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An Extension Principle for Closure Operators

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1. INTRODUCTION

The notions of closure system and closure operator are very useful tools in several areas of classical mathematics. As an example, we may quote the following closure systems (and the related closure operators):

—the class of closed subsets of a given topological space;

—the class of substructures of a given algebraic structure;

—the class of filters of a given Boolean algebra;

—the class of convex subsets of a given Euclidean space;

—the class of theories of a given deductive system.

This led several authors to investigate the closure systems and the closure operators in the framework of fuzzy set theory. The resulting researches give an elegant and powerful treatment of notions such as those of fuzzy subalgebras, necessity measures, and envelopes (for example, see [1–4] and [11]).

Let S be any set, and denote by $\mathscr{P}(S)$ and $\mathscr{F}(S)$ the class of the subsets and the class of the fuzzy subsets of S, respectively. In this paper we propose an "extension principle" enabling us to extend any classical operator J: $\mathscr{P}(S) \to \mathscr{P}(S)$ to an operator J^*: $\mathscr{F}(S) \to \mathscr{F}(S)$ in such a way that J^* is a closure operator if and only if J is a closure operator. Also, we

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examine a related ‘‘extension principle’’ to extend a class $\mathscr{C}$ of subsets of S in a class $\mathscr{C}^*$ of fuzzy subsets of S in such a way that $\mathscr{C}^*$ is a closure system if and only if $\mathscr{C}$ is a closure system. This enables us to restate in a uniform way several basic notions in fuzzy set theory, such as the natural fuzzy topologies, fuzzy subalgebras, necessity measures, and fuzzy convex subsets.

2. PRELIMINARIES

In the following we adopt the convention that, if $(S, \leq, 0, 1)$ is an ordered set with minimum 0 and maximum 1, then the least upper bound of the empty class is 0 and the greatest lower bound is 1, that is $\mathrm{Sup}(\emptyset) = 0$ and $\mathrm{Inf}(\emptyset) = 1$. We denote by U the real interval $[0, 1]$, and, if λ_1 and λ_2 are elements of U, we set

$$\lambda_1 \vee \lambda_2 = \mathrm{Max}\{\lambda_1, \lambda_2\} \qquad \text{and} \qquad \lambda_1 \wedge \lambda_2 = \mathrm{Min}\{\lambda_1, \lambda_2\}.$$

Given a set S, we call a *fuzzy subset* of S any map $s: S \to U$ and we denote by $\mathscr{F}(S)$ the class of fuzzy subsets of S (Zadeh [16]). We say that s is *crisp* provided that $s(x) \in \{0, 1\}$ for every $x \in S$. We identify the class $\mathscr{P}(S)$ of all subsets of S with the class of the crisp fuzzy subsets by associating to every $X \in \mathscr{P}(S)$ the related characteristic function χ_X. So, we identify S and $\emptyset$ with the map constantly equal to 1 and 0, respectively. Let s and s' be two fuzzy subsets; we then say that s is *contained* in s' or that s is a *part* of s' and we write $s \subseteq s'$ provided that $s(x) \leq s'(x)$ for any $x \in S$. Also, the *union* $s \cup s'$ is the fuzzy subset defined by setting $(s \cup s')(x) = s(x) \vee s'(x)$ and the *intersection* $s \cap s'$ by setting $(s \cap s')(x) = s(x) \wedge s'(x)$ for every $x \in S$. More generally, given a family $(s_i)_{i \in I}$ of fuzzy subsets of S, we set

$$\Big(\bigcup_{i \in I} s_i\Big)(x) = \mathrm{Sup}\{s_i(x) \mid i \in I\} \quad \text{and} \quad \Big(\bigcap_{i \in I} s_i\Big)(x) = \mathrm{Inf}\{s_i(x) \mid i \in I\}.$$

The *complement* $-s$ of s is defined by $-s(x) = 1 - s(x)$. The *support* $\mathrm{Supp}(s)$ of s is defined by $\mathrm{Supp}(s) = \{x \in S \mid s(x) \neq 0\}$ and s is *finite* if $\mathrm{Supp}(s)$ is finite. For every $\lambda \in U$ the subsets

$$C(s, \lambda) = \{x \in S \mid s(x) \geq \lambda\} \qquad \text{and} \qquad O(s, \lambda) = \{x \in S \mid s(x) > \lambda\}$$

are called the *closed* and the *open* λ*-cut* of s, respectively. We list below the main properties of the cuts

(a) $C(s \cup s', \mu) = C(s, \mu) \cup C(s', \mu)$; $O(s \cup s', \mu) = O(s, \mu) \cup O(s', \mu)$

(b) $C(s \cap s', \mu) = C(s, \mu) \cap C(s', \mu)$; $O(s \cap s', \mu) = O(s, \mu) \cap O(s', \mu)$

(c) $C(s, \mu) = \bigcap_{\lambda < \mu} O(s, \lambda)$; $O(s, \mu) = \bigcup_{\lambda > \mu} C(s, \lambda)$

(d) $C(s, \mathrm{Sup}_{i \in I} \lambda_i) = \bigcap_{i \in I} C(s, \lambda_i)$; $O(s, \mathrm{Inf}_{i \in I} \lambda_i) = \bigcup_{i \in I} O(s, \lambda_i)$

(e) $C(\bigcap s_i, \lambda) = \bigcap C(s_i, \lambda)$; $O(\bigcup s_i, \lambda) = \bigcup O(s_i, \lambda)$,

where $s, s' \in \mathscr{F}(S)$, $\lambda, \mu \in U$, $(\lambda_i)_{i \in I}$ is a family of elements of U and $(s_i)_{i \in I}$ is a family of fuzzy subsets. A fuzzy subset is characterized by the family of its cuts, indeed

$$s(x) = \mathrm{Sup}\{\lambda \in U \mid x \in C(s, \lambda)\} = \mathrm{Sup}\{\lambda \in U \mid x \in O(s, \lambda)\} \quad (2.1)$$

or, dually,

$$s(x) = \mathrm{Inf}\{\lambda \in U \mid x \notin C(s, \lambda)\} = \mathrm{Inf}\{\lambda \in U \mid x \notin O(s, \lambda)\} \quad (2.2)$$

In the following, given a subset X of S and $\lambda \in U$, we denote by $\lambda \wedge X$ and $\lambda \vee X$ the fuzzy subsets defined by

$$(\lambda \wedge X)(x) = \begin{cases} \lambda & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases}$$

and

$$(\lambda \vee X)(x) = \begin{cases} \lambda & \text{if } x \notin X \\ 1 & \text{otherwise} \end{cases}$$

respectively. By such notations we may rewrite (2.1) and (2.2) respectively by

$$s = \bigcup \lambda \wedge C(s, \lambda) = \bigcup \lambda \wedge O(s, \lambda) \quad (2.3)$$

and

$$s = \bigcap \lambda \vee C(s, \lambda) = \bigcap \lambda \vee O(s, \lambda). \quad (2.4)$$

Equation (2.1) suggests that we may associate to any order-reversing family $(C_\lambda)_{\lambda \in U}$ of subsets of S a fuzzy subset s as follows:

$$s(x) = \mathrm{Sup}\{\lambda \in U \mid x \in C_\lambda\}. \quad (2.5)$$

In other words, we set $s = \bigcup \lambda \wedge C_\lambda$. Likewise, (2.2) suggests setting $s = \bigcap \lambda \vee C_\lambda$. The following lemma shows some properties of s.

