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Extension principles for fuzzy set theory

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Abstract

In several cases we show that it is possible to extend a notion in classical mathematics by identifying each fuzzy subset with the continuous chain of its closed cuts and by applying this notion to these cuts. In particular this idea is applied to extend functions from subsets into subsets (for instance, closure operators) and functions from sets into real numbers (for instance, measures).

Keywords: Fuzzy sets; Extension principle; Closure operators

1. Introduction

Zadeh's extension principle is one of the most important tools in fuzzy set theory. It enables to extend any map $h: S_1 \times \cdots \times S_n \to S$, where $S_1, \ldots, S_n$ and S are sets, to a map $h^*: \mathscr{F}(S_1) \times \cdots \times \mathscr{F}(S_n) \to \mathscr{F}(S)$ where, for every set $X, \mathscr{F}(X)$ denotes the class of fuzzy subsets of X; namely the formula to define h^* is

$$h^*(s_1, \ldots, s_n)(x) = \sup\{s_1(x_1) \wedge \cdots \wedge s_n(x_n) / h(x_1, \ldots, x_n) = x\} \tag{1}$$

for $s_1 \in \mathscr{F}(S_1), \ldots, s_n \in \mathscr{F}(S_n)$ and $x \in S$. A characterization of this principle in terms of cylindrical extension and projection can be found in [1]. Another principle was proposed by Ramik [2,14] and successively applied by Gerla [3] and by Biacino and Gerla [4]; namely, given an operator $J: \mathscr{P}(S) \to \mathscr{P}(S)$, the *canonical extension* of J is the operator $J^*: \mathscr{F}(S) \to \mathscr{F}(S)$ defined by setting

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$$J^*(s)(x) = \sup\{\lambda \in [0,1]/x \in J(C(s,\lambda))\} \tag{2}$$

for every $s \in \mathscr{F}(S)$ and $x \in S$.

In this paper we show that, in accordance to Eq. (2), in several cases it is possible to extend a notion in classical mathematics by identifying each fuzzy subset with the continuous chain of its closed cuts and by applying this notion to these cuts. This enables to extend Ramik's and Zadeh's principles and also to define notions like the one of the diameter of a fuzzy set, distance between two fuzzy sets and so on.

More precisely, in Section 4 we show how to extend a set operator in several variables and we examine some properties of the resulting extension principle. In Section 5 we use this principle to restate some basic notion of fuzzy set theory. In Section 6 we extend compact operators, and in Section 7 we examine the properties that are preserved under the extension principle. In Section 8 we consider the closure operators. In Section 9 we substitute the minimum operation with a join-continuous triangular norm. In Section 10 we consider an extension principle for relations among sets. Finally, in Section 11, we propose a general way to extend a real-valued map (for instance, a measure, a probability, the distance between sets, the diameter of a set and so on).

2. Basic notions on fuzzy sets

We denote by U the real interval [0,1] and let $\lambda \vee \lambda' = \max\{\lambda, \lambda'\}$ and $\lambda \wedge \lambda' = \min\{\lambda, \lambda'\}$ for any $\lambda, \lambda' \in U$. Under these operations, U is a complete, completely distributive lattice whose minimum is 0 and maximum 1. Also, in U the *negation* operation $\sim : U \to U$ is defined by setting $\sim x = 1 - x$ for every $x \in U$. Given a set S, *a fuzzy subset* of S is any map from S into U and we denote by $\mathscr{F}(S)$, the class of all the fuzzy subsets of S (see [10,11]). The basic notions of set theory are extended to the fuzzy subsets as follows. The *inclusion* "$\subseteq$" is defined by setting, for any $s, s' \in \mathscr{F}(S)$

$$s \subseteq s' \iff s(x) \leqslant s'(x) \quad \text{for every } x \in S.$$

If $s \subseteq s'$, we say that s is *contained* in s' or that s is a *part* of s'. The *union* $s \cup s'$ and the *intersection* $s \cap s'$ of two fuzzy subsets s and s' are defined pointwise by

$$(s \cup s')(x) = s(x) \vee s'(x) \quad \text{and} \quad (s \cap s')(x) = s(x) \wedge s'(x),$$

respectively. More generally, given a family $(s_i)_{i\in I}$ of fuzzy subsets of S, the union $\bigcup_{i\in I} s_i$ and the intersection $\bigcap_{i\in I} s_i$ are defined by

$$\left(\bigcup_{i\in I} s_i\right)(x) = \sup\{s_i(x)/i \in I\} \quad \text{and}$$

$$\left(\bigcap_{i\in I} s_i\right)(x) = \inf\{s_i(x)/i \in I\}.$$

We define the *complement* $\sim s$ of s by $(\sim s)(x) = \sim s(x)$ for every $x \in S$.

Proposition 2.1. *$(\mathscr{F}(S), \cup, \cap, \sim)$ is a complete, completely distributive, lattice with an involution, i.e. the direct power, with index set S, of the structure $(U, \vee, \wedge, \sim)$. This structure extends the Boolean algebra $(\mathscr{P}(S), \cup, \cap, \sim)$, namely, the map associating to any subset X of S the related characteristic function χ_X is an embedding of $(\mathscr{P}(S), \cup, \cap, \sim)$ in $\mathscr{F}(S), \cup, \cap, \sim)$.*

As usually, we call *crisp* a fuzzy subset s such that $s(x) \in \{0, 1\}$ for every $x \in S$. Then Proposition 2.1 says that we can identify the classical subsets of S with the crisp fuzzy subsets of S (i.e. the characteristic functions). In particular, we identify S with the map constantly equal to 1 and $\emptyset$ with the map constantly equal to 0.

We conclude by recalling that further types of complement are possible. For instance, we define the *strong negation* as the function neg: $[0, 1] \to [0, 1]$ defined by setting $\text{neg}(\lambda) = 1$ if $\lambda \neq 1$ and $\text{neg}(\lambda) = 0$ if $\lambda = 1$. In correspondence, the *strong complement* of a fuzzy set s is the fuzzy set $-s$ defined by $(-s)(x) = \text{neg}(s(x))$ for every $x \in S$.

3. Fuzzy subsets as continuous chains of sets

As proposed in [5], we identify the fuzzy subsets with the continuous chains. In order to achieve this aim, we recall some definitions and results. Given $s \in \mathscr{F}(S)$, for every $\lambda \in [0, 1]$ the subsets

$$C(s, \lambda) = \{x \in S / s(x) \geqslant \lambda\} \quad \text{and} \quad O(s, \lambda) = \{x \in S / s(x) > \lambda\}$$

are called the *closed λ-cut* and the *open λ-cut* of s, respectively.

Proposition 3.1. *Let $s, s' \in \mathscr{F}(S)$. Then for every $\lambda \in U$*
(*a*) $C(s, 0) = S$, (*b*) $\lambda \leqslant \lambda' \Rightarrow C(s, \lambda) \supseteq C(s, \lambda')$.
(*c*) $s \subseteq s' \Rightarrow C(s, \lambda) \subseteq C(s', \lambda)$, (*d*) $C(s, \lambda) = \bigcap_{\mu < \lambda} O(s, \mu)$,
(*e*) $C(s \cup s', \lambda) = C(s, \lambda) \cup C(s', \lambda)$, (*f*) $C(s \cap s', \lambda) = C(s, \lambda) \cap C(s', \lambda)$.

The following proposition shows that every fuzzy subset can be defined by means of its cuts. Given $\lambda \in U$ and $X \subseteq S$, we denote by $\lambda \wedge X$ and $\lambda \vee X$ the fuzzy subset $\lambda \wedge \chi_X$ and $\lambda \vee \chi_X$, respectively.

