EFFECTIVENESS AND GODEL THEOREMS IN FUZZY LOGICS
by
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(A more complete version of this technical note can be find in Chapter 11 of my book in Fuzzy Logic by Kluwer Editor)

1. INTRODUCTION

It is well known that the notions of a "decidable subset" and "recursively enumerable subset" are basic one for classical logic. In particular, they are basic tools for the proof of the famous limitative theorems about the undecidability and incompleteness of first order logic. Then, the question of a suitable extension of such concepts to fuzzy set theory arises. A first proposal in such a direction was made by E. S. Santos in an interesting series of papers. Indeed, Santos, starting from an idea of L. Zadeh (Zadeh [1968]), proposed the notions of fuzzy Turing machine, Markov normal fuzzy algorithm and fuzzy program. Santos proved that all these definitions determine the same notion of computability for fuzzy maps (see Santos [1970] and Santos [1976]). As in the classical case, a corresponding definition of recursively enumerable fuzzy subset is obtained by calling recursively enumerable any fuzzy subset which is the domain of a computable fuzzy map. Successively, a notion of recursive enumerability was proposed in Harkleroad [1984] where a fuzzy subset s is said to be recursively enumerable if the restriction of s to its support is a partial recursive function.

In a large series of papers L. Biacino and the author proposed a definition of recursive enumerability which is a proper extension of both definitions of Santos and Harkleroad. In this paper, we will refer to the resulting theory. Background in recursion theory is required for understanding the arguments in this paper (see, for example, Rogers [1976]). A more complete version of this technical note can be find in Chapter 11 of my book in Fuzzy Logic by Kluwer Editor.

2. RECURSIVELY ENUMERABLE FUZZY SETS

We say that a set S admits a *coding* if there exists a (intuitively) computable one-one map $c : S \to \boldsymbol{N}$ from S onto the set $\boldsymbol{N}$ of positive natural numbers. For example, the set of rational numbers, the set of integers, and the set of formulas of a logic all admit a coding. We identify a set with a coding with $\boldsymbol{N}$ and this enables us to extend to this set all the notions of recursion theory usually defined in $\boldsymbol{N}$ (see Rogers [1976] page 27). As an example, assume that S_1 and S_2 are codified by the one-one computable functions $c_1 : S_1 \to \boldsymbol{N}$ and $c_2 : S_2 \to \boldsymbol{N}$. Then we say that a partial map $f : S_1 \to S_2$ is *partial recursive* provided that the map $f' : \boldsymbol{N} \to \boldsymbol{N}$ defined by setting,

$$f'(n) = c_2(f(c_1^{-1}(n)))$$

for any $n \in \boldsymbol{N}$ such that $c_1^{-1}(n)$ belongs to the domain of f, is a partial recursive function.

Let S be a codified set, then a subset X of S is called *recursively enumerable* if a partial recursive function $f : S \to \boldsymbol{N}$ exists whose domain is X. It is possible to prove that X is recursively enumerable iff either X is empty or a recursive sequence $f : \boldsymbol{N} \to S$ exists such that X is the codomain of f, that is

$$X = \{f(1), f(2), \dots\}.$$

The following proposition shows that we can define the recursive enumerability in terms of limit. Observe that the Cartesian product of two sets with a coding is a set with a coding. This means that, in particular, the notion of recursive map h from $S \times \boldsymbol{N}$ to $\{0,1\}$ is defined.

Proposition 2.1. *A subset X of S is recursively enumerable if and only if there exists a recursive map $h : S \times \boldsymbol{N} \to \{0,1\}$, increasing with respect to the second variable, such that, for any $x \in S$,*

$$c_X(x) = lim_{n \to \infty} h(x,n). \tag{2.1}$$

Let $U = [0,1]$ and denote by $\ddot{U}$ the set of rational numbers in U, i.e.,

$$\ddot{U} = \{x \in \boldsymbol{Q} : 0 \leq x \leq 1\}.$$

Then Proposition 2.1 suggests to extend the definition of recursive enumerability as follows (see Biacino and Gerla [1987] and [1988]):

Definition 2.2. A fuzzy subset $s : S \to U$ of S is *recursively enumerable* if a recursive map $h : S \times \boldsymbol{N} \to \ddot{U}$ exists such that, for every $x \in S$, $h(x,n)$ is increasing with respect to n and

$$s(x) = lim_{n \to \infty} h(x,n). \tag{2.2}$$

Observe that, since h is increasing with respect to n, (2.2) is equivalent to

$$s(x) = Sup\{h(x,n) : n \in \boldsymbol{N}\}.$$

As usual, we assume that a coding for all the partial recursive functions in S is given and we denote by $\phi_i : S \to S$ the partial recursive function in S whose code number is i. We denote the domain of ϕ_i by W_i obtaining in this way a coding for the class of recursively enumerable subsets of S. The following proposition gives some characterizations of the recursive enumerability:

Proposition 2.3. *For every fuzzy subset s of S the following are equivalent*:

(a) *s is recursively enumerable.*

(b) *The set $K(s) = \{(x,\lambda) \in S \times \ddot{U} : s(x) > \lambda\}$ is recursively enumerable.*

(c) *A recursive map $k : \ddot{U} \to \boldsymbol{N}$ exists such that, for every $\lambda \in \ddot{U}$,*

$$\mathrm{O}(s,\lambda) = W_{k(\lambda)}.$$

(d) *A recursive map $k : S \times \boldsymbol{N} \to \ddot{U}$ exists such that, for every $x \in S$,*

$$s(x) = Sup\{k(x,n) : n \in \boldsymbol{N}\}. \tag{2.3}$$

The characterization given by (b) shows that the notion of recursive enumerability fits the definition of fuzzy point proposed in Wong [1974] well. The characterization given by (c) shows that we can identify $\mathcal{F}_e(S)$ with the lattice of "effective" continuous co-chains of recursively enumerable subsets.

Also, we can give a definition of recursive enumerability in terms of finite fuzzy subsets. Let $\mathcal{F}_f(S)$ be the class of finite fuzzy subsets of S whose values belong to $\ddot{U}$. It is evident that a coding exists for such a set and, therefore, that the notion of recursively enumerable class of elements in $\mathcal{F}_f(S)$ is defined.

Proposition 2.4. *Given a fuzzy subset s the following are equivalent* :

(i) *s is recursively enumerable.*

(ii) *s is a union of a computable order-preserving sequence of elements in* $\mathcal{F}_f(S)$.

(iii) *s is a limit of a computable directed sequence of elements in* $\mathcal{F}_f(S)$.

(iv) *s is a union of a recursively enumerable class of subsets in* $\mathcal{F}_f(S)$.

Proof. (i) $\Rightarrow$ (ii). Let $k(x,n)$ be a computable map order-preserving with respect to n such that $s(x) = Sup\{k(x,n) : n \in \boldsymbol{N}\}$. Let $g : \boldsymbol{N} \to \mathcal{F}_f(S)$ be the function defined by setting, for any integer n, $g(n)(x) = k(x,n)$ if $x \leq n$ and $g(n)(x) = 0$ in the case $x>n$. Then g is computable and $(g(n))_{n\in N}$ is an order preserving sequence of elements in $\mathcal{F}_f(S)$ such that $s = \bigcup_{n\in N} g(n)$.
(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv). Straightforward.
(iv) $\Rightarrow$ (i). Assume that a computable map $g : \boldsymbol{N} \to \mathcal{F}_f(S)$ exists such that $s = \bigcup_{n\in N} g(n)$ and set $k(x,n) = g(n)(x)$. Then k is a computable map such that $s(x) = Sup\{k(x,n) : n \in \boldsymbol{N}\}$ and this proves that s is recursively enumerable. □

The following proposition shows that the intersection and the union of two recursively enumerable fuzzy subsets is a recursively enumerable fuzzy subset.

Proposition 2.5. *The class* $\mathcal{F}_e(S)$ *of recursively enumerable fuzzy subsets of S is closed under finite unions and intersections. More specifically,* $\mathcal{F}_e(S)$ *is a sublattice of the lattice* $\mathcal{F}(S)$ *of all fuzzy subsets of S extending the lattice of the* (*classical*) *recursively enumerable subsets.*

Proof. Let s_1 and s_2 be two recursively enumerable fuzzy subsets and $h_1 : S \times \boldsymbol{N} \to \ddot{U}$, $h_2 : S \times \boldsymbol{N} \to \ddot{U}$ two computable functions, increasing with respect to n, such that

$$s_1(x) = lim_{n\to\infty} h_1(x,n) \quad \text{and} \quad s_2(x) = lim_{n\to\infty} h_2(x,n)$$

for every $x \in S$. Then it is evident that

$$(s_1 \cup s_2)(x) = lim_{n\to\infty} (h_1(x,n) \vee h_2(x,n))$$

and

$$(s_1 \cap s_2)(x) = lim_{n\to\infty} (h_1(x,n) \wedge h_2(x,n)).$$

Furthermore, both the maps $h_1 \vee h_2$ and $h_1 \wedge h_2$ are computable and increasing with respect to n. Then, the first part of the proposition is proven.

Finally, observe that from Proposition 2.1 it follows that any recursively enumerable subset has a characteristic function which is a recursively enumerable fuzzy subset. □

The closed cuts of a recursively enumerable fuzzy subset are not necessarily recursively enumerable. In fact, the following theorem holds:

Theorem 2.6. *A subset of S is a closed cut of a recursively enumerable fuzzy subset iff it belongs to the Π_2-level of the arithmetical hierarchy.*

Proof. Let s be a recursively enumerable fuzzy subset and assume that $s(x) = lim_{n\to\infty}h(x,n)$, where $h(x,n)$ is a recursive map increasing with respect to n. Then, for any $\lambda \in Ü$,

$$x \in \mathrm{C}(s,\lambda) \Leftrightarrow s(x) \geq \lambda \Leftrightarrow lim_{n\to\infty}\, h(x,n) \geq \lambda$$
$$\Leftrightarrow \forall\, k\, \exists\, m \text{ such that } h(x,m) \geq \lambda - 1/k.$$

This proves that $\mathrm{C}(s,\lambda)$ belongs to the Π_2-level of the arithmetical hierarchy.

Let X be a subset of S belonging to the Π_2-level of the arithmetical hierarchy. By using a result of Hájek (see Hájek [1998], Theorem 6.3.4), it is possible to give an example of a recursively enumerable fuzzy subset s with a closed cut $\mathrm{C}(s,\mu)$ which is Π_2-complete. Then X is one-one reducible to $\mathrm{C}(s,\mu)$ by a recursive map $f : S \to S$, i.e., $x \in X$ iff $s(f(x)) \geq \mu$. This means that X is the cut of the recursively enumerable fuzzy subset $s \circ f$. □

The next theorem shows that $\mathcal{F}_e(S)$ is not related to the canonical extension of the class of recursively enumerable subsets.