LEMMA 2.1. *Let $(C_\lambda)_{\lambda \in U}$ be any order-reversing family of subsets of S, and define s by* (2.5); *then, for every $\mu \in U$,*

$$O(s, \mu) = \bigcup_{\lambda > \mu} C_\lambda \subseteq C_\mu \subseteq \bigcap_{\lambda < \mu} C_\lambda = C(s, \mu). \quad (2.6)$$

Proof. We have

$$x \in O(s, \mu) \Leftrightarrow s(x) > \mu \Leftrightarrow \exists \lambda > \mu \text{ such that } x \in C_\lambda \Leftrightarrow x \in \bigcup_{\lambda > \mu} C_\lambda.$$

Moreover,

$$x \in \bigcap_{\lambda<\mu} C_\lambda \Rightarrow \forall\lambda < \mu \quad x \in C_\lambda \Rightarrow s(x) \geq \operatorname{Sup}\{\lambda \mid \lambda < \mu\}$$
$$\Rightarrow s(x) \geq \mu \Rightarrow x \in C(s, \mu)$$

and

$$x \in C(s, \mu) \Rightarrow s(x) \geq \mu \Rightarrow \forall\lambda < \mu \, \exists\nu > \lambda \quad x \in C_\nu$$
$$\Rightarrow \forall\lambda < \mu \quad x \in C_\lambda \Rightarrow x \in \bigcap_{\lambda<\mu} C_\lambda. \quad \blacksquare$$

Note that if $(C_\lambda)_{\lambda \in U}$ is order-reversing then C_1 has no relevance in defining s. Indeed, s can be defined by the equality $\bigcap_{\lambda<\mu} C_\lambda = C(s, \mu)$ and in this equality C_1 does not occur. We call *continuous* any chain $(C_\lambda)_{\lambda \in U}$ of subsets of S such that

$$C_0 = S \qquad \text{and} \qquad C_\lambda = \bigcap_{x<\lambda} C_x$$

The following well known proposition follows from (2.6) and it shows that we may identify the fuzzy subsets of S with the continuous chains of subsets of S (see Negoita and Ralescu [12]).

PROPOSITION 2.2. *Given a fuzzy subset s, the family* $(C(s, \lambda))_{\lambda \in U}$ *of its closed cuts is a continuous chain. Conversely, given any continuous chain* $(C_\lambda)_{\lambda \in U}$ *of subsets of S define s by* (2.5). *Then s is a fuzzy subset such that* $C(s, \mu) = C_\mu$ *for every* $\mu \in U$.

3. FUZZY CLOSURE OPERATORS

Recall that, given a set S, a (classical) *closure operator* in $\mathscr{P}(S)$ is a map $J\colon \mathscr{P}(S) \to \mathscr{P}(S)$ such that, for every X and Y subset of S,

(i) $X \subseteq Y \Rightarrow J(X) \subseteq J(Y)$; (ii) $X \subseteq J(X)$; (iii) $J(J(X)) = J(X)$.

A collection $\mathscr{C}$ of subsets of S is a *closure system* if the intersection of any family of elements of $\mathscr{C}$ is an element of $\mathscr{C}$. In particular, since S is the intersection of the empty family, $S \in \mathscr{C}$. The extension of such concepts to fuzzy set theory is straightforward. We call *fuzzy operator*, in brief *operator*, any map J from $\mathscr{F}(S)$ to $\mathscr{F}(S)$ and we say that J is a *fuzzy closure operator*, in brief a *closure operator*, provided that

(i) $s \subseteq s' \Rightarrow J(s) \subseteq J(s')$; (ii) $s \subseteq J(s)$; (iii) $J(J(s)) = J(s)$.

Likewise, a class $\mathscr{C}$ of fuzzy subsets of S is called a *fuzzy closure system*, in brief a *closure system*, if the intersection of any family of elements of $\mathscr{C}$ is an element of $\mathscr{C}$. Now, it is well known that if J is a closure operator, then the set

$$\mathscr{C}_J = \{ X \mid J(X) = X \}$$

is a closure system and, if $\mathscr{C}$ is a closure system, then by setting

$$J_{\mathscr{C}}(X) = \bigcap \{ Y \in \mathscr{C} \mid Y \supseteq X \}$$

we obtain a closure operator $J_{\mathscr{C}}$. The following proposition shows that such a connection holds for the fuzzy closure operators and the fuzzy closure systems, too.

PROPOSITION 3.1. *Let $\mathscr{C}$ be a class of fuzzy subsets, then the operator $J_{\mathscr{C}}$ defined by*

$$J_{\mathscr{C}}(s) = \bigcap \{ s' \in \mathscr{C} \mid s' \supseteq s \} \tag{3.1}$$

is a fuzzy closure operator. Let J be a fuzzy operator satisfying (i) *and* (ii) *and set*

$$\mathscr{C}_J = \{ f \in \mathscr{F}(S) \mid J(f) = f \}, \tag{3.2}$$

then $\mathscr{C}_J$ is a closure system. Moreover, if J is a closure operator and $\mathscr{C}$ is a closure system, then

$$J_{\mathscr{C}_J} = J \qquad \text{and} \qquad \mathscr{C}_{J_{\mathscr{C}}} = \mathscr{C}. \tag{3.3}$$

Proof. The first part of the proposition is immediate. Let $(f_i)_{i \in I}$ be a family of elements of $\mathscr{C}_J$, then by (ii) $J(\bigcap f_i) \supseteq \bigcap f_i \supseteq \bigcap J(f_i) \supseteq J(\bigcap f_i)$ and therefore $J(\bigcap f_i) = \bigcap f_i$. This proves that $\mathscr{C}_j$ is a closure system. To prove that $J_{\mathscr{C}_J} = J$ it suffices to observe that, given a fuzzy set s, $J(s)$ is the least fixed point of J greater than or equal to s. Equation $\mathscr{C}_{J_{\mathscr{C}}} = \mathscr{C}$ is obvious. ∎

