based on a fuzzy inclusion using a *level mapping* $l : B_P \to \mathbb{N}_0$ can be taken under consideration. It is useful to consider the sets

$$I(X,k) = \{x \in X | l(x) < k\},$$

where $k \in \mathbb{N}$ and X is a subset of B_P.

Proposition 6.1. *Let $l : B_P \to \mathbb{N}_0$, be a level mapping and set $rl(x) = f(l(x))$, where $f : \mathbb{N}_0 \to [0,1]$ is an injective order-reversing map such that $f(0) = 1$ and $\lim_{n\to\infty} = 0$. Consider the fuzzy inclusion based on such a relevance function and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of subsets of B_P. Then, the followings are equivalent*:

(i) *for every $k \in \mathbb{N}$, there exists a natural number n_0 such that $I(X_n,k) \subseteq I(X_m,k)$ whenever $m \geqslant n \geqslant n_0$;*
(ii) *$(X_n)_{n\in\mathbb{N}}$ is forward Cauchy.*

Proof. Assume (i) and let $0 \leqslant \delta < 1$. Since $f(0) = 1$, the set $\{n : f(n) > \delta\}$ is nonempty and since $\lim_{n\to\infty} = 0$ this set is bounded. Then we can consider $m(\delta) = \max\{n : f(n) > \delta\}$. Then,

$$\begin{aligned} X_n \cap R_\delta &= \{x \in X_n | rl(x) > \delta\} = \{x \in X_n | f(l(x)) > \delta\} \\ &= \{x \in X_n | l(x) < m(\delta)\} = I(X_n, m(\delta)). \end{aligned}$$

This means that for every δ such that $0 \leqslant \delta < 1$, there is $k \in \mathbb{N}$ such that $I(X_n,k) = X_n \cap R_\delta$ and so (ii) is verified.

Conversely, assume (ii) and let k be an element of $\mathbb{N}$. If we set $\delta = f(k)$, we obtain that

$$X_n \cap R_\delta = \{x \in X_n | f(l(x)) > \delta\} = \{x \in X_n | l(x) < k\} = I(X_n,k),$$

where $\delta < 1$. So (i) is verified. $\square$

In other words, Proposition 6.1 says that $(X_n)_{n\in\mathbb{N}}$ is forward Cauchy provided that for every $k \in \mathbb{N}$, the set of facts whose level is less than k becomes stable from a suitable point on. As an example (see [8,9]) consider the program P

$$\begin{aligned} &\mathit{even}(0) \leftarrow \\ &\mathit{even}(s(x)) \leftarrow \mathit{even}(x) \end{aligned}$$

Then we can calculate the sequence

$$\begin{aligned} T_P(\emptyset) &= B_P \\ T_P^2(\emptyset) &= \{\mathit{even}(0)\} \\ T_P^3(\emptyset) &= \{\mathit{even}(0), \mathit{even}(s^2(0)), \mathit{even}(s^3(0)), \mathit{even}(s^4(0)), \mathit{even}(s^5(0)), \ldots\} \\ &= B_P - \{\mathit{even}(s(0))\} \\ T_P^4(\emptyset) &= \{\mathit{even}(0), \mathit{even}(s^2(0))\} \\ T_P^5(\emptyset) &= \{\mathit{even}(0), \mathit{even}(s^2(0)), \mathit{even}(s^4(0)), \mathit{even}(s^5(0)), \ldots\} \\ &= B_P - \{\mathit{even}(s(0)), \mathit{even}(s^3(0))\} \\ &\ldots \end{aligned}$$

By referring to the characteristic functions, we represent such a sequence by the following table:

It is evident that such a sequence is not monotone. Nevertheless, define the level mapping $l : B_P \to \mathbb{N}_0$, by setting $l(\mathit{even}(s^n(0))) = n$ and define the relevance measure $rl : B_P \to [0,1]$ by setting $rl(x) = 1/(l(x)+1)$. Then by referring to the fuzzy inclusion *Incl* based on *rl*, we can give the following proposition (for the proof see [9]).