Theorem 2.7. *A fuzzy subset exists which is not recursively enumerable in spite of the recursive enumerability of all its open and closed cuts. Moreover, a recursively enumerable fuzzy subset exists whose closed cuts are not recursively enumerable.*

Proof. Let μ be a real number which cannot be obtained as a limit of an increasing effectively computable sequence of rational numbers. Then the fuzzy subset s^{μ} constantly equal to μ is not recursively enumerable. Since a cut of s^{μ} coincides either with $\varnothing$ or with S, all the cuts of s^{μ} are recursively enumerable. The second part of the proposition was proved in Theorem 2.6. □

3. DECIDABILITY AND FUZZY COMPUTABILITY

In classical theory the notion of recursively enumerable subset enables us to define several basic concepts. For example, we can define *decidable* a subset X of $\boldsymbol{N}$ such that both X and its complement $-X$ are recursively enumerable. Moreover, we can define *computable* a function which is a recursively enumerable subset of $\boldsymbol{N} \times \boldsymbol{N}$.

Likewise, the proposed notion of recursively enumerable fuzzy subset enables us to obtain the definitions of decidable fuzzy subset and computable fuzzy functions.

Definition 3.1. A fuzzy subset *s of S* is *recursively co-enumerable* if its complement $-s$ is recursively enumerable. We say that s is *decidable* if it is both recursively enumerable and recursively co-enumerable (see Biacino and Gerla [1989]).

Proposition 3.2. *A fuzzy set s is recursively co-enumerable if and only if a recursive map* $k : S \times \boldsymbol{N} \to Ü$ *exists such that, for every* $x \in S$, $k(x,n)$ *is decreasing with respect to n and*

$$s(x) = lim_{n\to\infty} k(x,n).$$

Proof. Let $d : S \times \boldsymbol{N} \to Ü$ be a recursive function such that $d(x,n)$ is increasing with respect to n and $-s(x) = lim_{n\to\infty}\, d(x,n)$. Then, by setting $k(x,n) = 1 - d(x,n)$,

$$s(x) = 1 - lim_{n\to\infty}\, d(x,n) = lim_{n\to\infty}\, 1 - d(x,n) = lim_{n\to\infty}\, k(x,n),$$

where $k(x,n)$ is decreasing with respect to n. In the same way we can prove the converse implication. □

The following theorem, whose proof is trivial, gives a characterization of the decidable fuzzy subsets.

Theorem 3.3. *A fuzzy set s is decidable iff for every* $x \in S$, $s(x)$ *is the limit of an effectively computable nested sequence of intervals, i.e., iff two recursive maps* $h : S \times \boldsymbol{N} \to Ü$ *and* $k : S \times \boldsymbol{N} \to Ü$ *exist such that, for any* $x \in S$,

- $h(x,n)$ *is increasing and* $k(x,n)$ *is decreasing with respect to n,*
- *for every* $n \in \boldsymbol{N}$,

$$h(x,n) \le s(x) \le k(x,n),$$

- *and*

$$lim_{n\to\infty}\, h(x,n) = s(x) = lim_{n\to\infty}\, k(x,n).$$

We say that a recursive function $f : S \times \boldsymbol{N} \to Ü$ is *recursively convergent to s* if, for any $x \in S$, $s(x) = lim_{n\to\infty} f(x,n)$ and a recursive function $e : S \times \boldsymbol{N} \to \boldsymbol{N}$ exists such that, for every $x \in S$ and $p \in \boldsymbol{N}$,

$$|\, f(x,n) - f(x,m)\,| < 1/p, \ \text{ for any } \ n, m \ge e(x, p).$$

This notion enables us to obtain an interesting characterization of decidability.

Theorem 3.4. *A fuzzy subset s is decidable iff there exists a recursive function* $f : S \times \boldsymbol{N} \to Ü$ *recursively convergent to s.*

Proof. Assume that s is decidable and let h and k be as in Theorem 3.3. We claim that, for every $n, m \in \boldsymbol{N}$ and $x \in S$,

$$|\, h(x,n) - h(x,m)\,| \le k(x, n \wedge m) - h(x, n \wedge m).$$

Indeed, since

$$k(x, n \wedge m) \ge k(x,n) \ge h(x,n) \ \text{and} \ -h(x,m) \le -h(x,n \wedge m),$$

it is, for $n \geq m$

$$| h(x,n) - h(x,m) | = h(x,n) - h(x,m) \leq k(x, n \wedge m) - h(x,m)$$
$$\leq k(x, n \wedge m) - h(x,n \wedge m),$$

and the same holds for $m \geq n$. Set

$$e(x,p) = Min\{j \in \boldsymbol{N} : k(x,j) - h(x,j) < 1/p\}.$$

The map $e(x,p)$ is recursive and

$$| h(x,n) - h(x,m) | \leq k(x, n \wedge m) - h(x,n \wedge m) < 1/p \text{ for every } m, n \geq e(x,p),$$

and this proves that the recursive function h is recursively convergent to s.

Conversely, assume that f is a recursive function recursively convergent to s by the recursive function e and set, for every $n \in \boldsymbol{N}$, $m_n = e(x,n)$. We have

$$f(x,m_n) - 1/n \leq s(x) \leq f(x,m_n) + 1/n.$$

If we set

$$u(x,n) = Sup\{f(x,m_i) - 1/i : i = 1,...,n\} \quad \text{and} \quad v(x,n) = Inf\{f(x,m_i) + 1/i : i = 1,...,n\},$$

then u is recursive and increasing with respect to n, and v is recursive and decreasing with respect to n. Moreover, since

$$lim_{n\to\infty}(f(x,m_n) - 1/n) = s(x) = lim_{n\to\infty}(f(x,m_n) + 1/n),$$

it is

$$lim_{n\to\infty}u(x,n) = s(x) = lim_{n\to\infty}v(x,n).$$ □

From such a proposition it follows that a decidable fuzzy subset assumes only recursive real numbers as values. In particular, by recalling that, for any $\lambda \in U$, s^{λ} is the fuzzy subset constantly equal to λ,

$$s^{\lambda} \text{ is decidable } \Leftrightarrow \lambda \text{ is a recursive real number.}$$

Proposition 3.5. *The class $\mathcal{F}_d(S)$ of decidable fuzzy subsets is closed under finite unions, finite intersections and complements. Then, $\mathcal{F}_d(S)$ is a sublattice of the lattice $\mathcal{F}_e(S)$ of all recursively enumerable fuzzy sets extending the lattice of decidable sets.*

Proof. Let s be a decidable fuzzy set and let h and k be two functions as in Theorem 3.3. Then, by setting $h' = 1-k$ and $k' = 1-h$, we obtain two recursive sequences the first increasing the latter decreasing such that

$$-s(x) = lim_{n\to\infty} h'(x,n) = lim_{n\to\infty} k'(x,n).$$

This proves that $-s$ is decidable. Assume that s_1 and s_2 are decidable. Then, as s_1 and s_2 are recursively enumerable, $s_1 \cap s_2$ is recursively enumerable. Moreover, as $-s_1$ and $-s_2$ are recursively enumerable, the fuzzy subset $-(s_1 \cap s_2) = -s_1 \cup -s_2$ is recursively enumerable. Thus $s_1 \cap s_2$ is decidable. Finally, the fuzzy subset $s_1 \cup s_2$ is decidable since it is the complement of the decidable fuzzy subset $-s_1 \cap -s_2$. □

Also, the notion of recursive enumerability for fuzzy subsets enables us to define a notion of computability for fuzzy functions.

Definition 3.6. Let S_1 and S_2 be two sets which admit a coding. Then we say that a fuzzy function $f : S_1 \sim> S_2$ is *computable* or *partial recursive* if f is a recursively enumerable fuzzy subset of $S_1 \times S_2$.

4. ENUMERABILITY BY DISCRETE TOPOLOGY

In defining the notion of recursive enumerability we can also interpret the convergence in the equality $s(x) = lim_{n\to\infty} h(x,n)$ with respect to the discrete topology in U. In this way we get a different notion of recursive enumerability (see Biacino and Gerla [1987]).

Definition 4.1. A fuzzy subset $s : S \to U$ is *d-recursively enumerable* if a recursive map $h : S \times \boldsymbol{N} \to \ddot{U}$ exists, increasing with respect to the second variable, such that, for any $x \in S$,

$$s(x) = lim_{n\to\infty}\, h(x,n) \tag{4.1}$$

where the limit is taken with respect to the discrete topology.

By definition, (4.1) is equivalent to saying that

$$\forall x\ \exists m\ \forall\ n\ (n \geq m\ \Rightarrow\ h(x,n) = s(x)). \tag{4.2}$$

Consequently, the truth-values assumed by a d-recursively enumerable fuzzy subset are rational numbers. So, in this section we will consider only fuzzy subsets with values in $\ddot{U}$.