4. EXTENDING CLASSICAL CLOSURE OPERATORS AND SYSTEMS

In this section we will propose an extension principle for classical closure operators. Namely, given a classical operator $J: \mathscr{P}(S) \to \mathscr{P}(S)$ we extend it in a fuzzy operator $J^*: \mathscr{F}(S) \to \mathscr{F}(S)$ by setting, for every $s \in \mathscr{F}(S)$,

$$J^*(s)(x) = \operatorname{Sup}\{ \lambda \in U \mid x \in J(C(s, \lambda)) \}. \tag{4.1}$$

We call a *canonical extension* of J the operator J^*. From Lemma 2.1 it follows that, if J is order-preserving,

$$O(J^*(s), \mu) = \bigcup_{\lambda > \mu} J(C(s,\lambda)) \subseteq J(O(s,\mu)) \subseteq J(C(s,\mu)) \subseteq \bigcap_{\lambda < \mu} J(C(s,\lambda)) = C(J^*(s), \mu). \qquad (4.2)$$

PROPOSITION 4.1. *Let J be a classical operator, then J^* is an extension of J. Moreover, J^* is a closure operator if and only if J is a closure operator. In this case the closed cuts of J^* are fixed points for J.*

Proof. Assume that s is crisp, namely the characteristic function of the set X, then for every $\lambda \neq 0$, $C(s, \lambda) = X$ and therefore

$$x \in J(X) \Rightarrow x \in J(C(s,\lambda)) \quad \text{for every } \lambda \neq 0 \Rightarrow J^*(s)(x) = 1;$$

$$x \notin J(X) \Rightarrow x \notin J(C(s,\lambda)) \quad \text{for every } \lambda \neq 0 \Rightarrow J^*(s)(x) = 0.$$

This proves that $J^*(s)$ is the characteristic function of $J(X)$ and therefore that J^* is an extension of J. Assume that J is a closure operator, then it is immediate that J^* is increasing. To prove that $J^*(s) \supseteq s$, observe that, since $C(s, \lambda) \subseteq J(C(s, \lambda))$,

$$s(x) = \mathrm{Sup}\{\lambda \in U \mid x \in C(s,\lambda)\} \leq \mathrm{Sup}\{\lambda \in U \mid x \in J(C(s,\lambda))\} = J^*(s)(x).$$

To prove that $J^*(J^*(s)) = J^*(s)$, we observe that every cut $C(J^*(s), \lambda)$ is a fixed point for J. Indeed, the intersection of a class of fixed points is a fixed point and $C(J^*(s), \lambda) = \bigcap_{\mu < \lambda} J(C(s, \mu))$. Thus

$$J^*(J^*(s))(x) = \mathrm{Sup}\{\lambda \in U \mid x \in J(C(J^*(s),\lambda))\} = \mathrm{Sup}\{\lambda \in U \mid x \in C(J^*(s),\lambda)\} = J^*(s)(x). \quad \blacksquare$$

Note that there are extensions of a classical closure operator that are not canonical extensions. For example, let L be a subset of U closed with respect to the meets and containing 0 and 1, and consider the fuzzy operator H defined by $H(s)(x) = \mathrm{Inf}\{\lambda \in L \mid \lambda \geq s(x)\}$. Then, since the restriction J of H to $\mathscr{P}(S)$ is the identity map, H cannot be obtained by (4.1).

Given a class $\mathscr{C}$ of subsets of S, we set

$$\mathscr{C}^* = \{s \in \mathscr{F}(S) \mid C(s,\lambda) \in \mathscr{C} \text{ for every } \lambda \in U\}. \qquad (4.3)$$

So, in a sense, $\mathscr{C}^*$ coincides with the class of the continuous chains of elements of $\mathscr{C}$. It is easily proven that $\mathscr{C}^*$ is an extension of $\mathscr{C}$, namely that $\mathscr{C}$ coincides with the class of crisp elements of $\mathscr{C}^*$. We say that $\mathscr{C}^*$ is the *canonical extension* of $\mathscr{C}$.

PROPOSITION 4.2. *$\mathscr{C}^*$ is a fuzzy closure system if and only if $\mathscr{C}$ is a closure system.*

Proof. Let $\mathscr{C}$ be a closure system and $(s_i)_{i \in I}$ be any family of elements of $\mathscr{C}^*$, then being $C(\bigcap s_i, \lambda) = \bigcap C(s_i, \lambda) \in \mathscr{C}$, it is also $\bigcap s_i \in \mathscr{C}^*$. This proves that $\mathscr{C}^*$ is a closure system. The converse implication is trivial. ▮

The following proposition shows that the notions of canonical extension of a closure system and canonical extension of a closure operator are strictly related in accordance with the following diagrams

$$\begin{array}{ccc} \mathscr{C} & \to & J_{\mathscr{C}} \\ \downarrow & & \downarrow \\ \mathscr{C}^* & \to & J_{\mathscr{C}^*} \end{array} \qquad \begin{array}{ccc} J & \to & \mathscr{C}_J \\ \downarrow & & \downarrow \\ J^* & \to & \mathscr{C}_{J^*} \end{array}$$

PROPOSITION 4.3. *Let $\mathscr{C}$ be a closure system and $J_{\mathscr{C}}$ the closure operator associated with $\mathscr{C}$. Then*

$$J_{\mathscr{C}^*} = (J_{\mathscr{C}})^*. \tag{4.4}$$

Let J be a closure operator and $\mathscr{C}_J$ the related closure system, then

$$(\mathscr{C}_J)^* = \mathscr{C}_{J^*}. \tag{4.5}$$

Proof. Let s by any fuzzy subset of S, then $(J_{\mathscr{C}})^*(s)$ is an element of $\mathscr{C}^*$. Indeed, by (4.2),

$$C\big((J_{\mathscr{C}})^*(s), \mu\big) = \bigcap\big\{J_{\mathscr{C}}(C(s, \lambda)) \mid \lambda < \mu\big\} \in \mathscr{C}.$$

As a consequence, since $(J_{\mathscr{C}})^*(s) \supseteq s$, we have that $(J_{\mathscr{C}})^*(s) \supseteq J_{\mathscr{C}^*}(s)$. Conversely, for every $s' \in \mathscr{C}^*$ such that $s \subseteq s'$ and $\lambda \in U$ we have $J_{\mathscr{C}}(C(s, \lambda)) \subseteq J_{\mathscr{C}}(C(s', \lambda)) = C(s', \lambda)$ and therefore

$$\begin{aligned} (J_{\mathscr{C}})^*(s)(x) &= \operatorname{Sup}\big\{\lambda \in U \mid x \in J_{\mathscr{C}}(C(s, \lambda))\big\} \\ &\le \operatorname{Sup}\big\{\lambda \in U \mid x \in C(s', \lambda)\big\} = s'(x). \end{aligned}$$

This proves that $(J_{\mathscr{C}})^*(s) \subseteq s^*$ and therefore that $(J_{\mathscr{C}})^*(s) \subseteq J_{\mathscr{C}^*}(s)$.