Proposition 3.2. *For every $s \in \mathscr{F}(S)$:*

$$s = \bigcup_{\lambda \in U} (\lambda \wedge C(s, \lambda)), \tag{3}$$

$$s = \bigcup_{\lambda \in U} (\lambda \wedge O(s, \lambda)) \tag{4}$$

and, dually,

$$s = \bigcap_{\lambda \in U} (\lambda \vee O(s, \lambda)), \tag{5}$$

$$s = \bigcap_{\lambda \in U} (\lambda \vee C(s, \lambda)). \tag{6}$$

Observe that it is possible to rewrite (Eqs. (3)–(6)) in the following equalities:

$$s(x) = \sup\{\lambda \in U / x \in C(s, \lambda)\}, \tag{7}$$

$$s(x) = \sup\{\lambda \in U / x \in O(s, \lambda)\}, \tag{8}$$

$$s(x) = \inf\{\lambda \in U / x \notin O(s, \lambda)\}, \tag{9}$$

$$s(x) = \inf\{\lambda \in U / x \notin C(s, \lambda)\}, \tag{10}$$

where $x \in S$.

Definition 3.3. We define a *chain* in S any order-reversing family $(C_\lambda)_{\lambda \in U}$ of subsets of S such that $C_0 = S$ and we denote by $\mathrm{Ch}(S)$ the class of all the chains in S. We say that a chain $(C_\lambda)_{\lambda \in U}$ is *continuous* if

$$C_\lambda = \bigcap_{\mu < \lambda} C_\mu \tag{11}$$

for every $\lambda \in U$. We denote by $\mathrm{CCh}(S)$ the class of all the continuous chains in S.

The family $(C(s, \lambda))_{\lambda \in U}$ of the closed cuts of a given fuzzy subset s is a continuous chain. The following Lemma shows that given any chain of subsets, a fuzzy subset is defined in a natural way.

Lemma 3.4. *Let* $(C_\lambda)_{\lambda \in U}$ *be a chain of subsets of* S *and define* s *by*

$$s = \bigcup_{\lambda \in U} (\lambda \wedge C_\lambda). \tag{12}$$

Then we have also that

$$s = \bigcap_{\lambda \in U} (\lambda \vee C_\lambda), \tag{13}$$

$$O(s, \mu) = \bigcup_{\lambda > \mu} C_\lambda \subseteq C_\mu \subseteq \bigcap_{\lambda < \mu} C_\lambda = C(s, \mu). \tag{14}$$

We say that s is the fuzzy subset associated with $(C_\lambda)_{\lambda\in U}$. *Note that Eqs. (12) and (13) are equivalent to:*

$$s(x) = \sup\{\lambda \in U / x \in C_\lambda\}, \tag{15}$$

$$s(x) = \inf\{\lambda \in U / x \notin C_\lambda\}, \tag{16}$$

respectively. Also, observe that the subsets C_1 *and* C_0 *have no influence for defining the fuzzy subset s. In fact, the fuzzy subset* $0 \wedge C_0 = \emptyset$ *gives no contribution to the union in Eq. (12) and the fuzzy subset* $1 \vee C_1 = S$ *gives no contribution to the intersection in Eq. (13).*

In Ch(S), and therefore in CCh(S), an ordering is defined by setting,

$$(A_\lambda)_{\lambda\in U} \leqslant (B_\lambda)_{\lambda\in U} \quad \leftrightarrow \quad A_\lambda \subseteq B_\lambda \quad \textit{for every } \lambda \in U$$

for every pair of chains $(A_\lambda)_{\lambda\in U}$ *and* $(B_\lambda)_{\lambda\in U}$.

Proposition 3.5. *Let* $(C_\lambda)_{\lambda\in U}$ *be a chain and define* $(\bar{C}_\lambda)_{\lambda\in U}$ *by setting,*

$$\bar{C}_\lambda = \bigcap_{\mu<\lambda} C_\mu \tag{17}$$

for every $\lambda \in U$. *Then* $(\bar{C}_\lambda)_{\lambda\in U}$ *is the continuous chain generated by* $(C_\lambda)_{\lambda\in U}$, *i.e. the smallest continuous chain greater or equal* $(C_\lambda)_{\lambda\in U}$.

Proof. Since $C_\mu \supseteq C_\lambda$ for every $\lambda > \mu$, we have that $\bar{C}_\lambda \supseteq C_\lambda$. To prove that $(\bar{C}_\lambda)_{\lambda\in U}$ is a continuous chain, observe that

$$\bigcap_{\mu<\lambda} \bar{C}_\mu = \bigcap_{\mu<\lambda} \left(\bigcap_{v<\mu} C_v \right) = \bigcap_{v<\lambda} C_v = \bar{C}_\lambda.$$

Let $(A_\lambda)_{\lambda\in U}$ be a continuous chain containing $(C_\lambda)_{\lambda\in U}$. Then

$$A_\lambda = \bigcap_{\mu<\lambda} A_\mu \supseteq \bigcap_{\mu<\lambda} C_\mu = \bar{C}_\lambda. \qquad \square$$

As an immediate consequence of Eq. (12) we also have the following proposition whose meaning can be pictured as follows:

$$(C_\lambda)_{\lambda\in U} \rightarrow s \leftrightarrow (\bar{C}_\lambda)_{\lambda\in U}.$$

Proposition 3.6. *Let* $(C_\lambda)_{\lambda\in U}$ *be a chain and s the fuzzy subset obtained by Eq. (12). Then,* $C(s,\mu) = \bar{C}_\mu$, *i.e. the family* $(C(s,\mu))_{\mu\in U}$ *of cuts of s coincides with the continuous chain generated by* $(C_\lambda)_{\lambda\in U}$.

The following theorem shows that we can identify the fuzzy subsets with the continuous chains.

Theorem 3.7. *Let S be a set, then:*

1. *CCh(S) is a complete, completely distributive, lattice,*

2. *the correspondence* $H : \mathscr{F}(S) \to CCh(S)$ *defined by setting, for every* $s \in \mathscr{F}(S)$

$$H(s) = (C(s, \lambda))_{\lambda \in U} \tag{18}$$

is a lattice isomorphism between $\mathscr{F}(S)$ *and* $CCh(S)$,

3. *the inverse map* $H^{-1} : CCh(S) \to \mathscr{F}(S)$ *associates every continuous chain* $(C_\lambda)_{\lambda \in U}$ *with the fuzzy subset s defined by Eq. (12).*

4. An extension principle for operators

In this section we propose an extension principle generalizing a principle given by Ramik [2]. Let $S_1, \dots, S_n, S$ be sets and $J : \mathscr{P}(S_1) \times \cdots \times \mathscr{P}(S_n) \to \mathscr{P}(S)$ any operator. Then, given $s_1 \in \mathscr{F}(S_1), \dots, s_n \in \mathscr{F}(S_n)$, it is natural to proceed in the following way:

- identify $s_1, \dots, s_n$ with the continuous chains $(C(s_1, \lambda))_{\lambda \in U}, \dots, (C(s_n, \lambda))_{\lambda \in U}$,
- given $\lambda \in U$, apply J to the sets $C(s_1, \lambda), \dots, C(s_n, \lambda)$ obtaining the chain $(J(C(s_1, \lambda), \dots, (C(s_n, \lambda)))_{\lambda \in U}$,
- consider the continuous chain generated from this chain,
- assume as image of $s_1, \dots, s_n$ the fuzzy subset $J^*(s_1, \dots, s_n)$ corresponding to the continuous chain previously obtained.

Such a procedure can be pictured by the following diagram.

$$\begin{array}{ccc} (s_1, - \dots, s_n) \to (C(s_1, \lambda))_{\lambda \in U}, \dots, (C(s_n, \lambda))_{\lambda \in U} & \to & (J(C(s_1, \lambda), \dots, C(s_n, \lambda)))_{\lambda \in U} \\ \downarrow & & \downarrow \\ J^*(s_1, \dots, s_n) \leftrightarrow & & \overline{(J(C(s_1, \lambda), \dots, C(s_n, \lambda)))}_{\lambda \in U} \end{array}$$

As we have seen earlier in Section 3, we can obtain $J^*(s_1, \dots, s_n)$ in a more direct way using the following definition.