Proposition 6.2. *The sequence $(T_P^n(\emptyset))_{n\in\mathbb{N}}$ is forward Cauchy and it converges to the set* $\{\mathit{even}(s^n(0)) \in B_P | n \text{ is even}\}$ *and therefore this set is the least Herbrand model of the program P.*

Obviously, we can consider also a different relevance function. As an example we can set $rl(x) = 2^{-l(x)}$. In such a case we obtain a fuzzy inclusion *Incl*′ which is related with the quasi-metric $d' : P(B_P) \times P(B_P) \to [0,1]$ defined by Seda in [17] and [18].

	$even(0)$	$even(s(0))$	$even(s^2(0))$	...				
$T_P(\emptyset)$	1	1	1	1	1	1	1	...
$T_P{}^2(\emptyset)$	1	0	0	0	0	0	0	...
$T_P{}^3(\emptyset)$	1	0	1	1	1	1	1	...
$T_P{}^4(\emptyset)$	1	0	1	0	0	0	0	...
$T_P{}^5(\emptyset)$	1	0	1	0	1	1	1	...
$T_P{}^6(\emptyset)$	1	0	1	0	1	0	0	...
$T_P{}^7(\emptyset)$	1	0	1	0	1	0	1	...
$T_P{}^8(\emptyset)$	1	0	1	0	1	0	1	...
...	...	...	...	...	...	...	...	...

7. Fixed point theorems by fuzzy orders

Let us recall some well-known fixed point theorems for ordered sets and for metric spaces (see [7,16,19]). The first theorem refers to the notion of continuity in a partially ordered set $(S, \leqslant)$, saying that a map $f : S \to S$ is *continuous* if, for every order-preserving sequence $(x_n)_{n\in\mathbb{N}}$ having a supremum,

$$f(\sup_{n\in\mathbb{N}} x_n) = \sup_{n\in\mathbb{N}} f(x_n).$$

Theorem 7.1 (*Tarski*). *Let $(S, \leqslant)$ be a partially ordered set such that every countable chain has a supremum and let $f : S \to S$ be a continuous map. Assume that b is an element in S such that $b \leqslant f(b)$. Then $b_0 = \sup_{n\in\mathbb{N}} f^n(b)$ is the least fixed point of f greater or equal to b.*

In the next theorem we refer to the notion of contractive function [7]. Given a metric space (S, d), a map $f : (S, d) \to (S, d)$ is called *contractive* if for every $x, y \in X$, $d(f(x), f(y)) \leqslant Md(x, y)$, where M is a fixed constant $M < 1$.

Theorem 7.2 (*Banach*). *Let (S, d) be a complete metric space and let $f : S \to S$ be contractive. Then, f has a unique fixed point, which can be obtained as the limit of the sequence $(f^n(x))_{n\in\mathbb{N}}$, for any $x \in S$.*

Both the theorems can be extended into a unique theorem provided that we refer to the notion of quasi-metric. If (S, d) is a quasi-metric space, then an order relation $\leqslant_d$ is defined by setting $x \leqslant_d y$ if $d(x, y) = 0$. A map $f : S \to S$ is called *non-expansive* provided that $d(f(x), f(y)) \leqslant d(x, y)$ for every $x, y \in S$. The notion of continuity is given in a usual way.

Theorem 7.3 (*Rutten–Smith*). *Let (S, d) be a complete quasi-metric space and let $f : S \to S$ be a non-expansive map.*

- *If f is continuous and there exists $x \in S$ with $x \leqslant_d f(x)$, then f has a fixed point, which is the least fixed point above x;*
- *If f is continuous and contractive, then f has a unique fixed point.*

In accordance with the dualities shown in Section 3, such a theorem suggests that the notion of fuzzy order enables us to demonstrate theorems simultaneously generalizing the fixed point theorem of Tarski for ordered structures and the theorems for metric spaces [4].

Definition 7.4. Let ord be a fuzzy preorder on a set S. We say that x is a *fixed point* for f with respect to ord if $x \equiv f(x)$.

Obviously, in the case ord is a $\otimes$-fuzzy order, x is a fixed point of f if and only if $f(x) = x$.