To prove (4.5), observe that by setting in (4.4) $\mathscr{C} = \mathscr{C}_J$ we obtain $J_{\mathscr{C}_J^*} = J^*$. ▮

PROPOSITION 4.4. *Let $[a, b]$ be an interval contained in U, $f\colon U \to [a, b]$ a bijective continuous increasing map. Then $J^*(f \circ v) = f(J^*(v)) \cup J(\varnothing)$, that is*

$$J^*(f \circ v)(x) = \begin{cases} 1 & \text{if } x \in J(\varnothing) \\ f(J^*(v)(x)) & \text{otherwise.} \end{cases} \tag{4.6}$$

Proof. At first, observe that

$$C(f \circ v, \lambda) = \begin{cases} C(v, f^{-1}(\lambda)) & \text{if } \lambda \in [a, b] \\ \varnothing & \text{if } \lambda > b \\ S & \text{if } \lambda < a. \end{cases}$$

Indeed, if $\lambda \in [a, b]$ then

$$x \in C(f \circ v, \lambda) \Leftrightarrow f(v(x)) \geq \lambda \Leftrightarrow v(x) \geq f^{-1}(\lambda) \Leftrightarrow x \in C(v, f^{-1}(\lambda)),$$

while the remaining cases are immediate.

Now, if $x \in J(\varnothing)$ it is immediate that $J^*(f \circ v)(x) = 1$. If $x \notin J(\varnothing)$, then

$$\begin{aligned} J^*(f \circ v)(x) &= \mathrm{Sup}\{\lambda \in U \mid x \in J(C(f \circ v, \lambda))\} \\ &= \mathrm{Sup}(\{\lambda \in U \mid \lambda < a, x \in J(C(f \circ v, \lambda))\} \\ &\qquad \cup \{\lambda \in U \mid \lambda \in [a, b], x \in J(C(f \circ v, \lambda))\}) \\ &= \mathrm{Max}\{a, \mathrm{Sup}\{\lambda \in [a, b] \mid x \in J(C(v, f^{-1}(\lambda)))\}\} \\ &= \mathrm{Max}\{a, \mathrm{Sup}\{f(\lambda') \mid x \in J(C(v, \lambda'))\}\} \\ &= \mathrm{Max}\{a, f(\mathrm{Sup}\{\lambda' \mid x \in J(C(v, \lambda'))\})\} = f(J^*(v)(x)). \end{aligned}$$

∎

As an example, let $f(x) = \lambda \cdot x + \mu$ where $\lambda > 0$, $\mu \geq 0$, and $\lambda + \mu \leq 1$, then

$$J^*(\lambda \cdot v + \mu)(x) = \begin{cases} 1 & \text{if } x \in J(\varnothing) \\ \lambda \cdot J^*(v)(x) + \mu & \text{otherwise.} \end{cases} \tag{4.7}$$

COROLLARY 4.5. *For every subset X and $\lambda \in U$,*

$$J^*(\lambda \wedge X) = (\lambda \wedge J(X)) \cup J(\varnothing)$$

and, by setting $X = S$ and s^λ the map constantly equal to λ,

$$J^*(s^\lambda) = s^\lambda \cup J(\varnothing).$$

Proof. In (4.7) set v equal to the characteristic function of X, and $\mu = 0$. ∎

COROLLARY 4.6. *Let f: $U \to [a, 1]$ be a continuous bijective increasing map, then $J^*(f \circ v) = f(J^*(v))$. As a consequence, for every fixed point v of J^*, $f \circ v$ is a fixed point, too.*

Proof. Since $f(1) = 1$, for every $x \in J(\varnothing)$ we have that $f(J^*(v)(x)) = f(1) = 1$ and this proves that $J(\varnothing)$ is contained in $f(J^*(v))$. Let v be a fixed point, then $J^*(f \circ v) = f(J^*(v)) = f \circ v$. ∎

5. EXTENDING ALGEBRAIC CLOSURE OPERATORS

Recall that if $J\colon \mathscr{P}(S) \to \mathscr{P}(S)$ is a closure operator, then J is called *algebraic* if, for every subset X of S,

$$J(X) = \bigcup \left\{ J(X_f) \mid X_f \text{ is a finite part of } X \right\}.$$

Moreover, a closure system $\mathscr{C}$ of subsets of S is called *algebraic* if the union of every chain of elements of $\mathscr{C}$ belongs to $\mathscr{C}$. It is immediate to prove that, for every closure system $\mathscr{C}$,

$$J_{\mathscr{C}} \text{ is algebraic} \Leftrightarrow \mathscr{C} \text{ is algebraic}.$$

and that, for every closure operator J,

$$J \text{ is algebraic} \Leftrightarrow \mathscr{C}_J \text{ is algebraic}.$$

In this section we will examine the canonical extensions of the algebraic operators.

PROPOSITION 5.1. *Let $J\colon \mathscr{P}(S) \to \mathscr{P}(S)$ be algebraic then*

$$J^*(s)(x) = \begin{cases} 1 & \text{if } x \in J(\emptyset) \\ \operatorname{Sup}\{ s(x_1) \wedge \cdots \wedge s(x_n) \mid \\ \quad x \in J(\{x_1, \ldots, x_n\})\} & \text{otherwise}, \end{cases} \tag{5.1}$$

and

$$O(J^*(s), \mu) = J(O(s, \mu)). \tag{5.2}$$

Proof. If $x \in J(\emptyset)$ it is immediate that $J^*(s)(x) = 1$. Let $\lambda \in U$ and assume that $C(s, \lambda) \neq \emptyset$, then, since J is algebraic,

$$\begin{aligned} J(C(s, \lambda)) &= \bigcup \left\{ J(\{x_1, \ldots, x_n\}) \mid \{x_1, \ldots, x_n\} \subseteq C(s, \lambda) \right\} \\ &= \bigcup \left\{ J(\{x_1, \ldots, x_n\}) \mid s(x_1) \geq \lambda, \ldots, s(x_n) \geq \lambda \right\} \\ &= \bigcup \left\{ J(\{x_1, \ldots, x_n\}) \mid s(x_1) \wedge \cdots \wedge s(x_n) \geq \lambda \right\}. \end{aligned}$$

As a consequence, if $x \notin J(\emptyset)$, we have

$$\begin{aligned} J^*(s)(x) &= \operatorname{Sup}\{ \lambda \in U \mid x \in J(C(s, \lambda)) \} \\ &= \operatorname{Sup}\{ \lambda \in U \mid \exists x_1, \ldots, \exists x_n s(x_1) \wedge \cdots \wedge s(x_n) \geq \lambda \text{ and} \\ &\quad x \in J(\{x_1, \ldots, x_n\}) \} \\ &= \operatorname{Sup}\{ s(x_1) \wedge \cdots \wedge s(x_n) \mid x \in J(\{x_1, \ldots, x_n\}) \}. \end{aligned}$$

To prove (5.2) observe that, since the family $(C(s,\lambda))_{\lambda>\mu}$ is a chain

$$O(J^*(s),\mu) = \bigcup_{\lambda>\mu} J(C(s,\lambda)) = J\left(\bigcup_{\lambda>\mu} C(s,\lambda)\right)$$
$$= J(O(s,\mu)). \quad \blacksquare$$

6. EXAMPLES OF CANONICAL EXTENSIONS

In this section we will expose some examples of canonical extensions.