Definition 4.1 (*Extension-principle for operators*). If $J : \mathscr{P}(S_1) \times \cdots \times \mathscr{P}(S_n) \to \mathscr{P}(S)$ is an operator, then we define the *fuzzy operator* $J^* : \mathscr{F}(S_1) \times \cdots \times \mathscr{F}(S_n) \to \mathscr{F}(S)$, called the *canonical extension* of J, by setting for any $s_1 \in \mathscr{F}(S_1), \dots, s_n \in \mathscr{F}(S_n)$ and $x \in S$

$$J^*(s_1, \dots, s_n) = \bigcup (\lambda \wedge J(C(s_1, \lambda), \dots, C(s_n, \lambda))). \tag{19}$$

In terms of membership functions, we can read Eq. (19) as

$$J^*(s_1, \dots, s_n)(x) = \sup\{\lambda \in U / x \in J(C(s_1, \lambda), \dots, C(s_n, \lambda))\} \tag{20}$$

for every $x \in S$. The term "extension" is justified by the following proposition:

Proposition 4.2. *The fuzzy operator* J^* *is the extension of the operator* J, *i.e. for every* $X_1, \dots, X_n \in \mathscr{P}(S)$

$$J(X_1, \dots, X_n) = J^*(\chi_{X_1}, \dots, \chi_{X_n}).$$

Proof. Let $\chi_{X_1}, \ldots, \chi_{X_n}$ be the characteristic functions of the sets $X_1, \ldots, X_n$, respectively. Then for every $\lambda \neq 0, C(\chi_{X_i}, \lambda) = X_i$ for $i = 1, \ldots, n$. Therefore

$$\begin{aligned} x \in J(X_1, \ldots, X_n) &\Rightarrow x \in J(C(\chi_{X_1}, \lambda), \ldots, C(\chi_{X_n}, \lambda)) \quad \text{for every } \lambda \neq 0 \\ &\Rightarrow J^*(\chi_{X_1}, \ldots, \chi_{X_n})(x) = 1. \end{aligned}$$

Moreover,

$$\begin{aligned} x \notin J(X_1, \ldots, X_n) &\Rightarrow x \notin J(C(\chi_{X_1}, \lambda), \ldots, C(\chi_{X_n}, \lambda)) \quad \text{for every } \lambda \neq 0 \\ &\Rightarrow J^*(\chi_{X_1}, \ldots, \chi_{X_n})(x) = 0. \end{aligned}$$

Thus, $J^*(\chi_X) = \chi_{J(X)}$ and the restriction of J^* to the crisp subsets coincides with J. □

The following lemma relates the cuts of $J^*(s_1, \ldots, s_n)$ with cuts of $s_1, \ldots, s_n$.

Lemma 4.3. *Let $J : \mathscr{P}(S_1) \times \cdots \times \mathscr{P}(S_n) \to \mathscr{P}(S)$ be any order-preserving operator. Let $s_1 \in \mathscr{F}(S_1), \ldots, s_n \in \mathscr{F}(S_n)$. Then for every $\mu \in U$:*

$$\begin{aligned} &O(J^*(s_1, \ldots, s_n), \mu) \\ &\quad = \bigcup_{\lambda > \mu} J(C(s_1, \lambda), \ldots, C(s_n, \lambda)) \subseteq J(O(s_1, \mu), \ldots, O(s_n, \mu)) \\ &\quad \subseteq J(C(s_1, \mu), \ldots, C(s_n, \mu)) \subseteq \bigcap_{\lambda < \mu} J(C(s_1, \lambda), \ldots, C(s_n, \lambda)) \\ &\quad = C(J^*(s_1, \ldots, s_n), \mu). \end{aligned}$$

Proof. Since $C(s_i, \lambda) \subseteq O(s_i, \mu)$ for every $\lambda > \mu$ and $i = 1, \ldots, n$ and J is order-preserving, we have that $J(C(s_1, \lambda), \ldots, C(s_n, \lambda)) \subseteq J(O(s_1, \mu), \ldots, O(s_n, \mu))$. Then $J(O(s_1, \mu), \ldots, O(s_n, \mu)) \supseteq \bigcup_{\lambda > \mu} J(C(s_1, \lambda), \ldots, C(s_n, \lambda))$. The remaining part of the proposition is a consequence of Eq. (14) in Lemma 3.4 when we set $C_\lambda = J(C(s_1, \lambda), \ldots, C(s_n, \lambda))$. Therefore $s = J^*(s_1, \ldots, s_n)$. □

Note that, since the family $(J(C(s_1, \lambda), \ldots, C(s_n, \lambda)))_{\lambda \in U}$ is not necessarily continuous, we have that $J(C(s_1, \mu), \ldots, C(s_n, \mu)) \neq C(J^*(s_1, \ldots, s_n), \mu)$, in general.

Proposition 4.4. *Let $J: \mathscr{P}(S_1) \times \cdots \times \mathscr{P}(S_n) \to \mathscr{P}(S)$ be order-preserving. Then:*

$$J^*(s_1, \ldots, s_n) = \bigcap_{\lambda \in U} (\lambda \vee J(C(s_1, \lambda), \ldots, C(s_n, \lambda))), \tag{21}$$

$$J^*(s_1, \ldots, s_n) = \bigcup_{\lambda \in U} (\lambda \wedge J(O(s_1, \lambda), \ldots, O(s_n, \lambda))), \tag{22}$$

$$J^*(s_1, \ldots, s_n) = \bigcap_{\lambda \in U} (\lambda \vee J(O(s_1, \lambda), \ldots, O(s_n, \lambda))) \tag{23}$$

or, equivalently,

$$J^*(s_1,\ldots,s_n)(x) = \inf\{\lambda \in U/x \notin J(C(s_1,\lambda),\ldots,C(s_n,\lambda))\}. \tag{24}$$

$$J^*(s_1,\ldots,s_n)(x) = \sup\{\lambda \in U/x \in J(O(s_1,\lambda),\ldots,O(s_n,\lambda))\}. \tag{25}$$

$$J^*(s_1,\ldots,s_n)(x) = \inf\{\lambda \in U/x \notin J(O(s_1,\lambda),\ldots,O(s_n,\lambda))\}. \tag{26}$$

Proof. Equality (21) and the equivalence between Eqs. (22) and (23) follow from Lemma 3.4. To prove Eq. (22) observe that, since J is order preserving, $J(O(s_i,\lambda)) \subseteq J(C(s_i,\lambda))$ for $i = 1,\ldots,n$ and hence,

$$\begin{aligned}&\sup\{\lambda \in U/x \in J(O(s_1,\lambda),\ldots,O(s_n,\lambda))\}\\ &\quad\leqslant \sup\{\lambda \in U/x \in J(C(s_1,\lambda),\ldots,C(s_n,\lambda)\} = J^*(s_1,\ldots,s_n)(x)\end{aligned}$$

for every fuzzy subsets $s_1,\ldots,s_n$ and $x \in S$. Moreover, by Lemma 4.3, we also have $O(J^*(s_1,\ldots,s_n),\lambda) \subseteq J(O(s_1,\lambda),\ldots,O(s_n,\lambda))$ and therefore

$$\begin{aligned}J^*(s_1,\ldots,s_n)(x) &= \sup\{\lambda \in U/x \in O(J^*(s_1,\ldots,s_n),\lambda)\}\\ &\leqslant \sup\{\lambda \in U/x \in J(O(s_1,\lambda),\ldots,O(s_n,\lambda))\}.\end{aligned}$$

In a similar way one proves the remaining part of this proposition. □

5. Compactness

Now we examine the canonical extension of a classical operator J in the case in which the set $J(X)$ can be obtained by considering only some particular part of X. Recall that an operator $J: \mathscr{P}(S) \to \mathscr{P}(S)$ is called *compact* if

$$J(X) = \bigcup\{J(F)/F \subseteq X \text{ and } F \text{ finite}\}.$$

We extend such a notion as follows.