Proposition 4.2. *Every d-recursively enumerable fuzzy subset is recursively enumerable while a recursively enumerable fuzzy subset* $s : S \to \ddot{U}$ *exists which is not d-recursively enumerable.*

Proof. The first part of the proposition is trivial. To prove the second part, let $k : S \times \boldsymbol{N} \to \ddot{U}$ be the function defined by

$$k(x,n) = \begin{cases} n/(2n+1) & \text{if } \phi_{c(x)}(x) \text{ does not converge in less that } n \text{ steps,} \\ r/(2r+1) & \text{if } \phi_{c(x)}(x) \text{ converges in } r \text{ steps and } r \leq n \end{cases}$$

where $c : S \to \boldsymbol{N}$ is a coding of S. Then, since k is recursive and increasing with respect to n, the fuzzy subset

$$s(x) = lim_{n\to\infty} k(x,n) = \begin{cases} 1/2 & \text{if } \phi_{c(x)}(x) \text{ diverges,} \\ r/(2r+1) & \text{if } \phi_{c(x)}(x) \text{ converges in } r \text{ steps} \end{cases}$$

is recursively enumerable. Suppose that s is d-recursively enumerable, and let $h(x,n)$ be a recursive and increasing map satisfying (4.2). Then

$$\{x \in S : \phi_{c(x)}(x) \text{ divergent}\} = \{x \in S : s(x) = 1/2\}$$
$$= \{x \in S : n \text{ exists such that } h(x,n) = 1/2\}.$$

Hence, the set $\{x \in S : \phi_{c(x)}(x)$ divergent$\}$ is recursively enumerable and this contradicts the recursive unsolvability of the halting problem. □

Proposition 4.3. *For every fuzzy subset $s : S \to Ü$ the following are equivalent* :

(a) *s is d-recursively enumerable.*

(b) *The set $E(s) = \{(x,\lambda) \in S \times Ü : s(x) \geq \lambda\}$ is recursively enumerable.*

(c) *A recursive map $h : Ü \to \boldsymbol{N}$ exists such that, for any $\lambda \in Ü$,*

$$\mathrm{C}(s,\lambda) = W_{h(\lambda)}.$$

(d) *A recursive map $h : S \times \boldsymbol{N} \to Ü$ exists such that, for every $x \in S$,*

$$s(x) = Max\{h(x,n) : n \in \boldsymbol{N}\}. \tag{4.3}$$

Proof. (a) $\Rightarrow$ (b). Let $h(x,n)$ be a recursive map increasing with respect to n such that $s(x) = Max\{h(x,n) : n \in \boldsymbol{N}\}$ and define the partial recursive function $g : S \times Ü \to \boldsymbol{N}$ by setting

$$g(x,\lambda) = \begin{cases} 1 & \text{if } h(x,n) \geq \lambda \text{ for a suitable } n \in \boldsymbol{N}, \\ \text{divergent} & \text{otherwise.} \end{cases}$$

Then, since

$$(x,\lambda) \in E(s) \Leftrightarrow s(x) \geq \lambda \;\Leftrightarrow\; Max\{h(x,n) : n \in \boldsymbol{N}\} \geq \lambda$$
$$\Leftrightarrow \exists\, n \in \boldsymbol{N} \textit{ such that } h(x,n) \geq \lambda \;\Leftrightarrow\; (x,\lambda) \in Dom(g),$$

we have that $E(s) = Dom(g)$. This proves that $E(s)$ is recursively enumerable.

(b) $\Rightarrow$ (c). Let g be a partial recursive map whose domain is $E(s)$ and, by the *s-m-n*-theorem, let h be a recursive map such that $\phi_{h(\lambda)}(x) = g(x,\lambda)$. It is evident that

$$x \in \mathrm{C}(s,\lambda) \Leftrightarrow (x,\lambda) \in E(s) \Leftrightarrow g \text{ converges in } (x,\lambda)$$
$$\Leftrightarrow \phi_{h(\lambda)} \text{ converges in } x \;\Leftrightarrow\; x \in W_{h(\lambda)}.$$

(c) $\Rightarrow$ (d). By hypothesis,

$$s(x) = Max\{\lambda \in Ü : x \in \mathrm{C}(s,\lambda)\} = Max\{\lambda \in Ü : x \in W_{h(\lambda)}\}.$$

Set

$$g(x,j,\lambda) = \begin{cases} \lambda & \text{if } \phi_{h(\lambda)} \text{ converges in } x \text{ in fewer than } j \text{ steps,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $g(x,j,\lambda)$ is recursive and $s(x) = Max\{g(x,j,\lambda) : j \in \boldsymbol{N},\ \lambda \in Ü\}$. Let $\pi : \boldsymbol{N} \to \boldsymbol{N} \times Ü$ be any recursive one-one map and set $h(x,n) = g(x,\pi(n))$. Then h is total recursive and $s(x) = Max\{h(x,n) : n \in \boldsymbol{N}\}$.

(d) $\Rightarrow$ (a). Assume that h is a recursive map such that $s(x) = Max\{h(x,n) : n \in \boldsymbol{N}\}$ and set $k(x,n) = h(x,1) \vee ... \vee h(x,n)$. Then k is a recursive map increasing with respect to n such that $s(x) = Max\{k(x,n) : n \in \boldsymbol{N}\}$. □

The characterization given by (b) shows that the notion of *d*-recursive enumerability fits well the definition of fuzzy point which is proposed in Kerre [1980] and in Pu Pao-Mine and Liu Ying-Ming [1980]. Instead, recall that the notion of recursive enumerability fits the definition of fuzzy point proposed in Wong [1974] well. Also, observe that Proposition 4.2 entails the existence of a fuzzy subset s such that $K(s) = \{(x,\lambda) \in S \times \ddot{U} : s(x) > \lambda\}$ is recursively enumerable and $E(s) = \{(x,\lambda) \in S \times \ddot{U} : s(x) \geq \lambda\}$ is not recursively enumerable. The characterization given by (c) shows that we can identify the *d*-recursively enumerable fuzzy subsets with the "effective" continuous chains of recursively enumerable subsets.

Definition 4.4. We say that a fuzzy subset s is *d-recursively co-enumerable* if its complement $-s$ is *d*-recursively enumerable. We say that s is *d-decidable* if s is both *d*-recursively enumerable and *d*-recursively co-enumerable.

The following theorem shows that the notion of *d*-decidability is related to finite-steps computation processes and not necessarily to infinite (effective) approximation processes.

Theorem 4.5. *A fuzzy subset s is d-decidable iff s is a* (*classically*) *computable function from S to Ü.*

Proof. Assume that s is *d*-decidable and let h and k be two recursive functions such that

$$s(x) = Max\{h(x,n) : n \in \boldsymbol{N}\} = Min\{k(x,n) : n \in \boldsymbol{N}\}$$

where h is increasing and k is decreasing with respect to n. Then, to calculate $s(x)$, we have

- to generate the sequence $h(x,1)$, $k(x,1)$, $h(x,2)$, $k(x,2)$, ... until $h(x,i) \neq k(x,i)$
- to stop if i is the first integer such that $h(x,i) = k(x,i)$
- to give as output the number $h(x,i) = k(x,i)$.

This proves that s is a computable function from S to $\ddot{U}$.

Conversely, if s is computable, then also its complement $-s$ is computable. Then both s and $-s$ are *d*-recursively enumerable. □

The next proposition shows that decidable fuzzy sets whose degrees of membership belong to $\ddot{U}$ are not necessarily *d*-decidable.

Proposition 4.6. *A decidable but not d-decidable fuzzy set* $s : S \rightarrow \ddot{U}$ *with values in Ü exists.*

Proof. Let $k : S \times \boldsymbol{N} \rightarrow \ddot{U}$ be the map defined by setting

$$h(x,n) = \begin{cases} n/(n+1) & \text{if } \phi_{c(x)}(x) \text{ does not converge in fewer than } n \text{ steps,} \\ m/(m+1) & \text{if } \phi_{c(x)}(x) \text{ converges in } m \leq n \text{ steps.} \end{cases}$$

Then h is a computable function with rational values increasing with respect to n. As a consequence, the fuzzy subset s defined by

$$s(x) = lim_{n\to\infty}\, h(x,n) = \begin{cases} 1 & \text{if } \phi_{c(x)}(x) \text{ diverges,} \\ m/(m+1) & \text{if } \phi_{c(x)}(x) \text{ converges in } m \text{ steps.} \end{cases}$$

is recursively enumerable. Furthermore, s is decidable, indeed s is the limit of the decreasing, recursive function

$$k(x,n) = \begin{cases} 1 & \text{if } \phi_{c(x)}(x) \text{ does not converges in } n \text{ steps,} \\ m/(m+1) & \text{if } \phi_{c(x)}(x) \text{ converges in } m \text{ steps where } m \le n. \end{cases}$$

Since $s(x) \neq 1$ if $\phi_{c(x)}(x)$ converges and $s(x) = 1$ if $\phi_{c(x)}(x)$ diverges, the map s cannot be computable. □

5. GÖDEL NUMBERING AND CHURCH THESIS

In the following ψ_1, ψ_2, ... denotes a Gödel numbering of the partial recursive functions from $S \times \boldsymbol{N}$ to $Ü$.

Definition 5.1. We say that a recursive function $h : \boldsymbol{N} \to \boldsymbol{N}$ is a *Gödel numbering* for the class of the recursively enumerable fuzzy subsets if, for every $i \in \boldsymbol{N}$,

(a) $\psi_{h(i)} : S \times \boldsymbol{N} \to Ü$ is total and increasing with respect to the second variable,

(b) if s is any recursively enumerable fuzzy subset then, for any $x \in S$,

$$s(x) = lim_{n\to\infty}\, \psi_{h(i)}(x,n).$$

A Gödel numbering enables us to assign a code number to any recursively enumerable fuzzy subset. Indeed, given an index i, we denote by s_i the fuzzy subset defined by setting, for every $x \in S$,

$$s_i(x) = lim_{n\to\infty}\, \psi_{h(i)}(x,n). \tag{5.1}$$

Theorem 5.2. *A Gödel numbering $h : \boldsymbol{N} \to \boldsymbol{N}$ of the recursively enumerable fuzzy subsets exists such that if $\psi_i : S \times \boldsymbol{N} \to Ü$ is total and increasing with respect to the second variable, then, for any $x \in S$,*

$$lim_{n\to\infty}\, \psi_i(x,n) = lim_{n\to\infty}\, \psi_{h(i)}(x,n).$$

Proof. Let $\pi : \boldsymbol{N}^2 \to \boldsymbol{N}$ be a coding of $\boldsymbol{N}^2$ and ψ the function defined by

$$\psi(x,r,t,i) = \begin{cases} \psi_i(x,r) & \text{if } \psi_i(x,r) \text{ converges in fewer than } t \text{ steps,} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, set $\Psi^*(x,n,i) = Sup\{\psi(x,r,t,i) : \pi(r,t) \le n\}$. Then Ψ^* is recursive and increasing with respect to n. By the *s-m-n*-theorem a recursive map h exists such that $\psi_{h(i)}(x,n) = \Psi^*(x,n,i)$. In order to prove that

$$lim_{n\to\infty}\, \psi_i(x,n) = lim_{n\to\infty}\, \psi_{h(i)}(x,n),$$

where ψ_i is any total function increasing with respect to n, we must prove that

$$Sup\{\psi_i(x,r) : r \in \boldsymbol{N}\} = Sup\{\psi^*(x,n,i) : n \in \boldsymbol{N}\}. \tag{5.2}$$