The Natural Fuzzy Topologies. A typical example of a nonalgebraic classical closure system is furnished by the class $\mathscr{C}$ of the closed sets of a topological space. Namely, if (S,τ) is a topological space, we denote by $\mathscr{C}$ the class of the closed subsets in (S,τ) and therefore, for every subset X of S, by $J_{\mathscr{C}}(X)$ the topological closure of X. It is immediate to see that the canonical extension $\mathscr{C}^*$ of $\mathscr{C}$ coincides with the class of the upper semicontinuous fuzzy subsets. Now, the class $\tau^* = \{s \mid -s \in \mathscr{C}^*\}$ is a fuzzy topology that was examined in Conrad [6] under the name of *natural fuzzy topology*. So, $\mathscr{C}^*$ is the class of closed subsets of τ^* and, for every fuzzy subset s, $J_{\mathscr{C}^*}(s)$ is the topological closure $\bar{s}$ of s. Proposition 4.3 enables us to find a simple formula to calculate $\bar{s}$.

PROPOSITION 6.1. *Let (S,τ) be a Fréchet topological space and s a fuzzy subset of S, then the topological closure $\bar{s}$ of s in the natural fuzzy topology τ^* is given by*

$$\bar{s}(x) = \operatorname{Sup}\{\operatorname{Inf}_{n\in N} s(x_n) \mid (x_n)_{n\in N} \text{ is a sequence s.t. } x = \lim x_n\}. \quad (6.1)$$

Proof. By (4.4) $\bar{s} = (J_{\mathscr{C}})^*(s)$, that is,

$$\bar{s}(x) = \operatorname{Sup}\{\lambda \in U \mid x \text{ is adherent to } C(s,\lambda)\}.$$

Now, x is adherent to $C(s,\lambda)$ if and only if a sequence $(x_n)_{n\in N}$ exists such that $x = \lim x_n$ and $s(x_n) \geq \lambda$ for every $n \in N$. This completes the proof. $\blacksquare$

The natural fuzzy topologies enable us to show that in (4.2) we cannot set equality in the place of inclusion, in general. Indeed, let τ be the usual topology in the interval $[0,1]$ and let s: $[0,1] \to [0,1]$ be the fuzzy subset defined by setting $s(x) = x$ if $x \neq 1$ and $s(x) = 0$ if $x = 1$. Since, for every $\lambda \neq 1$, $J_{\mathscr{C}}(C(s,\lambda)) = J_{\mathscr{C}}([\lambda,1)) = [\lambda,1]$, we have that

$$(J_{\mathscr{C}})^*(s)(x) = \operatorname{Sup}\{\lambda \in U \mid x \in J_{\mathscr{C}}(C(s,\lambda))\}$$
$$= \operatorname{Sup}\{\lambda \in U \mid \lambda \leq x,\ \lambda \neq 1\} = x,$$

and therefore $(J_{\mathscr{C}})^*(s)$ is the identity map. Then, while $J_{\mathscr{C}}(C(s,1)) = J_{\mathscr{C}}(\emptyset) = \emptyset$, we have $C((J_{\mathscr{C}})^*(s),1) = \{1\}$ and while $J_{\mathscr{C}}(O(s,0)) = J_{\mathscr{C}}((0,1]) = [0,1]$ it is $O((J_{\mathscr{C}})^*(s),0) = (0,1]$. Thus

$$C((J_{\mathscr{C}})^*(s),1) \neq J_{\mathscr{C}}(C(s,1)) \quad \text{and} \quad O((J_{\mathscr{C}})^*(s),0) \neq J_{\mathscr{C}}(O(s,0)).$$

Rough Sets. Let $\equiv$ be an equivalence relation in a set S, and, for every $x \in S$, denote by $[x]$ the complete class of equivalence modulo $\equiv$. Then, the *upper approximation* of a subset X of S is the set

$$U(X) = \{x \in S \mid [x] \cap X \neq \emptyset\}.$$

The *lower approximation* is defined by setting

$$L(X) = \{x \in S \mid [x] \subseteq X\}.$$

It is immediate that $L(X) = -U(-X)$ and that U is a closure operator. Also, by setting, for every $X, Y \in \mathscr{P}(S)$,

$$X \equiv Y \Leftrightarrow U(X) = U(Y) \quad \text{and} \quad L(X) = L(Y)$$

we are able to extend $\equiv$ to $\mathscr{P}(S)$. We call *rough set* any equivalence class modulo $\equiv$ (see Pawlak [13]). Now it is immediate to prove that the canonical extension of U can be obtained by the formula

$$U^*(s)(x) = \operatorname{Sup}\{s(y) \mid y \equiv x\}.$$

On the other hand, it is natural to define L^* by setting $L^*(s)$ equal to the complement of the fuzzy subset $U^*(-s)$ and therefore to set

$$L^*(s)(x) = \operatorname{Inf}\{s(y) \mid y \equiv x\}.$$

Such definitions suggest the possibility of defining a theory of the fuzzy rough subsets. Indeed, we define in $\mathscr{F}(S)$ an equivalence relation by setting

$$s_1 \equiv s_2 \Leftrightarrow U^*(s_1) = U^*(s_2) \quad \text{and} \quad L^*(s_1) = L^*(s_2)$$

for every $s_1, s_2 \in \mathscr{F}(S)$ and we call *rough fuzzy subset* any class of equivalence.

Convex fuzzy subsets. Assume that $\mathscr{C}$ is the class of convex subsets of a Euclidean space $\mathbb{E}$ and therefore that, for every subset X of $\mathbb{E}$, $J_{\mathscr{C}}(X)$ is the convex envelope of X. Then, $\mathscr{C}^*$ is the class of *convex* fuzzy subsets as defined in Zadeh [16] and, for every fuzzy subset s of $\mathbb{E}$, $J_{\mathscr{C}^*}(s)$ is the convex envelope of s.