Definition 5.1. Let δ be a cardinal number. Then an operator $J: \mathscr{P}(S_1) \times \cdots \times \mathscr{P}(S_n) \to \mathscr{P}(S)$ is *δ-compact* if for every $X_1 \subseteq S_1,\ldots,X_n \subseteq S_n$

$$J(X_1,\ldots,X_n) = \bigcup\{J(F_1,\ldots,F_n) \mid F_i \subseteq S_i \text{ and card } (F_i) < \delta\}. \tag{27}$$

The ω-compact operators coincide with the compact operators. An example of $(\omega+1)$-compact operator is furnished by the topological closure in a Euclidean space. The following is a further extension of the notion of compactness.

Definition 5.2. Let $J: \mathscr{P}(S_1) \times \cdots \times \mathscr{P}(S_n) \to \mathscr{P}(S)$ be an operator and $\mathscr{C} = (\mathscr{C}_1,\ldots,\mathscr{C}_n)$, where $\mathscr{C}_i$ is a class of subsets of S_i. Then we say that J is *$\mathscr{C}$-compact* provided that

$$J(X_1,\ldots,X_n) = \bigcup\{J(F_1,\ldots,F_n) \mid F_i \subseteq S_i \text{ and } F_i \in \mathscr{C}_i\}. \tag{28}$$

If we set $\mathscr{C}_i = \{F_i \subseteq S_i/\text{card}(F_i) < \delta\}$, then the $\mathscr{C}$-compact operators coincide with the δ-compact operators.

Proposition 5.3. *Let $J : \mathscr{P}(S_1) \times \cdots \times \mathscr{P}(S_n) \to \mathscr{P}(S)$ be an operator. Then the following are equivalent:*
1. *J is $\mathscr{C}$-compact with respect to a suitable $\mathscr{C}$,*
2. *J is δ-compact with respect to a suitable cardinal number δ,*
3. *J is order-preserving.*

Proof. Obvious. □

In the following if $s : S \to U$ is a fuzzy subset and $F \subseteq S$, then we define sub(s, F) by

$$\text{sub}(F, s) = \begin{cases} \inf\{s(x)/x \in F\} & \text{if } F \neq \emptyset, \\ 1 & \text{if } F = \emptyset. \end{cases}$$

The number sub(F, s) is a multivalued valuation of the statement "for every $x \in F, x$ is an element of s" i.e. a measure of the degree of inclusion of F in s.

Proposition 5.4. *Let $J : \mathscr{P}(S_1) \times \cdots \times \mathscr{P}(S_n) \to \mathscr{P}(S)$ be a $\mathscr{C}$-compact operator. Then $J^*(s_1, \ldots, s_n)(x) =$*

$$\sup\{\text{sub}(F_1, s_1) \wedge \cdots \wedge \text{sub}(F_n, s_n)/x \in J(F_1, \ldots, F_n), F_i \in \mathscr{C}_i\}.$$

Proof. By Definitions 4.1 and 5.2 we have that

$$\begin{aligned} J^*(s_1, \ldots, s_n)(x) &= \sup\{\lambda \in U/x \in J(C(s_1, \lambda), \ldots, C(s_n, \lambda))\} \\ &= \sup\{\lambda \in U/F_1, \ldots, F_n \text{ exist such that} \\ &\qquad F_i \in \mathscr{C}_i, F_i \subseteq C(s_i, \lambda) \text{ and } x \in J(F_1, \ldots, F_n)\}. \end{aligned}$$

But $F_i \subseteq C(s_i, \lambda)$ means that $s_i(x) \geqslant \lambda$ for every $x \in F_i$ and hence sub$(F_i, s_i) \geqslant \lambda$ for $i = 1, \ldots, n$. Then,

$$\text{sub}(F_1, s_1) \wedge \cdots \wedge \text{sub}(F_n, s_n) \geqslant \lambda.$$

Thus, we can write

$$\begin{aligned} J^*(s_1, \ldots, s_n)(x) = \sup\{&\lambda \in U/\text{sub}(F_1, s_1) \wedge \cdots \wedge \text{sub}(F_n, s_n) \geqslant \lambda, \\ &x \in J(F_1, \ldots, F_n), F_i \in \mathscr{C}_i\} \end{aligned}$$

and this completes the proof. □

In particular, we have the following proposition.

Proposition 5.5. *If $J : \mathscr{P}(S) \to \mathscr{P}(S)$ is a compact operator, then $J^*(s)(x) = 1$ if $x \in J(\emptyset)$ and*

$$J^*(s)(x) = \sup\{s(x_1) \wedge \cdots \wedge s(x_n)/x \in J(\{x_1, \dots, x_n\})\}, \quad (29)$$

otherwise.

Eq. (29) was used in [3] to extend any crisp logic to a fuzzy logic. Indeed, one proves that if $\mathscr{D}$ is the deduction operator of any crisp logic, then there exists a fuzzy logic whose deduction operator is the canonical extension $\mathscr{D}^*$. Assume that $\alpha, \alpha_1, \dots, \alpha_n$ are formulas in such a logic and that s is a fuzzy set of axioms. Then if we write $\alpha_1, \dots, \alpha_n \vdash \alpha$ to denote that $\alpha \in \mathscr{D}(\{\alpha_1, \dots, \alpha_n\})$, we have that, $\mathscr{D}^*(s)(\alpha) = 1$ if α is a tautology and

$$J^*(s)(\alpha) = \sup\{s(\alpha_1) \wedge \cdots \wedge s(\alpha_n)/\alpha_1, \dots, \alpha_n \vdash \alpha\}, \quad (30)$$

otherwise.

Proposition 5.6. *Let* $J: \mathscr{P}(S_1) \times \cdots \times \mathscr{P}(S_n) \to \mathscr{P}(S)$ *be a compact operator. Then*

$$O(J^*(s_1, \dots, s_n), \lambda) = J(O(s_1, \lambda), \dots, O(s_n, \lambda)). \quad (31)$$

Proof. $J^*(s_1, \dots, s_n)(x) > \lambda \iff$ there exist finite sets $F_1, \dots, F_n$ such that $\mathrm{sub}(F_1, s_1) \wedge \cdots \wedge \mathrm{sub}(F_n, s_n) > \lambda$ and $x \in J(F_1, \dots, F_n) \iff$ there exist finite sets $F_1, \dots, F_n$ such that $F_1 \subseteq O(s_1, \lambda), \dots, F_n \subseteq O(s_n, \lambda)$ and $x \in J(F_1, \dots, F_n) \iff x \in J(O(s_1, \lambda), \dots, O(s_n, \lambda))$. □

Note that for the closed cuts is not possible to prove an analogous proposition.

6. Some examples

In the following proposition we show that the union, the intersection and the strong complement can be obtained by the proposed extension principle.

Proposition 6.1. *The canonical extensions of the union (intersection) in classical set theory is the usual union (intersection) in fuzzy set theory. Instead, the canonical extension of the classical complement is the strong complement.*

Proof. Let $J: \mathscr{P}(S) \times \mathscr{P}(S) \to \mathscr{P}(S)$ be the union operator, i.e. $J(X, Y) = X \cap Y$. Then

$$\begin{aligned} J^*(s_1, s_2)(x) &= \sup\{\lambda \in U/x \in C(s_1, \lambda) \cup C(s_2, \lambda)\} \\ &= \sup\{\lambda \in U/s_1(x) \geqslant \lambda \text{ or } s_2(x) \geqslant \lambda\} \\ &= \sup\{\lambda \in U/s_1(x) \vee s_2(x) \geqslant \lambda\} \\ &= s_1(x) \vee s_2(x). \end{aligned}$$

Let J be the intersection operator, i.e. $J(X, Y) = X \cap Y$. Then we have that

$$J^*(s_1,s_2)(x) = \sup\{\lambda \in U/x \in C(s_1,\lambda) \cap C(s_2,\lambda)\}$$
$$= \sup\{\lambda \in U/s_1(x) \geqslant \lambda \text{ and } s_2(x) \geqslant \lambda\} = s_1(x) \wedge s_2(x).$$

Finally, let $J: \mathscr{P}(S) \to \mathscr{P}(S)$ be the complement operator, i.e. $J(X) = S - X$. Then we have that if $s(x) \neq 1$,

$$J^*(s)(x) = \sup\{\lambda \in U/x \in J(C(s,\lambda))\} = \sup\{\lambda \in U/x \notin C(s,\lambda)\}$$
$$= \sup\{\lambda \in U/s(x) < \lambda\} = 1.$$

If $s(x) = 1$, it is immediate that $J^*(s)(x) = 0$. □

The following proposition shows that Zadeh's extension principle is generalized by the principle proposed here.