Now, for every $r \in \boldsymbol{N}$, let a_r be the number of the steps in which $\psi_i(x,r)$ converges and set $n = \pi(r,a_r)$. Then $\Psi^*(x,n,i) \geq \psi_i(x,r)$ and

$$Sup\{\Psi^*(x,n,i) \geq Sup\{\psi_i(x,r) : r \in \boldsymbol{N}\}. \tag{5.3}$$

On the other hand, for every $n \in \boldsymbol{N}$, either $\Psi^*(x,n,i) = 0$ or j, $t \in \boldsymbol{N}$ exist such that $\pi(j,t) \leq n$, $\psi_i(x,j)$ converges in fewer than t steps, and $\Psi^*(x,n,i) = \psi_i(x,r)$ for a suitable $r \in \boldsymbol{N}$. At any rate, for every $n \in \boldsymbol{N}$ there exists $r \in \boldsymbol{N}$ such that $\Psi^*(x,n,i) \leq \psi_i(x,r)$ and hence,

$$Sup\{\Psi^*(x,n,i) : n \in \boldsymbol{N}\} \leq Sup\{\psi_i(x,r) : r \in \boldsymbol{N}\}. \tag{5.4}$$

From (5.3) and (5.4) we get (5.2). Since (b) is evident, the proof is complete. □

We can try to define a *Gödel numbering for the d-recursively enumerable fuzzy subsets* as a recursive function $k : \boldsymbol{N} \rightarrow \boldsymbol{N}$ such that, for every $i \in \boldsymbol{N}$,

(a) $\psi_{k(i)} : S \times \boldsymbol{N} \rightarrow \ddot{U}$ is total, increasing with respect to the second variable, and convergent with respect to the discrete topology,

(b) for every recursively enumerable fuzzy subset $s : S \rightarrow U$, an index $i \in \boldsymbol{N}$ exists such that:

$$s(x) = lim_{n\rightarrow\infty}\, \psi_{k(i)}(x,n)$$

for any $x \in S$. In such a case it should be possible to represent the class of all the *d*-recursively enumerable fuzzy subsets by the sequence s_1, s_2, ... defined by setting, for every $i \in \boldsymbol{N}$ and $x \in S$,

$$s_i(x) = lim_{n\rightarrow\infty} \psi_{k(i)}(x,n).$$

Unfortunately, this is not the case as the following theorem proves:

Theorem 5.3. *No Gödel numbering exists for the d-recursively enumerable fuzzy subsets.*

Proof. Assume that such a numbering k exists. Let $f : \ddot{U} \rightarrow \ddot{U}$ be an order-preserving recursive map without fixed points. As an example, we can set $f(x) = x^2/2 + ¼$. Define the fuzzy subset s by setting

$$s(x) = lim_{n\rightarrow\infty} f(\psi_{k(x)}(x,n))$$

for any $x \in S$. It is evident that s is *d*-recursively enumerable and therefore that an index i exists such that $s = s_i$. Then,

$$f(lim_{n\rightarrow\infty}(\psi_{k(i)}(i,n))) = lim_{n\rightarrow\infty} f(\psi_{k(i)}(i,n)) = s(i) = s_i(i) = lim_{n\rightarrow\infty} \psi_{k(i)}(i,n).$$

This contradicts the hypothesis that no fixed point for f exists. □

Church's Thesis for fuzzy set theory. The claim that the notion of a partial recursive function (equivalently, the notion of a recursively enumerable subset) provides a satisfactory counterpart to the informal notion of computability is known as *Church Thesis* (see, e.g., Rogers [1967], pag. 20). Then, the following question arises:

does our definition of recursive enumerability give the correct formal counterpart of the intuition and experience of fuzzy people about fuzzy computability ?

We can call *Extended Church Thesis* the positive answer to this question. By this thesis we admit that a fuzzy algorithm is an infinitary step-by-step approximation process and therefore that fuzzy computability is related to recursive analysis rather than to recursive arithmetic. As in the classical case, it is not possible to give a proof for such a thesis. Nevertheless, several considerations and evidences exist in its favour. As an example, our definition is a proper extension of the definition proposed by E. S. Santos by its class of fuzzy Turing machines. Such an extension is necessary since if s is recursively enumerable in accordance with the definition of Santos, then the codomain of s is finite. Indeed, the truth values that s can assume belongs to the finite lattice generated by the set of truth values occurring in the machine. This is unsatisfactory since, for example, in the fuzzy logic proposed for the heap paradox the fuzzy set of theorems takes as values all the numbers of the sequence 0.9^n. Also, our definition extends the definition proposed in Harkleroad [1984] which appear to be too restrictive. As an example, the class of fuzzy subset which are recursively enumerable in accordance with such a definition is not closed with respect to finite unions and intersections (see Biacino and Gerla [1987]). This is an unsatisfactory departure from the classical theory of recursive enumerability.

Finally, observe that, while Theorem 5.2 gives a reason in favour of Extended Church Thesis, Theorem 5.3 shows that we cannot substitute recursive enumerability with discrete recursive enumerability in this thesis. In fact, Theorem 5.3 entails that no universal language or universal machine can exist for the whole class of the d-recursively enumerable fuzzy subsets.

6. REDUCIBILITY AND UNIVERSAL MACHINES

Given two subsets A and B of S, we say that A is *m-reducible* to B if a recursive map $d : S \to S$ exists such that $A = d^{-1}(B)$, that is

$$x \in A \iff d(x) \in B$$

(see Rogers [1967]). If d is a one-one mapping, then we say that A is *one-one reducible* to B. To extend these notions to the fuzzy subsets of S, we define the *inverse image* of a fuzzy subset s' of S via a map d as the fuzzy subset s such that,

$$s(x) = s'(d(x))$$

for every $x \in S$. In this case we write $s = d^{-1}(s')$.

Definition 6.1. We say that a fuzzy subset s is *m-reducible* to a fuzzy subset s', in brief $s \leq_m s'$, if s is the inverse image of s' via a recursive map $d : S \to S$. In the case in which d is one-one, we say that s is *one-one reducible* to s' and we write $s \leq_1 s'$.

The following propositions summarize the main properties of m-reducibility. The same properties are satisfied by one-one reducibility.

Proposition 6.2. *The m-reducibility is a preorder relation and*

$$s \leq_m s' \Rightarrow -s \leq_m -s'.$$

Moreover, if $s \leq_m s'$ *and* m *is recursively enumerable* (*d-recursively enumerable, decidable, d-decidable*)*, then* s *is recursively enumerable* (*d-recursively enumerable, decidable, d-decidable, respectively*).

Proposition 6.3. *Assume that* $s \leq_m s'$. *Then* :
(a) *For every* $\lambda \in U$, $\mathrm{C}(s,\lambda) \leq_m \mathrm{C}(s',\lambda)$.
(b) *For every* $\lambda \in U$, $\mathrm{O}(s,\lambda) \leq_m \mathrm{O}(s',\lambda)$.
(c) s' *is crisp* $\Rightarrow$ s *is crisp.*
(d) $s' \in \Sigma_n$, (Π_n *or* Δ_n) $\Rightarrow$ $s \in \Sigma_n$ (Π_n *or* Δ_n, *respectively*).

Proof. In order to prove (a) note that

$$x \in \mathrm{C}(s,\lambda) \Leftrightarrow s(x) \geq \lambda \Leftrightarrow s'(d(x)) \geq \lambda \Leftrightarrow d(x) \in \mathrm{C}(s',\lambda).$$

One demonstrates (b) in the same way. Property (c) is evident and (d) is a consequence of Proposition 5.2. □

In Rogers [1967] a subset K of S is called 1-1-*complete* if K is recursively enumerable and every recursively enumerable subset of S is one-one reducible to K. The existence of a complete subset is equivalent, in a sense, to the existence of a universal machine or to the existence of a universal programming language. The extension of such a notion to the fuzzy subsets is obvious.

Definition 6.4. We say that a fuzzy subset k of S is 1-1-*complete* if :
- k is recursively enumerable
- every recursively enumerable fuzzy subset is one-one reducible to k.

In other words, k is 1-1-complete if k is a greatest element of Σ_1 with respect to the relation $\leq_1$. The characteristic function c_K of a (classically) 1-1-complete subset K is not a 1-1-complete fuzzy subset. Indeed, it is obvious that only the crisp recursively enumerable fuzzy sets are one-one reducible to c_K. In the following we call *level set* of a fuzzy subset s the set $L(s,\lambda) = \{x \in S : s(x) = \lambda\}$.

Proposition 6.5. *Let* k *be an* 1-1-*complete fuzzy subset and* $\lambda \in$ Ü $-\{0\}$, *then* :
(a) *The level set* $L(k,\lambda)$ *is a* 1-1-*complete set.*
(b) *The closed cut* $\mathrm{C}(k,\lambda)$, $\lambda \neq 0$, *is a* 1-1-*complete set.*
(c) *The open cut* $\mathrm{O}(k,\lambda)$, $\lambda \neq 1$, *is a* 1-1-*complete set.*

Proof. In order to prove (a), let W be any recursively enumerable subset of S and define s by setting $s(x) = \lambda$ if $x \in W$ and $s(x) = 0$ if $x \notin W$. Then s is recursively enumerable and therefore $s \leq_1 k$, i.e., a recursive one-one map d exists such that $s(x) = k(d(x))$. Then

$$x \in W \Leftrightarrow s(x) = \lambda \Leftrightarrow k(d(x)) = \lambda \Leftrightarrow d(x) \in L(k,\lambda).$$

In order to prove (b), observe that, since the characteristic function c_W of W is a recursively enumerable fuzzy set, we have $c_W \leq_1 k$ via a suitable map d. Hence,

$$x \in W \Leftrightarrow c_W(x) = 1 \Leftrightarrow k(d(x)) = 1 \Leftrightarrow d(x) \in \mathrm{C}(k,\lambda).$$

One demonstrates (c) in the same way. □

In the following we denote by $s_1, s_2, \ldots$ the Gödel numbering of all the recursively enumerable fuzzy subsets of S obtained by Theorem 5.2. Moreover, we codify the elements of $S \times \boldsymbol{N}$ by the elements of S; that is, we consider a recursive one-one map $\pi : S \times \boldsymbol{N} \to S$ and two recursive maps $\pi' : S \to S$ and $\pi'' : S \to \boldsymbol{N}$ in such a way that $\pi'(\pi(x,i)) = x$ and $\pi''(\pi(x,i)) = i$ for all $x \in S$.