Now let us call an *almost* operator a linguistic modifier (see [3]) $Al : [0, 1] \to [0, 1]$, verifying

(1) for every $x \neq 0$, $x \neq 1$, $Al(x) > x$, while $Al(0) = 0$ and $Al(1) = 1$;
(2) Al is monotone and continue;
(3) for every $x \neq 0$ there is $\underline{x} \neq 0$ such that

$$Al^n(x) \otimes \cdots \otimes Al^{n-r+1}(x) \geqslant Al^n(\underline{x}) \quad \text{for every } n, r \in \mathbb{N}.$$

Then 0 and 1 are the only fixed points of Al and, for every $x \neq 0$,

$$\lim_{n\to\infty} Al^n(x) = 1.$$

In fact $(Al^n(x))_{n\in\mathbb{N}}$ is a strictly increasing sequence whose limit $\lim_{n\to\infty} Al^n(x) = \sup_{n\in\mathbb{N}} Al^n(x)$ is a fixed point of Al different from 0.

Example. Set $Al(x) = x^c$ with $0 < c < 1$ and assume that $\otimes$ is the usual product. Then Al verifies properties (1) and (2), trivially. Besides, since

$$c^n + c^{n+1} + \cdots + c^{n+r-1} = \frac{c^n - c^{n+r-1}}{1-c} = \frac{c^n(1-c^{r-1})}{1-c} \leqslant \frac{c^n}{1-c},$$

it is

$$x^{c^n} \cdot x^{c^{n+1}} \cdot \cdots \cdot x^{c^{n+r-1}} = x^{c^n + c^{n+1} + \cdots + c^{n+r-1}} \geqslant x^{c^n/(1-c)}$$

and (3) is satisfied by setting $\underline{x} = x^{1/(1-c)}$.

Definition 7.5. Let ord be a $\otimes$-fuzzy preorder on S and Al be an almost operator. Then we say that a map $f : S \to S$ is *contractive* if

$$\mathsf{ord}(f(x), f(y)) \geqslant Al(\mathsf{ord}(x, y)).$$

In other terms, a contraction is a map that strengthens the degree of relationship between two elements. The logical meaning is expressed by the formula

$$\underline{A}l(Ord(x, y)) \to Ord(f(x), f(y)),$$

where $\underline{A}l$ and *Ord* are expressions to denote Al and ord, respectively, saying that "if x is almost similar with (greater than) y, then $f(x)$ is similar with (greater than) $f(y)$". In the case ord is the characteristic function of an order relation, f is contractive if and only if it is order-preserving.

The following is an existence theorem for fixed points.

Theorem 7.6. *Let* (S, ord) *be a complete* $\otimes$*-fuzzy preorder, let* $f : S \to S$ *be a continuous and contractive map and* x_0 *an element in S such that* $\mathsf{ord}(x_0, f(x_0)) \neq 0$. *Then, the sequence* $(f^n(x_0))_{n\in\mathbb{N}}$ *converges and* $\lim_{n\to\infty} f^n(x_0) = l$ *is a fixed point for* f.

Proof. Firstly we will prove that the sequence $(f^n(x_0))_{n\in\mathbb{N}}$ is forward Cauchy. In fact, let Al be an almost operator for f, then

$$\mathsf{ord}(f^n(x_0), f^{n+1}(x_0)) \geqslant Al^n(\mathsf{ord}(x_0, f(x_0))).$$

Consequently, in account of $\otimes$-transitivity and (3), there is $\underline{x} \neq 0$ such that

$$\begin{aligned}
&\mathsf{ord}(f^n(x_0), f^{n+r}(x_0)) \\
&\quad \geqslant \mathsf{ord}(f^n(x_0), f^{n+1}(x_0)) \otimes \mathsf{ord}(f^{n+1}(x_0), f^{n+2}(x_0)) \otimes \cdots \otimes \mathsf{ord}(f^{n+r-1}(x_0), f^{n+r}(x_0)) \\
&\quad \geqslant Al^n(\mathsf{ord}(x_0, f(x_0))) \otimes Al^{n+1}(\mathsf{ord}(x_0, f(x_0))) \otimes \cdots \otimes Al^{n+r-1}\mathsf{ord}(x_0, f(x_0))) \\
&Al^n(\underline{x}),
\end{aligned}$$