Proposition 6.2. *Let $h: S_1 \times \cdots \times S_n \to S$ be a function and define J_h by setting*

$$J_h(X_1,\ldots,X_n) = \{h(x_1,\ldots,x_n)/x_1 \in X_1,\ldots,x_n \in X_n\}.$$

Then the canonical extension of J_h coincides with the Zadeh's extension h^* of h, i.e. $J_h^* = h^*$.

Proof. Since J_h is 2-compact,

$$J_h^*(s_1,\ldots,s_n)(x)$$
$$= \sup\{s_1(x_1) \wedge \cdots \wedge s_n(x_n)/x \in J_h(\{x_1\},\ldots,\{x_n\})\}$$
$$= \sup\{s_1(x_1) \wedge \cdots \wedge s_n(x_n)/h(x_1,\ldots,x_n) = x\} = h^*(s_1,\ldots,s_n)(x). \quad \square$$

Now we will show that some basic notions of fuzzy set theory can be obtained by the extension-principle for operators. Let S_1, S_2 and S_3 be sets. Then we recall that the composition $R \circ R'$ of two binary relations $R \subseteq S_1 \times S_2$ and $R' \subseteq S_2 \times S_3$ is the relation between S_1 and S_3 defined by setting

$$R \circ R' = \{(x_1,x_3)/(x_1,x_2) \in R \text{ and } (x_2,x_3) \in R' \text{ for some } x_2 \in S_2\}.$$

Proposition 6.3. *Denote by $\circ$ the canonical extension of the composition between two binary relations. Let S_1, S_2, S_3 be sets and $r \in \mathscr{F}(S_1 \times S_2), r' \in \mathscr{F}(S_2 \times S_3)$ fuzzy relations. Then we have*

$$(r \circ r')(x_1,x_3) = \sup\{r(x_1,x_2) \wedge r'(x_2,x_3)/x_2 \in S_2\}. \tag{32}$$

Proof. Define $J: \mathscr{P}(S_1 \times S_2) \times \mathscr{P}(S_2 \times S_3) \to \mathscr{P}(S_1 \times S_3)$ by setting $J(R,R') = R \circ R'$ for every pair of relations R and R'. Then, since J is a 2-compact operator,

$$J^*(r,r')(x_1,x_3)$$
$$= \sup\{r(x,y) \wedge r'(y,z)/(x_1,x_3) \in J(\{(x,y)\},\{(y,z)\})\}$$
$$= \sup\{r(x,y) \wedge r'(y,z)/(x_1,x_3) = (x,z)\}$$
$$= \sup\{r(x_1,x_2) \wedge r'(x_2,x_3)/x_2 \in S_2\}.$$

Proposition 6.4. *Denote by $\times$ the canonical extension of the Cartesian product of two subsets. Then, for every $s_1 \in \mathscr{F}(S_1)$ and $s_2 \in \mathscr{F}(S_2)$, $s_1 \times s_2$ is the fuzzy subset of $S_1 \times S_2$ defined by*

$$(s_1 \times s_2)(x_1, x_2) = s_1(x_1) \wedge s_2(x_2). \tag{33}$$

Proof. Let $J: \mathscr{P}(S_1) \times \mathscr{P}(S_2) \to \mathscr{P}(S_1 \times S_2)$ be the operator defined by setting $J(X, Y) = X \times Y$ for any $X \in \mathscr{P}(S_1)$ and $Y \in \mathscr{P}(S_2)$. Since J is a 2-compact operator, we have

$$\begin{aligned} J^*(s_1, s_2)(x_1, x_2) &= \sup\{s_1(x) \wedge s_2(y)/(x_1, x_2) \in J(\{x\}, \{y\})\} \\ &= \sup\{s_1(x) \wedge s_2(y)/(x_1, x_2) = (x, y)\} \\ &= s_1(x_1) \wedge s_2(x_2). \qquad \square \end{aligned}$$

We recall that if $R \in S_1 \times S_2$ is a relation and $X \subseteq S_1$, then the image $R(X)$ of a subset X by R is defined by

$$R(X) = \{x_2 \in S_2/x_1 R x_2 \text{ for some } x_1 \in X\}.$$

Proposition 6.5. *Let $r \in \mathscr{F}(S_1, S_2)$ be a fuzzy relation, s a fuzzy subset of S_1 and let $r(s)$ be the image of s by r (as the canonical extension of the classical notion of image). Then*

$$r(s)(x_2) = \sup\{r(x_1, x_2) \wedge s(x_1)/x_1 \in S_1\} \tag{34}$$

for every $x_2 \in S_2$.

Proof. Define $J: \mathscr{P}(S_1 \times S_2) \times \mathscr{P}(S_1) \to \mathscr{P}(S_2)$ by setting $J(R, X) = R(X)$. Let $r \in F(S_1 \times S_2)$ be a fuzzy relation and s a fuzzy subset of S_1. Then

$$\begin{aligned} J^*(r, s)(x_2) &= \sup\{r(x, y) \wedge s(z)/x_2 \in J(\{(x, y)\}, \{z\}\} \\ &= \sup\{r(x, x_2) \wedge s(x)/x_2 \in J(\{(x, x_2)\}, \{x\}\}. \qquad \square \end{aligned}$$

Proposition 6.6. *Let V be a linear space and, for every subset X of V, let $J(X)$ be the subspace generated by X. Then J is $\mathscr{C}$-compact, where $\mathscr{C}$ is the class of independent finite systems of vectors. Consequently, for every $x \in V$*

$$\begin{aligned} J^*(s)(x) = \sup\{s(x_1) \wedge \cdots \wedge s(x_n)/x_1, \ldots, x_n &\text{ are independent} \\ \text{and } x \text{ is a linear combination of } &x_1, \ldots, x_n\}. \end{aligned} \tag{35}$$

7. Some properties preserved by the extension principle

In this section we will examine some properties that are preserved under the extension principle.

Proposition 7.1. *If* $J : \mathscr{P}(S_1) \times \cdots \times \mathscr{P}(S_n) \to \mathscr{P}(S)$ *is an operator, then J order preserving* $\iff J^*$ *is order-preserving.*

Proof. Assume that J is order preserving and that $s_i \subseteq v_i$ for every i. Then, since $C(s_i, \lambda) \subseteq C(v_i, \lambda)$, we have

$$\begin{aligned} J^*(s_1, \dots, s_n)(x) &= \sup\{\lambda \in U / x \in J(C(s_1, \lambda), \dots, C(s_n, \lambda))\} \\ &\leqslant \sup\{\lambda \in U / x \in J(C(v_1, \lambda), \dots, C(v_n, \lambda))\} \\ &= J^*(v_1, \dots, v_n)(x). \end{aligned}$$

The converse implication is obvious. □

Proposition 7.2. *Assume that* J_1 *and* J_2 *are two order-preserving operators. Then:*

$$(J_1 \vee J_2)^* = J_1^* \vee J_2^*, \tag{36}$$

$$J_2 \geqslant J_1 \;\Rightarrow J_2^* \geqslant J_1^*, \tag{37}$$

$$(J_1 \wedge J_2)^* = J_1^* \wedge J_2^*, \tag{38}$$

$$(J_1 \circ J_2)^* \leqslant (J_1^* \circ J_2^*). \tag{39}$$

Proof. Let $(s_1, \dots, s_n) \in \mathscr{F}(S_1) \times \cdots \times \mathscr{F}(S_n)$ and $x \in S$. In order to prove Eq. (36), we observe that the following equalities hold:

$$\begin{aligned} &(J_1 \vee J_2)^*(s_1, \dots, s_n)(x) \\ &\quad = \sup\{\lambda \in U / x \in (J_1 \vee J_2)(C(s_1, \lambda), \dots, C(s_n, \lambda))\} \\ &\quad = \sup\{\lambda \in U / x \in J_1(C(s_1, \lambda), \dots, C(s_n \lambda)) \cup J_2(C(s_1, \lambda), \dots, C(s_n, \lambda))\} \\ &\quad = \sup\{\sup\{\lambda \in U / x \in J_1(C(s_1, \lambda), \dots, C(s_n, \lambda))\}\}, \\ &\qquad\quad \sup\{\lambda \in U / x \in J_2(C(s_1, \lambda), \dots, C(s_n, \lambda))\} \\ &\quad = J_1^*(s_1, \dots, s_n)(x) \vee J_2^*(s_1, \dots, s_n)(x). \end{aligned}$$

In order to prove Eq. (37), we observe that if $J_1 \leqslant J_2$, we have that $J_1 \vee J_2 = J_2$. So, as consequence of Eq. (36), we have that $J_2^* = (J_1 \vee J_2)^* = J_1^* \vee J_2^*$ and then $J_2^* \geqslant J_1^*$.