Theorem 6.6. *Let k_0 be the fuzzy set defined by setting, for every $x \in S$,*

$$k_0(x) = s_{\pi''(x)}(\pi'(x)). \tag{6.1}$$

Then k_0 is a 1-1-*complete fuzzy subset of S.*

Proof. By definition, we have

$$k_0(x) = lim_{n\to\infty}\psi_{h(\pi''(x))}(\pi'(x),n)$$

and the function $\psi(x,n) = \psi_{h(\pi''(x))}(\pi'(x),n)$ is recursive and increasing with respect to n. It follows that $k_0 \in \Sigma_1$. Moreover, let $s_i \in \Sigma_1$ and set $d(x) = \pi(x,i)$. Then, since $d : S \to S$ is recursive and injective and, for all $x \in S$,

$$k_0(d(x)) = lim_{n\to\infty}\psi_{h(i)}(x,n) = s_i(x),$$

we have that $s_i \leq_1 k_0$. □

Theorem 6.7. *Let k be the fuzzy subset of S defined by setting, for every $x \in S$*

$$k(x) = s_{c(x)}(x), \tag{6.2}$$

then k is 1-1-*complete.*

Proof. It is evident that $k \in \Sigma_1$. In order to prove that k is 1-1-complete we prove that $k_0 \leq_1 k$. To simplify the proof we assume that $S = \boldsymbol{N}$ and therefore that $c(x) = x$. Let $g : \boldsymbol{N}^3 \to \boldsymbol{N}$ be the recursive function defined by setting

$$g(x,y,n) = \psi_{h(\pi''(x))}(\pi'(x),n).$$

Note that such a function does not depend on the variable y. Also, by the *s-m-n-*theorem there is a recursive one-one map f such that

$$\psi_{f(x)}(y,n) = g(x,y,n). \tag{6.3}$$

Since $\psi_{f(x)}(y,n)$ is increasing with respect to n,

$$s_{f(x)}(y) = lim_{n\to\infty}\psi_{h(f(x))}(y,n) = lim_{n\to\infty}\psi_{f(x)}(y,n).$$

Since $s_{f(x)}$ is a constant map,

$$k_0(x) = s_{\pi''(x)}(\pi'(x)) = lim_{n\to\infty}\ \psi_{h(\pi''(x))}(\pi'(x),n) = lim_{n\to\infty} g(x,y,n)$$
$$= lim_{n\to\infty}\psi_{f(x)}(y,n) = s_{f(x)}(y) = s_{f(x)}(f(x)) = k(f(x)).$$

This proves that $k_0 \leq_1 k$ via f. □

Both Theorems 6.6 and 6.7 suggest that a universal machine and a universal programming language for the class of recursively enumerable fuzzy subsets can exist. It is an interesting open question to show concrete examples of such a machine or such a language.

7. EFFECTIVE ABSTRACT FUZZY LOGIC

It is natural to require that the deduction operator of a fuzzy logic satisfies some kind of "*computability*" property besides continuity. To this purpose, we start from the definition of enumeration operator given in Rogers [1967] for crisp operators. We expose such a definition in terms of sequents. First, notice that, since $\mathbb{F}$ is codified, both the class of finite subsets $\mathcal{P}_f(\mathbb{F})$ of $\mathbb{F}$ and the class $SEQ_f = \mathcal{P}_f(\mathbb{F})\times\mathbb{F}$ of finite sequents can be codified. Consequently, the notion of recursively enumerable subset of SEQ_f is defined.

Definition 7.1. A consequence relation $\vdash$ is *recursively enumerable* if $\vdash$ is the compact extension of a recursively enumerable consequence relation $W \subseteq SEQ_f$. An operator $H : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F})$ is an *enumeration operator* if H is the operator associated with a recursively enumerable consequence relation.

In other words, $H : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F})$ is an enumeration operator if a recursively enumerable subset W of SEQ_f exists such that, for any $X \in \mathcal{P}(\mathbb{F})$,

$$H(X) = \{x \in \mathbb{F} : X_f \text{ exists such that } (X_f,x) \in W \text{ and } X_f \subseteq X\}. \tag{7.1}$$

It is easy to verify that an enumeration operator H is compact and that

$$X \text{ recursively enumerable } \Rightarrow H(X) \text{ recursively enumerable.}$$

Proposition 7.2. *An operator $H : \mathcal{P}(\mathbb{F}) \to \mathcal{P}(\mathbb{F})$ is an enumeration operator iff H is compact and*

$$W_H = \{(X_f,x) \in SEQ_f : x \in H(X_f)\}$$

is a recursively enumerable relation.

Proof. Assume that H is an enumeration operator and let W be as in Definition 7.1. Then, for any $X_f \in \mathcal{P}_f(\mathbb{F})$,

$$H(X_f) = \{x \in \mathbb{F} : X'_f \text{ exists such that } (X'_f,x) \in W \text{ and } X'_f \subseteq X_f\}.$$

Let g be a computable function whose codomain is W. Then, since

$$(X_f,x) \in W_H \Leftrightarrow n\in \boldsymbol{N} \text{ exists such that } g(n) = (X'_f,x) \in W \text{ and } X'_f \subseteq X_f,$$

W_H is recursively enumerable. The converse implication is self-evident. □

To extend Definition 7.1 to the fuzzy operators, we set $\underline{SEQ_f} = \mathcal{F}_f(\mathbb{F})\times\mathbb{F}$ and we call *finite fuzzy sequents* the elements of $\underline{SEQ_f}$. Moreover, given any fuzzy set $w : \underline{SEQ_f} \to U$ of sequents and $(X,\alpha) \in \underline{SEQ_f}$, we write $w(X \vdash \alpha)$ to denote the value $w(X,\alpha)$. Since $\underline{SEQ_f}$ can be codified, the notion of recursively enumerable fuzzy subset of $\underline{SEQ_f}$ is well defined.

Definition 7.3. We say that a fuzzy operator $H : \mathcal{F}(\mathbb{F}) \to \mathcal{F}(\mathbb{F})$ is an *enumeration fuzzy operator* if a recursively enumerable fuzzy subset $w : \underline{SEQ_f} \to U$ exists such that

$$H(s)(x) = Sup\{w(s_f \vdash x) : s_f \ll s\}. \tag{7.2}$$

Obviously, H is an enumeration fuzzy operator if a recursive map $k : \underline{SEQ_f} \times \boldsymbol{N} \to \ddot{U}$ exists such that, for any $x \in \mathbb{F}$ and $s \in \mathcal{F}(\mathbb{F})$,

$$H(s)(x) = Sup\{k(s_f, x, m) : m \in \boldsymbol{N} \text{ and } s_f \ll s\}.$$

Theorem 7.4. *Any enumeration fuzzy operator $H : \mathcal{F}(\mathbb{F}) \to \mathcal{F}(\mathbb{F})$ is continuous. Moreover,*

s recursively enumerable $\Rightarrow$ $H(s)$ recursively enumerable.

Proof. Let $(s_i)_{i\in I}$ be a directed family of fuzzy subsets. In order to prove that $H(\cup_{i\in I} s_i) = \cup_{i\in I} H(s_i)$, recal that, if s_f is an element in $\mathcal{F}_f(S)$ such that $s_f \ll \cup_{i\in I} s_i$ then $i \in I$ exists such that $s_f \subseteq s_i$. Then,

$$H(\cup_{i\in I} s_i)(x) = Sup\{w(s_f \vdash x) : s_f \ll \cup_{i\in I} s_i\} = Sup\{w(s_f \vdash x) : s_f \ll s_i,\ i \in I\}$$
$$= Sup_{i\in I} H(s_i)(x).$$

Assume that s is a recursively enumerable fuzzy subset and therefore that $s(x) = lim_{n\to\infty} h(x,n)$, where $h : \mathbb{F} \times \boldsymbol{N} \to \ddot{U}$ is a recursive map increasing with respect to the second variable. Moreover, let $k : \underline{SEQ_f} \times \boldsymbol{N} \to \ddot{U}$ a recursive map such that

$$w(s_f \vdash x) = Sup_{m\in \boldsymbol{N}} k(s_f,x,m)$$

for any $x \in \mathbb{F}$ and $s_f \in \mathcal{F}_f(\mathbb{F})$. For any $n \in \boldsymbol{N}$, let h_n be the fuzzy subset of $\mathbb{F}$ defined by setting $h_n(x) = h(x,n)$ for any $x \in \mathbb{F}$. Then, since H is continuous and $(h_n)_{n\in \boldsymbol{N}}$ is a directed family, we have

$$H(s)(x) = H(\cup_{n\in \boldsymbol{N}} h_n)(x) = Sup_{n\in \boldsymbol{N}} H(h_n)(x).$$

By observing that $H(h_n)(x) = Sup\{w(s_f \vdash x) : s_f \ll h_n\}$,we get

$$H(s)(x) = Sup\{k(s_f,x,m) : s_f \ll h_n\ \ n, m \in \boldsymbol{N}\}.$$

Define r by setting $r(x,i,n,m) = k(s_f,x,m)$ if i is the code number of s_f and $s_f \ll h_n$ and otherwise, $r(x,i,n,m) = 0$. Then r is a recursive map and

$$H(s)(x) = Sup_{i\in \boldsymbol{N}} Sup_{n\in \boldsymbol{N}} Sup_{m\in \boldsymbol{N}} r(x,i,n,m).$$

This demonstrates that $H(s)$ is recursively enumerable. □

Theorem 7.5. *Let $H : \mathcal{F}(\mathbb{F}) \to \mathcal{F}(\mathbb{F})$ be a fuzzy operator and define the fuzzy relation $w_H : \underline{SEQ_f} \to U$ by setting, for any $x \in \mathbb{F}$ and $s_f \in \mathcal{F}_f(\mathbb{F})$,*

$$w_H(s_f \vdash x) = H(s_f)(x). \tag{7.3}$$

Then H is an enumeration operator iff H is continuous and w_H is a recursively enumerable fuzzy relation.

Proof. Let H be an enumeration operator and let $k : \underline{SEQ_f} \times \boldsymbol{N} \to \ddot{U}$ be a recursive map such that

$$H(s)(x) = Sup\{k(s_f, x, m) : m \in \boldsymbol{N} \text{ and } s_f \ll s\}$$

for any $x \in \mathbb{F}$ and $s \in \mathcal{F}(\mathbb{F})$. Then

$$w_H(s_f \vdash x) = H(s_f)(x) = Sup\{k(s'_f,x, m) : s'_f \ll s_f,\ m \in \boldsymbol{N}\}.$$

Define the function h by setting $h(s'_f,x,s_f,m) = k(s'_f,x,m)$ if $s'_f \ll s_f$ and otherwise, $h(s'_f,x,s_f,m) = 0$. Then

$$w_H(s_f \vdash x) = Sup\{h(s'_f,x,s_f,m) : s'_f \in \mathcal{F}_f(\mathbb{F}) \text{ and } m \in \boldsymbol{N}\},$$

and this proves that w_H is recursively enumerable.