PROPOSITION 6.2. *For every fuzzy subset s of a Euclidean space, the convex envelope $J_{\mathscr{C}*}(s)$ of s is given by*

$$J_{\mathscr{C}*}(s)(x) = \operatorname{Sup}\{s(x_1) \wedge \cdots \wedge s(x_n) \mid x = \lambda_1 x_1 + \cdots + \lambda_n x_n, \\ \lambda_1, \ldots, \lambda_n \in U, \lambda_1 + \cdots + \lambda_n = 1\}. \quad (6.2)$$

Proof. By Proposition 4.3 $J_{\mathscr{C}*}(s) = (J_{\mathscr{C}})^*(s)$ and therefore, since $J_{\mathscr{C}}$ is algebraic, we may apply Proposition 5.1 to compute $J_{\mathscr{C}*}(s)$. Thus (6.2) is an immediate consequence of equality $J_{\mathscr{C}}(\emptyset) = \emptyset$ and of the fact that $x \in J_{\mathscr{C}}(\{x_1, \ldots, x_n\})$ if and only if $\lambda_1, \ldots, \lambda_n \in U$ exist such that $\lambda_1 + \cdots + \lambda_n = 1$ and $x = x_1\lambda_1 + \cdots + \lambda_n x_n$. ∎

If $\mathbb{E}$ is the real line, the convex fuzzy subsets are known under the name of *convex fuzzy numbers*. Simple calculations enable us to prove that the convex fuzzy number $J_{\mathscr{C}*}(s)$ generated by s is given by

$$\begin{aligned} J_{\mathscr{C}*}(s)(x) &= \operatorname{Sup}\{s(x_1) \wedge s(x_2) \mid x_1 \le x \le x_2\} \\ &= (\operatorname{Sup}\{s(x_1) \mid x_1 \le x\}) \wedge (\operatorname{Sup}\{s(x_2) \mid x \le x_2\}). \end{aligned}$$

Note that, $J_{\mathscr{C}}$ being algebraic, $O(J_{\mathscr{C}}^*(v), \lambda) = J_{\mathscr{C}}(O(v, \lambda))$, but $C(J_{\mathscr{C}}^*(v), \lambda) \neq J_{\mathscr{C}}(C(v, \lambda))$, in general. Indeed, define a fuzzy subset s of the real line by setting $s(x) = -|x| + 1$ if $x \in [-1, 1] - \{0\}$ and $s(x) = 0$ otherwise. Then, for every $\lambda \neq 0$, $C(s, \lambda) = [\lambda - 1, 1 - \lambda] - \{0\}$ and therefore $J_{\mathscr{C}}(C(s, \lambda)) = [\lambda - 1, 1 - \lambda]$. Since

$$C(J_{\mathscr{C}}^*(s), 1) = \bigcap_{\lambda < 1} J_{\mathscr{C}}(C(s, \lambda)) = \{0\} \qquad \text{and}$$

$$J_{\mathscr{C}}(C(s, 1)) = J_{\mathscr{C}}(\emptyset) = \emptyset$$

we may conclude that $C(J_{\mathscr{C}}^*(v), 1) \neq J_{\mathscr{C}}(C(v, 1))$.

Generalized Necessities. Let B be a Boolean algebra with minimum **0** and maximum **1** and assume that $\mathscr{C}$ is the class of filters **B** and therefore, for every $X \subseteq \mathbf{B}$, that $J_{\mathscr{C}}(X)$ is the filter generated by X. It is well known that $\mathscr{C}$ is an algebraic closure system and therefore that $J_{\mathscr{C}}$ is an algebraic closure operator. We call *fuzzy filters* the elements of $\mathscr{C}^*$ and therefore, for every fuzzy subset s of **B**, $J_{\mathscr{C}*}(s)$ is the *fuzzy filter generated by s*. Now, recall that a *generalized necessity* is any map $n\colon \mathbf{B} \to U$ such that

$$n(\mathbf{1}) = 1; \qquad n(x \wedge y) = n(x) \wedge n(y),$$

for every $x, y \in \mathbf{B}$ (see Biacino and Gerla [3]). The name "generalized necessity" is justified by the fact that the generalized necessities n for which $n(\mathbf{0}) = 0$ are known in literature under the name of *necessities* (see Dubois and Prade [8]).

Proposition 6.3. *The fuzzy filters coincide with the generalized necessities.*

Proof. Let n be a fuzzy filter, then, since $C(n,1)$ is a filter, $\mathbf{1} \in C(n,1)$ and therefore $n(\mathbf{1}) = 1$. Also, given $x, y \in \mathbf{B}$, set $\lambda = n(x) \wedge n(y)$, then, since x and y are elements of the filter $C(n, \lambda)$, $x \wedge y \in C(n, \lambda)$ and therefore $n(x \wedge y) \geq \lambda = n(x) \wedge n(y)$. On the other hand, if we set $\lambda = n(x \wedge y)$, since $x \geq x \wedge y$ and $x \wedge y \in C(n, \lambda)$, $x \in C(n, \lambda)$, and therefore $n(x) \geq \lambda = n(x \wedge y)$. Likewise one proves that $n(y) \geq n(x \wedge y)$ and therefore $n(x) \wedge n(y) \geq n(x \wedge y)$.

Conversely, assume that n is a generalized necessity, then, since $n(\mathbf{1}) = 1$, $\mathbf{1} \in C(n, \lambda)$ for every $\lambda \in U$. Also, if $x, y \in C(n, \lambda)$, then $n(x \wedge y) = n(x) \wedge n(y) \geq \lambda$ and therefore $x \wedge y \in C(n, \lambda)$. Finally, if $x \in C(n, \lambda)$ and $y \geq x$, then $n(x) = n(x \wedge y) = n(x) \wedge n(y)$ and therefore $n(y) \geq n(x)$. Then $y \in C(n, \lambda)$ and therefore $C(n, \lambda)$ is a filter. This proves that n is a fuzzy filter. ∎

Proposition 6.4. *For every fuzzy subset s the fuzzy filter (that is, the generalized necessity) $J_{\mathscr{C}*}(s)$ generated by s is given by*

$$J_{\mathscr{C}*}(s)(x) = \begin{cases} \operatorname{Sup}\{s(x_1) \wedge \cdots \wedge s(x_m) \mid x_1 \wedge \cdots \wedge x_m \leq x\} & \text{if } z \neq \mathbf{1} \\ 1 & \text{if } z = \mathbf{1}. \end{cases} \tag{6.3}$$

If s is consistent, that is, a necessity containing s exists, then the above formula gives the necessity generated by s.

Proof. Note that $J_{\mathscr{C}}$ is algebraic, that $J_{\mathscr{C}}(\emptyset) = \{\mathbf{1}\}$, and that $x \in J_{\mathscr{C}}(\{x_1, \ldots, x_m\})$ if and only if $x_1 \wedge \cdots \wedge x_m \leq x$. Then, on account of (4.4), (6.3) is a consequence of Proposition 5.1. Let n be a necessity containing s, then it is immediate that $J_{\mathscr{C}*}(s)(\mathbf{0}) \leq n(\mathbf{0}) = 0$ and therefore that $J_{\mathscr{C}*}(s)$ is a necessity. ∎

Other interesting applications can be obtained by applying the extension principle to the deduction operators of the classical deductive systems. The resulting fuzzy logics are extensively examined in [10].