In order to prove Eq. (38), assume that $J_1^*(s_1, \dots, s_n)(x) \leqslant J_2^*(s_1, \dots, s_n)(x)$. Then

$$\begin{aligned} &\{\lambda \in U / x \in J_1(C(s_1, \lambda), \dots, C(s_n, \lambda))\} \\ &\subseteq \{\lambda \in U / x \in J_2(C(s_1, \lambda), \dots, C(s_n, \lambda))\}. \end{aligned}$$

But it is easily seen that the following equalities hold:

$$\begin{aligned} &(J_1 \wedge J_2)^*(s_1, \dots, s_n)(x) \\ &\quad = \sup\{\lambda \in U / x \in J_1(C(s_1, \lambda), \dots, C(s_n, \lambda)) \cap J_2(C(s_1, \lambda), \dots, C(s_n, \lambda))\} \\ &\quad = \sup\{\lambda \in U / x \in J_1(C(s_1, \lambda), \dots, C(s_n, \lambda))\} = J_1^*(s_1, \dots, s_n)(x) \\ &\quad = J_1^*(s_1, \dots, s_n)(x) \wedge J_2^*(s_1, \dots, s_n)(x). \end{aligned}$$

Finally, observe that Lemma 4.3 implies

$$J_2(C(s_1,\lambda),\dots,C(s_n,\lambda)) \subseteq C(J_2^*(s_1,\dots,s_n),\lambda).$$

Then Eq. (39) follows from the following inequality:

$$\begin{aligned} J_1^*(J_2^*(s_1,\dots,s_n))(x) &= \sup\{\lambda \in U/x \in J_1(C(J_2^*(s_1,\dots,s_n),\lambda))\} \\ &\geqslant \sup\{\lambda \in U/x \in J_1(J_2(C(s_1,\lambda),\dots,C(s_n,\lambda)))\} \\ &= (J_1 \circ J_2)^*(s_1,\dots,s_n)(x). \qquad \square \end{aligned}$$

Corollary 7.3. *The map $J \to J^*$ is an embedding of the lattice of the order-preserving operators into the lattice of the fuzzy order-preserving operators.*

Proof. Eqs. (36) and (38) show that the lattice operations are preserved. Proposition 4.2 entails that the map is injective. □

8. Closure operators

The case $n = 1$ is particularly interesting and extensively examined in [3,4]. In this section we confine ourselves to recall some basic results without the proofs. Recall that if S is any nonempty set, then a *closure operator* in S is any map $J: \mathscr{P}(S) \to \mathscr{P}(S)$ such that, for $X, Y \in \mathscr{P}(S)$,

$$X \subseteq Y \;\Rightarrow\; J(X) \subseteq J(Y);\; X \subseteq J(X); \quad J(J(X)) = J(X).$$

If we have also that

$$J(X \cup Y) = J(X) \cup J(Y) \quad \text{and} \quad J(\emptyset) = \emptyset,$$

then J is called a *topological* closure operator. Finally, if J is compact, then we say that J is an *algebraic* closure operator. We will extend such a definition as follows.

Definition 8.1. If S is a set, then a *fuzzy closure operator* in S is any operator $J: \mathscr{F}(S) \to \mathscr{F}(S)$ satisfying the following properties:
1. $s_1 \leqslant s_2 \;\Rightarrow\; J(s_1) \leqslant J(s_2)$ (monotony),
2. $s \leqslant J(s)$ (inclusion),
3. $J(J(s)) = J(s)$ (idempotence). If we have also that
4. $J(s_1 \cup s_2) = J(s_1) \cup J(s_1)$,
5. $J(\emptyset) = \emptyset$, then we say that J is a *fuzzy topological closure operator.*

A *fixed point* of a fuzzy operator J is a fuzzy subset s such that $J(s) = s$. The proof of the following propositions is immediate.

Proposition 8.2. *The class of the fixed points of a (fuzzy) closure operator is closed with respect to the intersection.*

Proposition 8.3. *The class of fixed points of a (fuzzy) topological closure operator is the class of closed (fuzzy) subsets of a (fuzzy) topology.*

We have the following Proposition.

Proposition 8.4. *Let $J: \mathscr{P}(S) \to \mathscr{P}(S)$ be an operator and J^* the related extension. Then*

J^* is a closure operator $\Longleftrightarrow$ J is a closure operator.

J^* is a topological closure operator $\Longleftrightarrow$ J is a topological closure operator.

Definition 8.5 (*Extension-principle for fixed points*). Let $\mathscr{C}$ be the class of fixed points of an order-preserving operator $J: \mathscr{P}(S) \to \mathscr{P}(S)$. Let $\mathscr{C}^*$ be the class of fixed points of J^*. Then we say $\mathscr{C}^*$ to be the *canonical extension* of $\mathscr{C}$.

In the following we give two applications of the extension-principle for fixed points [4]. The first one is related with the notion of natural fuzzy topology as defined in [6].

Proposition 8.6. *Let J be the closure operator of a topological space (S, τ) and let $\mathscr{C}$ be the related class of closed subsets. Then the extension $\mathscr{C}^*$ of $\mathscr{C}$ is the class of closed subsets of the natural fuzzy topology associated with τ. If (S, τ) is a Fréchet space and s a fuzzy subset of S, then the topological closure $\bar{s} = J^*(s)$ of s is given by*

$$\bar{s}(x) = \sup\{\mathrm{sub}((x_n)_{n\in N}, s)/(x_n)_{n\in N} \text{ is a sequence converging to } x\}.$$

Equivalently,

$$\bar{s}(x) = \overline{\lim_{y\to x}}\, s(y).$$

Proof. We observe only that J is $\mathscr{C}$-compact where $\mathscr{C}$ is the class of the converging sequences. □

The second application is related with the notion of fuzzy subalgebras as defined by Rosenfeld [7].