Conversely, if w_H is recursively enumerable and H is continuous, then

$$H(s)(x) = Sup\{H(s_f)(x) : s_f \ll s\} = Sup\{w(s_f \vdash x) : s_f \ll s\}.$$

This proves that H is an enumeration operator. □

We reach the main definition in this section.

Definition 7.6. We call *effective abstract fuzzy logic* an abstract fuzzy logic $(F, \mathcal{D})$ such that $\mathcal{D}$ is an enumeration fuzzy operator.

8. FUZZY LOGIC = ENUMERATION FUZZY CLOSURE OPERATOR

It is natural to require some kind of "*effectiveness*" to a fuzzy deduction apparatus and therefore to assume that the fuzzy subset of logical axioms is recursively enumerable and that the fuzzy inference rules are "computable". Also, as showed in Biacino and Gerla [2000]a, in the case of infinite inference rules we have to require that the algorithms for these rules are given in a *uniform* way. So, we propose the following definition. We assume that:

- r'_1, r'_2, ... is an effective coding of all the partial recursive functions from a Cartesian product $\mathbb{F}^n$ of $\mathbb{F}$ to $\mathbb{F}$ (where n varies in $\boldsymbol{N}$),
- d_1, d_2, ... is an effective coding of all the partial recursive functions from a Cartesian product $\ddot{U}^n$ of $\ddot{U}$ to $\ddot{U}$ (where n varies in $\boldsymbol{N}$).

Moreover, we set

$$r''_i(x_1,...,x_n) = Sup\{d_i(\lambda_1,...,\lambda_n) : \lambda_1<x_1,..., \lambda_n<x_n, \text{ and } (\lambda_1,...,\lambda_n)\in Dom(d_i)\}. \quad (8.1)$$

Then each r''_i is a total function satisfying the continuity condition. All the usual triangular norms can be obtained in such a way, i.e., extending by (8.1) a recursive operation defined on the rational numbers.

Definition 8.1. A fuzzy H-system $(a, \mathbb{R})$ is *effective* provided that :

(a) two recursive maps $h : \boldsymbol{N} \to \boldsymbol{N}$ and $k : \boldsymbol{N} \to \boldsymbol{N}$ exist such that $d_{k(i)} : \ddot{U}^n \to \ddot{U}$ is a total function satisfying the continuity condition and

$$\mathbb{R} = \{(r'_{h(i)},r''_{k(i)}) : i \in \boldsymbol{N}\},$$

(b) the fuzzy set a of logical axioms is recursively enumerable.

Observe that each $r''_{k(i)}$ is an extension of $d_{k(i)}$ and therefore $r''_i(\lambda_1,...,\lambda_n)$ is a rational number whenever $\lambda_1,...,\lambda_n$ are rational numbers.

The constructive point of view imposes a more precise definition of a proof. Indeed, recall that the justification of a formula α_j in a proof $\alpha_1,...,\alpha_n$ is one of the following claims:

i) *α_j is a logical axiom*

ii) *α_j is a hypothesis*

iii) *α_j is obtained by the n-ary rule $r_{h(i)}$ applied to the formulas $\alpha_{s(1)},...\alpha_{s(n)}$.*

If we represent the cases i), ii) and iii) by *la*, *hy* and $(i, s(1),...,s(n))$, respectively, then a justification is an element in the set

$$\{la, hy\} \cup \{(i, s(1),...,s(n)) : r_{h(i)} \text{ is } n\text{-ary}\},$$

and therefore we can assign a code number to any justification. In accordance, a *proof* is a sequence $<\alpha_1,i_1>,...,<\alpha_n,i_n>$ of elements in $\mathbb{F} \times \boldsymbol{N}$ such that, for $j = 1,...,n$, if i_j is the code number of a justification like $(i,s(1),...,s(n))$, then $s(1)<j,..., s(n)<j$ and $\alpha_j = r'_{h(i)}(\alpha_{s(1)},...,\alpha_{s(n)})$. It is evident that a partial recursive function exists such that, given any sequence $<\alpha_1,i_1>,...,<\alpha_n,i_n>$, the function converges if such a sequence is a proof, it diverges otherwise. Then the set of proofs is recursively enumerable. If the domain of each rule $r'_{h(i)}$ is recursive and a is decidable, then the set of proofs is decidable.

The following theorem shows that the theory of effective abstract deduction systems coincides with the theory of effective fuzzy Hilbert systems, i.e.,

effective approximate reasoning = theory of enumeration fuzzy closure operators.

Theorem 8.2. *The deduction operator $\mathcal{D}$ of an effective fuzzy Hilbert system $\mathcal{S}$ is an enumeration fuzzy closure operator. Conversely, let $\mathcal{D}$ be an enumeration fuzzy closure operator. Then an effective fuzzy Hilbert system exists whose deduction operator coincides with $\mathcal{D}$.*

Proof. Let $\mathcal{D}$ be the deduction operator of an effective Hilbert system $(a, \mathbb{R})$ and define $w : \mathbb{F} \times \mathcal{F}_f(\mathbb{F}) \to U$ by setting $w(s_f \vdash x) = \mathcal{D}(s_f)(x)$ for any $x \in \mathbb{F}$ and $s_f \in \mathcal{F}_f(\mathbb{F})$. Then, because of the continuity of $\mathcal{D}$, we have only to prove that w is recursively enumerable. Indeed, let $a' : \mathbb{F} \times \boldsymbol{N} \to \ddot{U}$ be a computable function such that, for every formula x,

$$a(x) = Sup\{a'(x, n) : n \in \boldsymbol{N}\}, \tag{8.2}$$

and, for every $n \in \boldsymbol{N}$, define the fuzzy subset a_n by setting $a_n(x) = a'(x,n)$. Moreover, given a proof π, let $Val(\pi, s_f, n)$ be the valuation of π in the fuzzy H-system $(a_n, \mathbb{R})$ obtained by assuming a_n instead of a as a fuzzy set of logical axioms. Since the values of s_f and a_n are rational numbers, $Val(\pi, s_f, n)$ is a rational number. We will prove that

$$Val(\pi,s_f) = Sup\{Val(\pi, s_f, n) : n \in \boldsymbol{N}\} \tag{8.3}$$

by induction on the length $l(\pi)$ of $\pi = \alpha_1, \alpha_2, ...,\alpha_m$. In fact, in the case $l(\pi) = 1$, and, more generally, in the case that the last formula α in π is assumed either as a hypothesis or as a logical axiom, (8.3) is evident. Suppose that α is obtained by an inference rule r, namely that $\alpha = r'(\alpha_{s(1)},...,\alpha_{s(p)})$; then

$$\begin{aligned}Val(\pi,s_f) &= r''(Val(\pi_{s(1)},s_f),...,Val(\pi_{s(p)},s_f))\\ &= r''(Sup\{Val(\pi_{s(1)}, s_f, n) : n \in \boldsymbol{N}\},...,Sup\{Val(\pi_{s(p)}, s_f, n) : n \in \boldsymbol{N}\})\\ &= Sup\{r''(Val(\pi_{s(1)}, s_f, n_1),...,Val(\pi_{s(p)}, s_f, n_p) : n_1,...,n_p \in \boldsymbol{N}\}\\ &= Sup\{r''(Val(\pi_{s(1)}, s_f, n),...,Val(\pi_{s(p)}, s_f, n) : n \in \boldsymbol{N}\}\end{aligned}$$

$= Sup\{Val(\pi, s_f, n) : n \in \boldsymbol{N}\}$,
where we used the inductive hypothesis, the fact that r'' preserves the joins and the fact that the quantities $Val(\pi_{s(j)}, s_f, n)$ are increasing with respect to n. From (8.3) it follows that

$$\mathcal{D}(s_f)(\alpha) = Sup\{Val(\pi,s_f) : \pi \text{ is a proof of } \alpha\}$$
$$= Sup\{Val(\pi, s_f, n) : \pi \text{ is a proof of } \alpha \text{ and } n \in \boldsymbol{N}\}.$$

Let $\pi_1, \pi_2, \ldots$ be an effective enumeration of all the proofs and define the function $h : \mathcal{F}_f(\mathbb{F}) \times \mathbb{F} \times \boldsymbol{N} \times \boldsymbol{N} \to \ddot{U}$ by setting $h(s_f, \alpha, i, n) = Val(\pi_i, s_f, n)$ if π_i is a proof of α and otherwise, $h(s_f,\alpha,i,n) = 0$. Then h is a computable map and

$$\mathcal{D}(s_f)(\alpha) = Sup\{h(s_f,\alpha,i,n) : i \in \boldsymbol{N} \text{ and } n \in \boldsymbol{N}\}.$$

This proves that w is recursively enumerable and therefore that $\mathcal{D}$ is an enumeration fuzzy operator.

Let $\mathcal{D}$ be an enumeration fuzzy closure operator and let h: $\mathcal{F}_f(\mathbb{F}) \times \mathbb{F} \times \boldsymbol{N} \to \ddot{U}$ be a recursive map such that, for any $s_f \in \mathcal{F}_f(\mathbb{F})$ and $x \in \mathbb{F}$,

$$\mathcal{D}(s_f)(x) = Sup\{h(s_f,x,n) : n \in \boldsymbol{N}\}.$$

To define a suitable Hilbert system, we associate with any $\alpha \in \mathbb{F}$, $m \in N$ and $s_f \in \mathcal{F}_f(\mathbb{F})$, $s_f \neq \emptyset$, the fuzzy rule (r',r'') defined as follows. Let $\alpha_1,\ldots,\alpha_n$ be the formulas in $Supp(s_f)$, then we set

$$r'(x_1,\ldots,x_n) = \begin{cases} \alpha & \text{if } x_1 = \alpha_1,\ldots, x_n = \alpha_n, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Also, define $d : \ddot{U}^n \to \ddot{U}$ by setting

$$d(\lambda_1,\ldots,\lambda_n) = \begin{cases} h(s_f,\alpha,m) & \text{if } \lambda_1 > s_f(\alpha_1), \ldots, \lambda_n > s_f(\alpha_n), \\ 0 & \text{otherwise.} \end{cases}$$