for every $r \in \mathbb{N}$. Since $\lim_{n\to\infty} Al^n(\underline{x}) = 1$, for every $\varepsilon < 1$ there exists a natural number n_0 such that ord $(f^n(x_0), f^{n+r}(x_0)) \geqslant Al^n(\underline{x}) \geqslant \varepsilon$, for every $n \geqslant n_0$ and $r \in \mathbb{N}$. This proves that $(f^n(x_0))_{n\in\mathbb{N}}$ is forward Cauchy and by the completeness a limit l of such a sequence exists. Also, since f is continuous, $f(l) \equiv \lim_{n\to\infty} f(f^n(x_0)) = \lim_{n\to\infty} f^{n+1}(x_0) = \lim_{n\to\infty} f^n(x_0) \equiv l$, and this proves that l is a fixed point for f. $\square$

Theorem 7.7. *Let* (S, ord) *be a complete* $\otimes$*-fuzzy preorder, let* $f : S \to S$ *be a continuous and contractive map and let* x_0 *be an element in S such that* $\mathsf{ord}(x_0, f(x_0)) \neq 0$. *Then, if l and* l' *are two fixed points for f,* $\mathsf{ord}(l, l')$ *is either* 0 *or* 1. *Moreover, if l is the fixed point* $\lim_{n\to\infty} f^n(x_0)$, *then*

$$l' \text{ fixed point and } x_0 \leqslant l' \Rightarrow l \leqslant l'.$$

Proof. Let *Al* be an almost operator for *f*. Then, since *f* is a contractive map,

$$\mathsf{ord}(l, l') = \mathsf{ord}(f(l), f(l')) \geqslant Al(\mathsf{ord}(l, l')).$$

Hence, $\mathsf{ord}(l, l')$ is a fixed point for *Al*. This entails that either $\mathsf{ord}(l, l') = 1$ or $\mathsf{ord}(l, l') = 0$. To prove the second part of the proposition, observe that if l' is a fixed point, then

$$\mathsf{ord}(l', f^n(l')) \otimes \mathsf{ord}(f^n(l'), l') = 1,$$

i.e. l' is a fixed point for every natural number *n*. Indeed, for every natural number *h*,

$$\mathsf{ord}(f^{h-1}(l'), f^h(l')) \geqslant Al^h(\mathsf{ord}(l', f(l')) = 1$$

and

$$\mathsf{ord}(l', f^n(l')) \geqslant \mathsf{ord}(l', f(l')) \otimes \mathsf{ord}(f(l'), f^2(l')) \otimes \cdots \otimes \mathsf{ord}(f^{n-1}(l'), f^n(l')).$$

Assume that $x_0 \leqslant l'$, i.e. that $\mathsf{ord}(x_0, l') = 1$. Then, since

$$\mathsf{ord}(f^n(x_0), f^n(l')) \geqslant Al^n(\mathsf{ord}(x_0, l')) = Al^n(1) = 1,$$

we have that $\mathsf{ord}(f^n(x_0), f^n(l')) = 1$, for every *n*. Since l' is a fixed point,

$$\mathsf{ord}(f^n(x_0), l') \geqslant \mathsf{ord}(f^n(x_0), f^n(l')) \otimes \mathsf{ord}(f^n(l'), l') = 1.$$

Thus

$$\mathsf{ord}(l, l') = \lim_{n\to\infty} \mathsf{ord}(f^n(x_0), l') = 1. \qquad \square$$

Let us observe that if ord is a fuzzy order corresponding to a distance *d*, then $\mathsf{ord}(x, y) \neq 0$. Hence, $\mathsf{ord}(l, l') = \mathsf{ord}(l', l) = 1$ and, therefore, by antisymmetry, $l = l'$. So, in such a case a contractive map has a unique fixed point. If ord is the characteristic function of a crisp order, Theorem 7.3 says that $\lim_{n\to\infty} f^n(x_0) = \sup\{f^n(x_0) | n \in \mathbb{N}\}$ is the least fixed point greater or equal to x_0.

8. Future work

This is an exploratory paper and our researches on the convergence associated with fuzzy orders are at a very initial state. So, there are a lot of open questions. First it should be opportune to analyze fixed point theorems with respect to the different classes of fuzzy orders defined in literature. For example, in [2] some methods for representation and construction of fuzzy weak orders are given.