Proposition 8.7. *Let $\mathscr{A}$ be an algebraic structure and define J by setting $J(X) = \langle X \rangle$, for any $X \subseteq \mathscr{A}$, i.e. the algebraic substructure of $\mathscr{A}$ generated by X. Then the extension of the class $\mathscr{C}$ of the subalgebras of $\mathscr{A}$ is the class $\mathscr{C}^*$ of the fuzzy algebras of $\mathscr{A}$. Given a fuzzy subset s, the fuzzy subalgebra $\langle s \rangle = J^*(s)$ generated by s can be obtained by setting*

$$\langle s \rangle(x) = \begin{cases} \sup\{s(x_1) \wedge \cdots \wedge s(x_n)/p(x_1, \ldots, x_n) = x, p \in \mathrm{pol}(\mathscr{A})\} & \text{if } x \notin \langle \mathbb{C} \rangle \\ 1 & \text{if } x \in \langle \mathbb{C} \rangle \end{cases}.$$

where $\mathbb{C}$ is the set of constants and pol $(\mathscr{A})$ *the set of polynomial functions of $\mathscr{A}$.*

9. Extending by a join-continuous triangular norm

If $J: \mathscr{P}(S) \to \mathscr{P}(S)$ is a compact operator, then we can obtain J^* by using Eq. (29). In this section we will attempt to substitute both the lattice [0,1] and the operation $\wedge$ in such a formula by using join-continuous triangular norm. Assume that U is a complete lattice equipped with a binary operation $\otimes$ that we suppose a *join-continuous triangular norm* in U. This means that $\otimes$ is a binary continuous operation such that

- $(x \otimes y) \otimes z = x \otimes (y \otimes z)$ (associativity),
- $x \otimes y = y \otimes x$ (commutativity),
- $x \otimes 1 = x$ (1 is a neutral element),
- $(\sup_{i \in I} x_i) \otimes y = \sup_{i \in I}(x_i \otimes y)$,

where x, y and z are elements of U and $(x_i)_{i \in I}$ is any family of elements in U. For instance, we can assume that $\otimes$ is the ordinary product in the interval $U = [0, 1]$. We call *U-subset* of S any map s from S into U and we denote by U^S the class of U-subsets of S. It is immediate that U^S is a complete lattice. The notion of closure operator in U^S is obvious. The following definition enables to extend any operator $J : \mathscr{P}(S) \to \mathscr{P}(S)$ to an operator $J : U^S \to U^S$.

Definition 9.1. For every operator J, the function $J_\otimes$ defined by

$$J_\otimes(s)(x) = \begin{cases} 1 & \text{if } x \in J(\emptyset), \\ \sup\{s(x_1) \otimes \cdots \otimes s(x_n)\} / x \in J(\{x_1, \ldots, x_n\}) & \text{otherwise} \end{cases} \tag{40}$$

is called *the canonical extension of J* via $\otimes$.

Proposition 9.2. *Let J be an algebraic closure operator. Then $J_\otimes$ is a closure operator in U^S extending J. Let $U = [0, 1]$ and let J^* be the extension of J via the meet operation. Then $J_\otimes \leqslant J^*$* and *$J_\otimes \neq J^*$, in general.*

Proof. Firstly, observe that, since $x \otimes y \leqslant x \otimes 1 = x$, we have $x \otimes y \leqslant x \wedge y$ and, in particular, that $x \otimes x \leqslant x$. It is immediate that $J_\otimes(s) \supseteq s$ and that, since $\otimes$ is order-preserving, $J_\otimes$ is order-preserving. So we have only to prove that, for every $x \in S$, $J_\otimes(s)(x) \geqslant J_\otimes(J_\otimes(s))(x)$. Now, this inequality is immediate if $x \in J(\emptyset)$, while, in the case $x \notin J(\emptyset)$, we have to prove that for every $x \in S$ and $x \in J(\{x_1, \ldots, x_n\})$

$$J_\otimes(s)(x_1) \otimes \cdots \otimes J_\otimes(s)(x_n) \leqslant \sup\{s(z_1) \otimes \cdots \otimes s(z_h) / x \in J(\{z_1, \ldots, z_h\})\}. \tag{41}$$

Let, for $i = 1, \ldots, n$

$$J_{\otimes}(s)(x_i) = \sup\{s(z_1^i) \otimes \cdots \otimes s(z_{k(i)}^i)/x_i \in J(\{z_1^i, \ldots, z_{k(i)}^i\})\}.$$

Then, since the least upper bounds are preserved by $\otimes$, we obtain that

$$\begin{aligned} &J_{\otimes}(s)(x_1) \otimes \cdots \otimes J_{\otimes}(s)(x_n) \\ &\quad = \sup\left\{s(z_1^1) \otimes \cdots \otimes s\left(z_{k(1)}^1\right) \otimes \cdots \otimes s(z_1^n) \otimes \cdots \otimes s\left(z_{k(n)}^n\right) \Big/ \right. \\ &\qquad \left. x_i \in J\left(\left\{z_1^i, \ldots, z_{k(i)}^i\right\}\right) \quad \text{for } i = 1, \ldots, n\right\}. \end{aligned}$$

By noting that, from $x_i \in J(\{z_1^i, \ldots, z_{k(i)}^i\})$ for $i = 1, \ldots, n$ and $x \in J(\{x_1, \ldots, x_n\})$, it follows that $x \in J(\{z_1, \ldots, z_h\})$, where $z_1, \ldots, z_h$ are the (distinct) elements of the sequence $z_1^1, \ldots, z_{k(i)}^1, \ldots, z_{k(n)}^n$, and that

$$s(z_1^1) \otimes \cdots \otimes s\left(z_{k(1)}^1\right) \otimes \cdots \otimes s(z_1^n) \otimes \cdots \otimes s\left(z_{k(n)}^n\right) \leqslant s(z_1) \otimes \cdots \otimes s(z_h),$$

we obtain Eq. (41). In order to prove that $J_{\otimes}$ is an extension of J, let s be a crisp subset, namely the characteristic function of the subset X. Then it is immediate that $J_{\otimes}(s)(x) = 1$ iff either $x \in J(\emptyset)$ or $x_1, \ldots, x_n$ exist in X such that $x \in J(\{x_1, \ldots, x_n\})$. Since J is algebraic, this is equivalent to say that $x \in J(X)$. Thus $J_{\otimes}(s)$ is the characteristic function of $J(X)$. □

10. Extension-principle for relations

Together with the extension-principle for operators on subsets we can consider the following very simple extension-principle of relations among subsets.

Definition 10.1 (*extension-principle for relations*). Let $\mathscr{R} \subseteq \mathscr{P}(S_1) \times \cdots \times \mathscr{P}(S_n)$ be an n-ary relation between subsets. Then the *canonical extension* of $\mathscr{R}$ is the (classical) relation $\mathscr{R}^* \subseteq \mathscr{F}(S_1) \times \cdots \times \mathscr{F}(S_n)$ among fuzzy subsets defined by setting

$$\begin{aligned} &(s_1, \ldots, s_n) \in \mathscr{R}^* \\ &\quad \Longleftrightarrow \quad (C(s_1, \lambda), \ldots, C(s_1, \lambda), \ldots, C(s_n, \lambda)) \in \mathscr{R} \quad \text{for every } \lambda \in U. \end{aligned}$$

For instance, if $\mathscr{R}$ is the usual inclusion between sets, then it is easy to prove that $\mathscr{R}^*$ is the inclusion between fuzzy subsets. In the case $n = 1$, this principle is very simple and useful. For instance, we can define *bounded* (*measurable, compact, convex* and so on) a fuzzy subset s of the Euclidean plane provided that every closed cut of s is bounded (measurable, compact, convex, respectively). Also, in the case $n = 1$ the extension-principle for relations is strictly related with the Extension-principle for fixed points.

Proposition 10.2. *Let $\mathscr{C}$ be the class of fixed points of a closure operator J and let $\mathscr{C}^*$ be the extension of $\mathscr{C}$ by the extension-principle for fixed points. Then $\mathscr{C}^*$ is the extension of $\mathscr{C}$ in accordance with the extension-principle for relations, i.e.*

$$\mathscr{C}^* = \{s \in S/C(s,\lambda) \in \mathscr{C} \text{ for every } \lambda \in U\}.$$

Proof. Let s be a fixed point of J^*. Then, since each $J(C(s,\mu))$ is a fixed point for J and the intersection of a family of fixed points is a fixed point, we have

$$C(s,\lambda) = C(J^*(s),\lambda) = \bigcap\{J(C(s,\mu)) \mid \mu < \lambda\} \in \mathscr{C}.$$