Such a map satisfies the continuity condition and therefore the map r'' defined by (8.1) is an extension of d. Also, both r' and d are partial recursive functions whose algorithms depend uniformly on α, s_f, m. Consequently, two recursive functions $h : \boldsymbol{N} \to \boldsymbol{N}$ and $k : \boldsymbol{N} \to \boldsymbol{N}$ exist such that $r'_{h(i)} = r'$ and $d_{k(i)} = d$ where i is the code number of (α, s_f, m). We indicate by $\mathcal{S}$ the effective Hilbert fuzzy system whose fuzzy set of logical axioms is $\mathcal{D}(\emptyset)$ and such that $\mathbb{R} = \{(r'_{h(i)},r''_{k(i)}) : i \in \boldsymbol{N}\}$. To prove that $\mathcal{D}$ is the deduction operator of $\mathcal{S}$, we prove that a fuzzy set of formulas τ is a theory of $\mathcal{S}$ iff τ is a fixed point of $\mathcal{D}$, i.e. $\tau \supseteq \mathcal{D}(s_f)$ for any $s_f \in \mathcal{F}_f(\mathbb{F})$ such that $s_f \ll \tau$. Indeed, let τ be a theory. Then in the case $s_f = \emptyset$ we have that $\tau \supseteq \mathcal{D}(s_f)$ by hypothesis. If $s_f \neq \emptyset$, let $\alpha_1,\ldots,\alpha_n$ be the elements in $Supp(s_f)$, $m \in \boldsymbol{N}$, $\alpha \in \mathbb{F}$ and let (r',r'') be the inference rule associated with α, s_f and m. Then,

$$\tau(\alpha) = \tau(r'(\alpha_1,\ldots,\alpha_n)) \geq r''(\tau(\alpha_1),\ldots,\tau(\alpha_n))$$

$= Sup\{d(\lambda_1,...,\lambda_n) : \lambda_1<\tau(\alpha_1),...,\lambda_n<\tau(\alpha_n)\}$.

Since $\lambda_1, ...,\lambda_2$ exist such that $\tau(\alpha_1) > \lambda_1 > s_f(\alpha_1), ..., \tau(\alpha_n) > \lambda_n > s_f(\alpha_n)$, we have

$\tau(\alpha) \geq d(\lambda_1,...,\lambda_n) = h(s_f,\alpha,m)$.

Consequently, $\tau(\alpha) \geq Sup\{h(s_f,\alpha,n) : n\in \boldsymbol{N}\} = \mathcal{D}(s_f)(\alpha)$ and $\tau \supseteq \mathcal{D}(s_f)$.

Let τ be a fixed point of $\mathcal{D}$, then $\tau = \mathcal{D}(\tau) \supseteq \mathcal{D}(\emptyset)$. Moreover, let (r',r'') be any rule and assume that (r',r'') is defined by $s_f \neq \emptyset$, $m\in N$ and $\alpha \in \mathbb{F}$. We claim that

$$\tau(r'(\alpha_1,...,\alpha_n)) \geq d(\lambda_1,...,\lambda_n)$$

for any $\lambda_1< \tau(\alpha_1),..., \lambda_n< \tau(\alpha_n)$ and therefore that

$$\tau(r'(\alpha_1,..., \alpha_n)) \geq r''(\tau(\alpha_1),..., \tau(\alpha_n)),$$

where $\alpha_1,..., \alpha_n$ are the elements in $Supp(s_f)$. Indeed, if $d(\lambda_1,...,\lambda_n) \neq 0$ then $\lambda_1 > s_f(\alpha_1),...,\lambda_n > s_f(\alpha_n)$. Consequently,

$$\tau(r'(\alpha_1,..., \alpha_n)) = \tau(\alpha) = \mathcal{D}(\tau)(\alpha) \geq \mathcal{D}(s_f)(\alpha) \geq h(s_f,\alpha,m) = d(\lambda_1,...,\lambda_n).$$

This means that τ is a theory of $\mathcal{S}$. □

We call *axiomatizable* a fuzzy theory admitting a decidable fuzzy subset of axioms. The next theorem extends a basic feature of classical logic to fuzzy logic.

Theorem 8.3. *Consider an effective fuzzy Hilbert logic. Then any axiomatizable theory is recursively enumerable. If the logic is with negation, and v is a decidable fuzzy set of axioms, then $\mathcal{D}(v)$ is recursively enumerable and $\mathcal{D}(v)^{\perp}$ is recursively co-enumerable. Moreover, any axiomatizable and complete theory is decidable.*

Proof. Theorem 8.2 proves that any axiomatizable theory is recursively enumerable. Assume that the fuzzy logic under consideration is with negation and that v is decidable. Then, a recursive map $h : \mathbb{F} \times \boldsymbol{N} \to \ddot{U}$ exists such that h is increasing with respect to the second variable and $\mathcal{D}(v)(\alpha) = lim_{n\to\infty} h(\alpha,n)$ for any formula α. As a consequence,

$$\mathcal{D}(v)^{\perp}(\alpha) = 1-lim_{n\to\infty} h(\neg\alpha,n) = lim_{n\to\infty} -h(\neg\alpha, n) = lim_{n\to\infty} k(\alpha,n)$$

where we have set $k(\alpha,n) = 1-h(\neg\alpha,n)$. The map k is computable and increasing with respect to the second variable. Then, $\mathcal{D}(v)^{\perp}$ is recursively enumerable.

Assume that v is complete, then, since $\mathcal{D}(v) = \mathcal{D}(v)^{\perp}$ the fuzzy set $\mathcal{D}(v)$ is both recursively enumerable and recursively co-enumerable. □

It is easy to prove that in the logics with optimal proofs the axiomatizable theories are d-recursively enumerable. The following theorem is a consequence of Theorem 6.3.4 in Hájek [1998] (see also Scarpellini [1962]):

Theorem 8.4. An effective Hilbert system exists with an axiomatizable theory τ whose cut $C(\tau,1)$ is Π_2-complete and therefore not recursively enumerable.

Obviously, such a theorem does not contradict Theorem 8.3. It means only that, given any formula α, while we are able to produce an increasing sequence of rational numbers converging to $\tau(\alpha)$, we are not able to decide if $\tau(\alpha)$ is equal to 1 or not. This phenomenon is not a characteristic of fuzzy logic. It arises whenever a constructive approach for a theory involving real numbers is proposed. Indeed, it is not decidable if two recursive real numbers are equal or not. Then it is not surprising that we know an algorithm to compute the real number r and that, at the same time, we are not able to decide if r is equal to 1 or not.

Remark. The set of signed formulas of an axiomatizable theory τ is not necessarily recursively enumerable. Indeed, Proposition 4.3 says that this set is recursively enumerable iff τ is d-recursively enumerable.
<u>*This fact is an argument against the reduction of fuzzy logic to a (crisp) calculus of signed formulas.*</u>
Indeed, any logic is an effective process to generate information from a piece of available information. So, a reduction of a logic $\mathcal{L}_1$ to a logic $\mathcal{L}_2$ needs to take into account the effectiveness of the corresponding deduction apparatus. On the other hand, the notion of effectiveness in fuzzy logic is related to the continuous structure [0,1] and, therefore, to the idea of an (effective) infinite approximation process. Instead, in any crisp logic the notion of effectiveness is related to the discrete structure {0,1} and, therefore, to the idea of a finite-steps and terminating computation. Then, in our opinion, no reduction of fuzzy logic to crisp logic is possible.

We conclude this section by giving an example of a continuous truth-functional modal semantics which is axiomatizable by a Hilbert system but not axiomatizable by an effective Hilbert system (see Biacino and Gerla [2000]a).

Theorem 8.5. *A continuous truth-functional fuzzy semantics exists which is not effectively axiomatizable.*

Proof. Consider any continuous truth-functional fuzzy semantics $\mathcal{M}$ such that $\neg$ is interpreted by an injective computable map and add a new unary connective $\Diamond$ to the language of $\mathcal{M}$. We interpret $\Diamond$ by a unary function $f: U \to U$ by obtaining a new semantics $\mathcal{M}_\Diamond$ for a fuzzy modal logic in which, as usual, $\Diamond(\alpha)$ means *"α is possible"*. We define f as follows: Let W be a subset of $\boldsymbol{N}$ and assume that W is not recursively enumerable. Then f is the continuous function such that, for any $i \in \boldsymbol{N}$

$f(x) = (2i+1)/(2i^2+2i)$ if $x = 1/(i+1)$ and $i \in W$,

$f(x) = x$ if either $x = 0$ or $x = 1$ or $x = 1/(i+1)$ and $i \notin W$,

f is linear in each interval $[1/(i+1), 1/i]$.

Since $(2i+1)/(2i^2+2i)$ is the average between $1/(i+1)$ and $1/i$, we have that

$$1/(i+1) < (2i+1)/(2i^2+2i) < 1/i.$$

Then, it is easy to verify that $f(1/(i+1)) < f(1/i)$ for any $i \in \boldsymbol{N}$ and therefore, that f is an injective, order-preserving mapping. The function f is not computable. Since f is order-preserving, $f(0) = 0$, $f(1) = 1$ and $f(\lambda) \geq \lambda$ for every $\lambda \in U$, f represents a plausible interpretation of $\Diamond$ in a modal logic. Consider now the initial valuation $v : \mathbb{F} \to U$ defined by setting:

$$v(x) = \begin{cases} 1/(i+1) & \text{if } x = p_i\,, \\ \sim(1/(i+1)) & \text{if } x = \neg\, p_i\,, \\ 0 & \text{otherwise.} \end{cases}$$

Such a valuation has only a model m_v, namely the truth-functional model such that $m_v(p_i) = 1/(i+1)$. Consequently, for any formula α,

$$Lc(v)(\alpha) = m_v(\alpha).$$

Since,

$$m_v(\Diamond(p_i)) = f(m_v(p_i)) = f(1/(i+1)),$$

we have

$$Lc(v)(\Diamond(p_i)) = \begin{cases} 1/(i+1) & \text{if } i \notin W, \\ \\ (2i+1)/(2i^2+2i) & \text{if } i \in W. \end{cases}$$

Assume that $\mathcal{M}_\Diamond$ is effectively axiomatizable, i.e. that there is a suitable *H-system* whose deduction operator $\mathcal{D}$ coincides with the logical consequence operator Lc of $\mathcal{M}_\Diamond$. In such a case the decidability of v implies that the fuzzy subset $\mathcal{D}(v) = Lc(v)$ is recursively enumerable. Then, a recursive function $h : \mathbb{F} \times \boldsymbol{N} \to \ddot{U}$ exists such that h is increasing with respect to the second variable and

$$Lc(v)(\alpha) = \mathcal{D}(v)(\alpha) = lim_{n\to\infty}\, h(\alpha,n)$$

for any formula α. In particular,

$$lim_{n\to\infty}\, h(\Diamond(p_i),n) = 1/(i+1) \qquad \text{if } i \notin W,$$
$$lim_{n\to\infty}\, h(\Diamond(p_i),n) = (2i+1)/(2i^2+2i) \qquad \text{if } i \in W.$$

This means that W is the set $\{i \in \boldsymbol{N} : \exists\, n\ h(\Diamond(p_i),n) > 1/(i+1)\}$ and therefore, that W is the projection of the decidable relation

$$R(i,n) \equiv\ h(\Diamond(p_i),n) > 1/(i+1).$$

This contradicts the hypothesis that W is not recursively enumerable. □

9. GÖDEL-LIKE THEOREMS

In Rogers [1976] a subset X of S is called *productive* if a partial recursive function $e : S \to S$ exists such that

$$W_i \subseteq X \;\Rightarrow\; e(i) \in X\text{-}W_i.$$

In other words, X is productive if we can prove in an effective way that X is different from any recursively enumerable subset of S. A subset X is *creative* if X is recursively enumerable and its complement is productive. The notions of productive subset and creative subset are very important in first order logic. Indeed, we can summarize two basic limitative theorems as follows.