7. ANOTHER EXAMPLE: THE FUZZY SUBALGEBRAS

In the following $\mathscr{A} = (\mathbb{A}, \mathbb{H}, \mathbb{C})$ denotes an algebraic structure, where $\mathbb{A}$ is the domain, $\mathbb{H}$ is the set of operations on $\mathbb{A}$, and $\mathbb{C} \subseteq \mathbb{A}$ is the set of constants. Assume that $\mathscr{C}$ is the class of subalgebras of $\mathscr{A}$, where, if there is no constant the empty subset is considered as a subalgebra, then $\mathscr{C}$ is an algebraic closure system and, for every subset X of $\mathbb{A}$, $J_{\mathscr{C}}(X)$ is the subalgebra of $\mathscr{A}$ generated by X. In accordance with the literature, we write $\langle X \rangle$ instead of $J_{\mathscr{C}}(X)$. Then, the elements of $\mathscr{C}^*$ are the fuzzy subsets whose closed cuts are subalgebras of $\mathscr{A}$ and are well known in literature under the name of *fuzzy subalgebras* (Rosenfeld [14] and Di Nola and Gerla [7]). Moreover, for every fuzzy subset s of $\mathbb{A}$, $J_{\mathscr{C}^*}(s)$ is the fuzzy subalgebra generated by s and we denote it by $\langle s \rangle$.

PROPOSITION 7.1. *Denote by* Pol($\mathscr{A}$) *the set of polynomial functions of* $\mathscr{A}$, *then, for every fuzzy subset s of* $\mathbb{A}$, *the fuzzy subalgebra generated by s is given by*

$$\langle s \rangle(x) = \begin{cases} \operatorname{Sup}\{s(x_1) \wedge \cdots \wedge s(x_n) \mid \\ \quad p(x_1, \ldots, x_n) = x,\, p \in \operatorname{Pol}(\mathscr{A})\} & \text{if } x \notin \langle \mathbb{C} \rangle \\ 1 & \text{if } x \in \langle \mathbb{C} \rangle. \end{cases} \tag{7.1}$$

Proof. $J_{\mathscr{C}}$ is algebraic and $J_{\mathscr{C}}(\emptyset) = \langle \mathbb{C} \rangle$. Moreover, recall that $x \in J_{\mathscr{C}}(\{x_1, \ldots, x_n\})$ if and only if an $n$ary $p \in \operatorname{Pol}(\mathscr{A})$ exists such that $x = p(x_1, \ldots, x_n)$. ∎

Formula (7.1) becomes very simple if we consider classes of algebraic structures in which the polynomial function can be reduced to a canonical form. As an example, if $\mathscr{A}$ is a semigroup then the fuzzy subsemigroup $\langle s \rangle$ generated by s is obtained by

$$\langle s \rangle(x) = \begin{cases} \operatorname{Sup}\{s(x_1) \wedge \cdots \wedge s(x_n) \mid x_1 \ldots x_n = x\} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1. \end{cases}$$

If $\mathscr{A}$ is a group then the fuzzy subgroup $\langle s \rangle$ generated by s is given by

$$\langle s \rangle(x) = \begin{cases} \operatorname{Sup}\{s(x_1) \wedge \cdots \wedge s(x_n) \mid \\ \quad x_1^{i_1} \cdots x_n^{i_n} = x,\, i_1, \ldots, i_n \in \{1, -1\}\} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1. \end{cases}$$

(see Biacino and Gerla [2]). Given a free semigroup $\mathscr{A}$, we obtain further examples of fuzzy closure operators by considering the classes of the free, pure, very pure, left unitary, right unitary, unitary fuzzy subsemigroups of $\mathscr{A}$ (see Gerla [9]).

8. FUZZY CLOSURE SYSTEM ASSOCIATED WITH A FAMILY OF CLOSURE SYSTEMS

Very simple examples of closure systems in $\mathscr{P}(S)$ are obtained by considering the principal filters in the lattice $\mathscr{P}(S)$, that is, classes such as

$$\mathscr{C}_A = \{X \in \mathscr{P}(S) \mid X \supseteq A\}$$

where A is any fixed subset of S. Likewise, examples of fuzzy closure systems are furnished by the principal filters in the lattice $\mathscr{F}(S)$, that is classes as

$$\mathscr{C}_v = \{s \in \mathscr{F}(S) \mid s \supseteq v\}$$

where v is any fixed fuzzy subset of S. Now, *are such fuzzy closure systems canonical extensions of a classical one*? As a matter of fact, the answer is negative everywhere v is not crisp. Indeed, the class of crisp elements of $\mathscr{C}_v$ coincides with $\mathscr{C} = \{X \in \mathscr{P}(S) \mid X \supseteq \mathrm{Supp}(v)\}$ and since $\mathrm{Supp}(v)$ is not contained in $C(v, 1)$, $v \notin \mathscr{C}^*$. In spite of that, it is possible to obtain $\mathscr{C}_v$ by a formula very like to formula (4.3). In fact, if we set $\mathscr{C}_\lambda = \{X \mid X \supseteq C(v, \lambda)\}$ then $(\mathscr{C}_\lambda)_{\lambda \in U}$ is an (order-preserving) family of closure systems such that

$$\mathscr{C}_v = \{s \in \mathscr{F}(S) \mid C(s, \lambda) \in \mathscr{C}_\lambda \text{ for every } \lambda \in U\}.$$

This suggests the following definition.

DEFINITION 8.1. Let $(\mathscr{C}_\lambda)_{\lambda \in U}$ be a family of closure systems and set

$$\mathscr{C} = \{s \in \mathscr{F}(S) \mid C(s, \lambda) \in \mathscr{C}_\lambda \text{ for every } \lambda \in U\} \tag{8.1}$$

then we say that $\mathscr{C}$ is the *fuzzy closure system associated* to $(\mathscr{C}_\lambda)_{\lambda \in U}$.

This terminology is justified by the following proposition showing that $\mathscr{C}$ is a closure system.

PROPOSITION 8.2. *The class $\mathscr{C}$ of fuzzy subsets associated with a family $(\mathscr{C}_\lambda)_{\lambda \in U}$ of closure systems is a fuzzy closure system.*

Proof. Let $(s_i)_{i \in I}$ be a family of elements of $\mathscr{C}$, then, since $C(\bigcap s_i, \lambda) = \bigcap C(s_i, \lambda) \in \mathscr{C}_\lambda$ for every $\lambda \in U$, we have $\bigcap s_i \in \mathscr{C}$. ∎

Obviously, (8.1) generalizes the formula for the canonical extension of a classical closure system.

PROPOSITION 8.3. *Let $\mathscr{C}$ be a fuzzy closure system, and set, for every $\lambda \in U$,*

$$\mathscr{H}(\mathscr{C}, \lambda) = \{C(s, \lambda) \mid s \in \mathscr{C}\}. \tag{8.2}$$

Then $(\mathscr{H}(\mathscr{C}, \lambda))_{\lambda \in U}$ is a family of closure systems.