Conversely, assume that every cut of s belongs to $\mathscr{C}$. Then

$$J^*(s)(x) = \sup\{\lambda \in U \mid x \in J(C(s,\lambda))\}$$
$$= \sup\{\lambda \in U \mid x \in C(s,\lambda)\} = s(x).$$

and therefore s is a fixed point for J^*. □

Note. An operator $J: \mathscr{P}(S_1) \times \cdots \times \mathscr{P}(S_n) \to \mathscr{P}(S)$ can be regarded as a relation $\mathscr{R}_J \subseteq \mathscr{P}(S_1) \times \cdots \times \mathscr{P}(S_n) \times \mathscr{P}(S)$, where one puts

$$(X_1,\ldots,X_n,X) \in \mathscr{R}_J \iff J(X_1,\ldots,X_n) = X.$$

Now, in general we have that J^* is different from $(\mathscr{R}_J)^*$ and it is also possible that $(\mathscr{R}_J)^*$ is the empty relation. In fact, if $s_1,\ldots,s_n$, s are fuzzy subsets,

$$(s_1,\ldots,s_n,s) \in (\mathscr{R}_J)^*$$
$$\iff (C(s_1,\lambda),\ldots,C(s_n,\lambda),C(s,\lambda)) \in \mathscr{R}_J \quad \text{for every } \lambda \in U$$
$$\iff J(C(s_1,\lambda),\ldots,C(s_n,\lambda)) = C(s,\lambda) \quad \text{for every } \lambda \in U.$$

So, in the case that $(J(C(s_1,\lambda),\ldots,C(s_n,\lambda)))_{\lambda \in U}$ is not a continuous chain, no fuzzy subset s can exist for which $(s_1,\ldots,s_n,s) \in (\mathscr{R}_J)^*$.

11. Extending real-valued maps

Let $f: \mathscr{P}(S_1) \times \cdots \times \mathscr{P}(S_n) \to R$ be a real-valued map. Then we can extend f to a map $f^*: \mathscr{F}(S_1) \times \cdots \times \mathscr{F}(S_n) \to R$ as follows:
- we identify $s_1,\ldots,s_n$ with the continuous chains $(C(s_1,\lambda))_{\lambda \in U},\ldots,(C(s_n,\lambda))_{\lambda \in U}$;
- for every $\lambda \in U$, we apply f to $C(s_1,\lambda),\ldots,C(s_n,\lambda)$ by obtaining the family $(f(C(s_1,\lambda),\ldots,C(s_n,\lambda)))_{\lambda \in U}$ of real numbers;
- we assume as image of $s_1,\ldots,s_n$ the mean of the obtained values.

Then, we have the following definition:

Definition 11.1 (*Extension-principle for real-valued maps*). Let D be a subset of $\mathscr{P}(S_1) \times \cdots \times \mathscr{P}(S_n)$ and $f: D \to R$ a monotone map. Then the *canonical extension* of f is the map $f^*: D^* \to R$ defined by setting,

$$f^*(s_1,\ldots,s_n) = \int_{[0,1]} f(C(s_1,\lambda),\ldots,C(s_n,\lambda))\mathrm{d}\lambda. \tag{42}$$

for every $(s_1,\ldots,s_n) \in D$.

Note that the function $g(\lambda) = f(C(s,\lambda), C(s',\lambda))$ is monotone and therefore the right side of Eq. (42) is always defined (although it could be either finite or infinite). Such a procedure can be pictured by the following diagram:

$$\begin{array}{ccc} (s_1,\ldots,s_n) & \rightarrow & (C(s_1,\lambda))_{\lambda\in U},\ldots,(C(s_n,\lambda))_{\lambda\in U} \\ \downarrow & & \downarrow \\ f^*(s_1,\ldots,s_n) & \leftarrow & (f(C(s_1,\lambda),\ldots,C(s_n,\lambda)))_{\lambda\in U} \end{array}$$

The term "extension" is justified by the following obvious proposition:

Proposition 11.2. *The function f^* is the extension of f, i.e. $f(X_1,\ldots,X_n) = f^*(\chi_{X_1},\ldots,\chi_{X_n})$, for every $X_1 \in \mathscr{P}(S_1),\ldots,X_n \in \mathscr{P}(S_n)$.*

The proof of the following proposition is immediate:

Proposition 11.3. *Let $f: D \rightarrow R$ be a monotone function. Then:*
1. *f order – preserving $\Rightarrow f^*$ order – preserving,*
2. *f order – reversing $\Rightarrow f^*$ order – reversing,*
3. *the codomain of f^* is contained in the smallest interval containing the codomain of f,*
4. *f finitely additive $\Rightarrow f^*$ finitely additive.*

Examples (See [8,9]). (a) Consider a classical measure, for instance, a finitely additive probability $\mu: D \rightarrow [0,1]$, where $D \subseteq \mathscr{P}(S)$. Then D^* is the class of measurable fuzzy subsets and, by Proposition 11.1, the extension $\mu^*: \mathscr{F}(S) \rightarrow [0,1]$ is a finitely additive fuzzy measure. We have that

$$\mu^*(s) = \int_S s\,\mathrm{d}\mu.$$

(b) Let (M,d) be a metric space. Then the distance between two nonempty subsets X and Y is the number $\delta(X,Y)$ defined by the formula

$$\delta(X,Y) = \inf\{\mathrm{d}(x,y)/x \in X, y \in Y\}. \tag{43}$$

Since δ is defined in the class $D = \{(X,Y)/X \neq \emptyset \text{ and } Y \neq \emptyset\}$, δ^* is defined in the class

$$D^* = \{(s,s')/s(x) = 1 \text{ and } s'(x') = 1 \text{ for some } x,x' \in M\}.$$

Moreover,

$$\delta^*(s,s') = \int_{[0,1]} \delta(C(s,\lambda), C(s',\lambda))\,\mathrm{d}\lambda. \tag{44}$$

(c) Recall that the *diameter* of a nonempty subset X of M is defined by

$$\mathrm{diam}(X) = \sup\{\mathrm{d}(x,y)/x,y \in X\}. \tag{45}$$

In this case:

$$D = \{X/X \text{ is bounded}\}, \quad D^* = \{s/C(s,\lambda) \text{ is bounded for every } \lambda \neq 0\}$$

and, then

$$\mathrm{diam}^*(s) = \int_{[0,1]} \mathrm{diam}\,(C(s,\lambda))\,\mathrm{d}\lambda \tag{46}$$

for every $s \in D^*$.

Note. We can extend a real-valued map $f : \mathscr{P}(S_1) \times \cdots \times \mathscr{P}(S_n) \to R$ also via the extension-principle for operators. Indeed, we can apply such a principle to the operator $f : \mathscr{P}(S_1) \times \cdots \times \mathscr{P}(S_n) \to \mathscr{P}(R)$ defined by setting $\hat{f}(x_1,\ldots,x_n) = \{f(x_1,\ldots,x_n)\}$. We obtain the function $\hat{f}\mathscr{F}(S_1) \times \cdots \times \mathscr{F}(S_n) \to \mathscr{F}(R)$ where

$$\hat{f}^*(s_1,\ldots,s_n)(x) = \sup\{\lambda \in U/x = f(C(s_1,\lambda),\ldots,C(s_n,\lambda))\}. \tag{47}$$

Then, $\hat{f}^*(s_1,\ldots,s_n)$ is a fuzzy real number and not a real number as in Eq. (42). For instance, if we consider a finitely additive probability $\mu : \mathscr{P}(S) \to [0,1]$, then the extension $\hat{\mu}^*$ is the map $\hat{\mu}^* : \mathscr{F}(S) \to \mathscr{F}(R)$ defined by

$$\hat{\mu}^*(s)(x) = \sup\{\lambda \in U/x = \mu(C(s,\lambda))\}. \tag{48}$$

Then, while the extension-principle for real-valued maps extends a probability in a map assuming values in [0,1], the extension-principle for operators extends a probability to a map assuming values in the class $\mathscr{F}([0,1])$ of the "fuzzy numbers" of the interval [0,1]. In the literature, sometimes one prefers Eqs. (47) and (42), e.g. Zadeh [10] defined the cardinality $|s|$ of a fuzzy subset s (whose cuts are finite sets) as the fuzzy number $|s|$ obtained by setting

$$|s|(n) = \sup\{\lambda \in U/n = |C(s,\lambda)|\}. \tag{49}$$

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