(i) *The set V of true formulas of elementary arithmetic is a productive set. Consequently, V cannot be axiomatized.*
(ii) *The set T of theorems of Peano arithmetic is creative. Consequently, T is undecidable and incomplete.*

A first step to extend such results to fuzzy logic is to give a suitable definition of productive and creative fuzzy subset.

Definition 9.1. We define *productive* a fuzzy set s for which a partial recursive function $e : S \to S$ exists such that

$$s_i \subseteq s \;\Rightarrow\; s(e(i)) \neq s_i(e(i)).$$

We say that s is *creative* if s is recursively enumerable and $-s$ productive.

In other words, s is productive if we can prove in an effective and uniform way that, given any recursively enumerable fuzzy set s_i, s is different from s_i.

Lemma 9.2. *Let $f : U \to U$ be any computable upper-semicontinuous function such that*
- *$f(x) \neq 1- x$ for every $x \in U$,*
- *$f(x)$ is rational for x rational,*
- *$f : \ddot{U} \to \ddot{U}$ is computable.*

Let k be the 1-1-complete fuzzy subset given in Theorem 7.7. Then the fuzzy subset $s = f \circ k$ is creative.

Proof. k is recursively enumerable and therefore there exists a computable map $h : S \times \boldsymbol{N} \to \ddot{U}$ which is increasing with respect to n and such that $k(x) = Sup\{h(x,n) : n \in \boldsymbol{N}\}$ for every $x \in S$. Then, since

$$s(x) = f(k(x)) = f(Sup\{h(x,n) : n \in \boldsymbol{N}\}) = Sup\{f(h(x,n)) : n \in \boldsymbol{N}\},$$

s is a recursively enumerable fuzzy subset. In order to prove that $-s$ is productive, assume that $s_i \subseteq -s$ and let $c : S \to \boldsymbol{N}$ a coding and $t : \boldsymbol{N} \to S$ its inverse. Then,

$$s_i(t(i)) \leq -s(t(i)) = 1- f(s_i(t(i))).$$

Since by hypothesis $s_i(t(i)) \neq 1-f(s_i(t(i)))$, we have that $s_i(t(i)) \neq -s(t(i))$. This demonstrates that s is creative. □

Lemma 9.2 entails the existence of creative fuzzy sets.

Proposition 9.3. *If s, p and c are fuzzy subsets, then*:
(i) *p productive and $p \leq_1 s \quad\Rightarrow\quad s$ productive,*
(ii) *c creative, s recursively enumerable and $c \leq_1 s \quad\Rightarrow\quad s$ is creative.*

Proof. (i) Assume that $p(x) = s(f(x))$ for every $x \in S$, where $f : S \to S$ is total and recursive and let $e : S \to S$ be as in Definition 9.1. Then from $s_i \subseteq s$ it follows that $s_i(f(x)) \leq s(f(x)) = p(x)$. By recalling that $s_i(f(x)) = lim_{n\to\infty}\psi_{h(i)}(f(x),n)$, we define the recursive function $\phi(i,x,n)$ by setting $\phi(i,x,n) = \psi_{h(i)}(f(x),n)$ and by the *s-m-n*-theorem we set $\psi_{g(i)}(x,n) = \phi(i,x,n)$ where $g : \boldsymbol{N} \to \boldsymbol{N}$ is a suitable recursive function. Since $\psi_{g(i)}(x,n)$ is recursive and increasing with respect to n,

$$s_{g(i)}(x) = lim_{n\to\infty}\psi_{h(g(i))}(x,n) = s_i(f(x))$$

and this means that $s_i \circ f = s_{g(i)}$. Then $s_i \subseteq s$ implies $s_{g(i)} \subseteq p$, i.e., $e(g(i))$ is convergent and $p(e(g(i))) > s_{g(i)}(e(g(i))$. Therefore $f(e(g(i)))$ is convergent and $s(f(e(g(i)))) > s_i(f(e(g(i))))$. Thus, s is productive via the function $f \circ e \circ g$.

(ii). Observe that from $c \leq_1 s$ it follows that $-c \leq_1 -s$. Since $-c$ is productive, by (i), $-s$ is productive. Thus, *s is creative.* □

In recursion theory one proves that a subset X is creative iff it is 1-1-complete. The next proposition shows that we cannot extend such a result to fuzzy subsets.

Proposition 9.4. *Every* 1-1-*complete fuzzy set is creative while a creative fuzzy subset exists which is not* 1-1-*complete.*

Proof. Let k be a 1-1-complete fuzzy subset and c a creative fuzzy subset. Then, since $c \leq_1 k$, by Proposition 9.3 k is creative. Let f be defined by setting $f(x) = 1$ if $x \neq 0$ and $f(x) = 0$ if $x = 0$ and let $c = f \circ k$ be the creative fuzzy subset defined in Lemma 9.2. Then, c coincides with the characteristic function of the open cut O$(k,0)$. Since c is crisp, a non-crisp fuzzy subset cannot be one-one reducible to c. This shows that c cannot be 1-1-complete (in the whole class of fuzzy sets). □

Consider a fuzzy logic such that $\mathbb{F}$ is the set of formulas of a first order language with *"numerals"*, i.e., with a name $\underline{n}$ for every integer n. Given a model m, we say that a fuzzy subset s of $\boldsymbol{N}$ *is represented in* m by the formula α if, for every $n \in \boldsymbol{N}$, $s(n) = m(\alpha(\underline{n}))$. Likewise, given a fuzzy theory τ, we say that a fuzzy subset s of $\boldsymbol{N}$ *is represented in* τ *by* α if, for every $n \in \boldsymbol{N}$, $s(n) = \tau(\alpha(\underline{n}))$.

Theorem 9.5. *Consider a fuzzy Hilbert logic with a model m able to represent a productive fuzzy subset. Then m is productive and no axiomatization for it exists. Namely, given any recursively enumerable fuzzy system of axioms v for m, a formula* α *exists such that* $\mathcal{D}(v)(\alpha) < m(\alpha)$.

Proof. Assume that a productive fuzzy set c is represented in m by the formula α. Then c is one-one reducible to m via the function associating every $n \in \boldsymbol{N}$ with the formula $\alpha(\underline{n})$. By Proposition 9.3 this proves that m is productive. □

Theorem 9.6. *Let* τ *be an axiomatizable fuzzy theory of an effective fuzzy H-system with a negation and assume that* τ *is able to represent a creative fuzzy subset. Then* τ *is creative and, hence, undecidable and incomplete.*

Proof. From Proposition 9.3 it follows that τ is creative and, hence, undecidable. From Proposition 9.3 it follows that τ is incomplete. □

10. SHARPENED AND SHADED VERSIONS: LIMITATIVE THEOREMS

Given two elements λ_1 and λ_2 in U, we set $\lambda_2 \preceq \lambda_1$ provided that

$$\lambda_1 > 1/2 \;\Rightarrow\; \lambda_2 \geq \lambda_1 \quad \text{and} \quad \lambda_1 < 1/2 \;\Rightarrow\; \lambda_2 \leq \lambda_1.$$

In particular, this means that $\lambda \preceq 0.5$ for any $\lambda \in U$. Moreover, if s and s' are two fuzzy subsets of a set S, then we set $s \preceq s'$ if $s(x) \preceq s'(x)$ for every $x \in S$. In such a case we say that s is a *sharpened version* of s' or that s' is a *shaded version* of s (see A. De Luca and S. Termini [1972]). The fuzzy set $s^{0.5}$ constantly equal to 0.5 is the greatest element of U with respect to the sharpness relation, while the crisp subsets are the minimal elements.

Let's examine the following question:

given an undecidable fuzzy subset k of S, is it always possible to modify k so as to get a decidable (*or a recursively enumerable*) *sharpened version of k ?*

Now, if k is crisp, since there is no proper sharpened version of k, the answer is negative in a trivial way. Then, we are only interested in the case in which k is not crisp. In fact, in this case we can try to modify k in order to obtain a decidable sharpened version. The more favorable case is to have infinite x such that $k(x) = 1/2$. We say that a fuzzy subset s of a set S is *infinitely undetermined* if $L(s,1/2) = \{x \in S : s(x) = 1/2\}$ is infinite. Then, we can reformulate the above question as:

is it always possible to modify an infinitely undetermined fuzzy subset k to get a decidable (*or a recursively enumerable*) *sharpened version of k ?*

Proposition 10.1 and Theorem 10.2 give a negative answer:

Proposition 10.1. *Let k be the* 1-1-*complete fuzzy subset defined in Proposition 7.7. Then, k is a recursively enumerable infinitely undetermined fuzzy subset such that no sharpened version of k is decidable.*

Proof. At first, observe that k is infinitely undetermined. Indeed, let $s^{1/2}$ be the fuzzy set constantly equal to 1/2. Then, since $s^{1/2} \leq_1 k$, a recursive one-one map h exists such that $k(h(x)) = 1/2$ for every $x \in S$. Let s be a sharpened version of k and W any recursively enumerable set which is not decidable. Since k is 1-1-complete, a recursive map d exists such that $c_W(x) = k(d(x))$ for every $x \in S$. We now have that $s(y) = k(y)$ everywhere $k(y) \in \{0,1\}$ and this entails that $c_W(x) = s(d(x))$. Then W is one-one reducible to s. Thus, since W is not decidable, we can conclude that s is not decidable. □

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