Besides, it is not clear whether the fuzzy orders associated with a relevance measure are complete or not. Also it is not explored the convergence associated with a very interesting notion of excess in a metric space (a canonical example of quasi-metric).

Another direction we will explore is in the framework of fuzzy logic programming [10]. In fact, also in this case of monotony, fixed point theorems based on fuzzy orders should be useful. Indeed, it is again true that Tarski fixed point theorem works taking in account the Zadeh inclusion between fuzzy subsets. Nevertheless, the process to obtain such a fixed point happens in a continuous environment and it cannot finish by giving the exact output. Rather it is an endless approximation process and we have to consider sufficient a suitable approximation after a finite number of step. From here the need arises to define someway the notion of “approximate fixed point” and to calculate the related degree of approximation. We argue that the notion of convergence associated with a fuzzy order gives useful tools to do this.

References

[1] U. Bodenhofer, Representations and constructions of similarity-based fuzzy orderings, Fuzzy Sets and Systems 137 (2003) 113–136.
[2] U. Bodenhofer, B. De Baets, J. Fodor, A compendium of fuzzy weak orders: representations and constructions, Fuzzy Sets and Systems 158 (2007) 811–829.
[3] B. Bouchon-Meunier, C. Marsala, Linguistic modifiers and measures of similarity or resemblance, in: 9th IFSA World Congr. 2001, Vancouver, 4, 2001, pp. 2195–2199.
[4] C. Coppola, Distances and closeness measures in information spaces, Ph.D. Thesis, University Federico II of Naples, Italy, 2006.
[5] B. De Baets, R. Mesiar, Metrics and $\mathscr{T}$-equalities, J Math. Anal. Appl. 267 (2002) 531–547.
[6] D. Dubois, H. Prade, Fuzzy Sets and Systems: Theory and Applications, Academic Press, New York, 1980.
[7] J. Dugundji, A. Granas, Fixed Point Theory, PWN-Polish Scientific Publishers, 1982.
[8] M. Fitting, Metric methods: three examples and a theorem, J. Logic Programming 21 (3) (1994) 113–127.
[9] M. Fitting, Fixpoint semantics for logic programming—a survey, Theoret. Comput. Sci. 278 (2002) 25–51.
[10] G. Gerla, Fuzzy Logic: Mathematical Tools for Approximate Reasoning, Kluwer Academic Publishers, Dordrecht, 2001.
[11] P. Hájek, Metamathematics of Fuzzy Logic, Kluwer Academic Publishers, Dordrecht, 1998.
[12] U. Höhle, N. Blanchard, Partial ordering in L-undeterminate sets, Inform. Sci. 35 (1985) 133–144.
[13] J.W. Lloyd, Foundations of Logic Programming, Springer, Berlin, 1984.
[14] V. Novak, I. Perfilieva, J. Mockor, Mathematical Principles of Fuzzy Logic, Kluwer Academic Publishers, London, 1999.
[15] S. Priess-Crampe, P. Ribenboim, Logic Programming and Ultrametric Spaces, Rendiconti di Matematica Serie VII, Vol. 19, Roma, 1999, pp. 155–176.
[16] J.J.M.M. Rutten, Elements of generalized ultrametric domain theory, Theoret. Comput. Sci. 170 (1996) 349–381.
[17] A.K. Seda, Quasi-metrics and fixed points in computing, Bull. Eur. Assoc. Theor. Comput. Sci. 60 (1996) 154–163.
[18] A.K. Seda, Quasi-metrics and the semantics of logic programs, Fund. Inform. 29 (2) (1997) 97–117.
[19] M.B. Smith, Quasi uniformities: reconciling domains with metric spaces, in: M. Main, A. Melton, M. Mislove, D. Schmidt (Eds.), Mathematical Foundations of Programming Language Semantics, Lecture Notes in Computer Science, Vol. 298, Springer, Berlin, 1988, pp. 236–253.
[20] L. Valverde, On the structure of F-indistinguishability operators, Fuzzy Sets and Systems 17 (1985) 313–328.
[21] L.A. Zadeh, Similarity relations and fuzzy orderings, Inform. Sci. 3 (1971) 177–200.