Proof. Let $(X_i)_{i \in I}$ be a family of elements of $\mathscr{H}(\mathscr{C}, \lambda)$, then a family $(s_i)_{i \in I}$ of elements of $\mathscr{C}$ exists such that $X_i = C(s_i, \lambda)$. Since $\bigcap X_i = C(\bigcap s_i, \lambda)$, $\bigcap X_i$ belongs to $\mathscr{H}(\mathscr{C}, \lambda)$. ∎

DEFINITION 8.4. Given a fuzzy closure system $\mathscr{C}$, we denote by $\mathscr{C}^*$ the fuzzy closure system associated with the family $\mathscr{C}_\lambda = \mathscr{H}(\mathscr{C}, \lambda)$ and we say that $\mathscr{C}^*$ is the *fuzzy closure system associated with $\mathscr{C}$*. In other words,

$$\mathscr{C}^* = \{s \in \mathscr{F}(S) \mid C(s, \lambda) \in \mathscr{H}(\mathscr{C}, \lambda) \text{ for every } \lambda \in U\}. \tag{8.3}$$

The following obvious proposition shows that such a notation is in accordance with formula (4.3).

PROPOSITION 8.5. *If* $\mathscr{C}$ *is a classical closure system then the fuzzy closure system associated with* $\mathscr{C}$ *by* (8.3) *coincides with the canonical extension of* $\mathscr{C}$ *defined by* (4.3).

PROPOSITION 8.6. *Let* $\mathscr{C}$, $\mathscr{C}_1$, *and* $\mathscr{C}_2$ *be fuzzy closure systems*; *then*

$$\text{(i)}\ \mathscr{C} \subseteq \mathscr{C}^*; \qquad \text{(ii)}\ \mathscr{C}_1 \subseteq \mathscr{C}_2 \Rightarrow \mathscr{C}_1^* \subseteq \mathscr{C}_2^*; \qquad \text{(iii)}\ (\mathscr{C}^*)^* = \mathscr{C}^*.$$

Moreover,

$$\mathscr{C} = \mathscr{C}^* \Leftrightarrow \mathscr{C} \text{ is associated with a family of closure systems}.$$

Proof. Properties (i), (ii), and (iii) are obvious. Assume that $\mathscr{C}$ is associated with the family $(\mathscr{C}_\lambda)_{\lambda \in U}$ of closure systems, then if $s \in \mathscr{C}^*$, for every $\lambda \in U$ an element s_λ of $\mathscr{C}$ exists such that $C(s, \lambda) = C(s_\lambda, \lambda)$. On the other hand, since $s_\lambda \in \mathscr{C}$, $C(s_\lambda, \lambda) \in \mathscr{C}_\lambda$ and this proves that $s \in \mathscr{C}$. Thus $\mathscr{C}^* \subseteq \mathscr{C}$ and by (i) we may conclude that $\mathscr{C}^* = \mathscr{C}$. The converse part is immediate. ∎

In the following, given any class $\mathscr{D}$ of fuzzy subsets, we denote by $\overline{\mathscr{D}}$ the fuzzy closure system generated by $\mathscr{D}$, that is,

$$\begin{aligned}\overline{\mathscr{D}} &= \bigcap\{\mathscr{C} \mid \mathscr{C} \text{ is a fuzzy closure system containing } \mathscr{D}\} \\ &= \left\{s \in \mathscr{F}(S) \mid s = \bigcap s_i \text{ where } (s_i)_{i \in I} \text{ is a family of elements of } \mathscr{D}\right\}.\end{aligned}$$

Also, we define the operator Q by setting

$$Q(\mathscr{C}) = \{\lambda \vee C(s, \lambda) \mid s \in \mathscr{C},\ \lambda \in U\}. \tag{8.4}$$

PROPOSITION 8.7. *Let* $\mathscr{C}$ *be a fuzzy closure system, then* $\mathscr{C}^* \subseteq \overline{Q(\mathscr{C})}$ *and, if* $(\mathscr{H}(\mathscr{C}, \lambda))_{\lambda \in U}$ *is an order-preserving family, we have* $\mathscr{C}^* = \overline{Q(\mathscr{C})}$, *that is,* $\mathscr{C}^*$ *is equal to the fuzzy closure system generated by* $Q(\mathscr{C})$.

Proof. Recall that

$$\mathscr{C}^* = \left\{s \in \mathscr{F}(S) \mid \text{ for every } \lambda \in U, C(s, \lambda) = C(s_\lambda, \lambda) \text{ for a suitable } s_\lambda \in \mathscr{C}\right\}.$$

Then, for every $s \in \mathscr{C}^*$, since by formula (2.4) $s = \bigcap \lambda \vee C(s, \lambda) = \bigcap \lambda \vee C(s_\lambda, \lambda)$, $s_\lambda \in \mathscr{C}$, we have that $s \in \overline{Q(\mathscr{C})}$. Assume that $(\mathscr{H}(\mathscr{C}, \lambda))_{\lambda \in U}$ is an order-preserving family, then since

$$C(\lambda \vee \chi_{C(s,\lambda)}, \mu) = \begin{cases} S & \text{if } \mu \le \lambda \\ C(s, \lambda) & \text{if } \mu > \lambda. \end{cases}$$

and $C(s, \lambda) \in \mathscr{H}(\mathscr{C}, \lambda) \subseteq \mathscr{H}(\mathscr{C}, \mu)$ for every $\mu > \lambda$, we may conclude that $Q(\mathscr{C}) \subseteq \mathscr{C}^*$. ∎

PROPOSITION 8.8. *If $\mathscr{C}$ is any fuzzy closure system then*

$$\mathscr{C} = \overline{Q(\mathscr{C})} \Rightarrow \mathscr{C} = \mathscr{C}^*.$$

If $(\mathscr{H}(\mathscr{C}, \lambda))_{\lambda \in U}$ is an order-preserving family, then

$$\mathscr{C} = \overline{Q(\mathscr{C})} \Leftrightarrow \mathscr{C} = \mathscr{C}^*.$$

Proof. Assume $\mathscr{C} = Q(\mathscr{C})$, then, by Propositions 8.6 and 8.7, $\mathscr{C} \subseteq \mathscr{C}^* \subseteq \overline{Q(\mathscr{C})} = \mathscr{C}$ and therefore $\mathscr{C} = \mathscr{C}^*$. Assume that $(\mathscr{H}(\mathscr{C}, \lambda))_{\lambda \in U}$ is order-preserving and that $\mathscr{C} = \mathscr{C}^*$, then by Proposition 8.7 $\mathscr{C} = \overline{Q(\mathscr{C})}$. ∎

PROPOSITION 8.9. *Assume that the closure system, $\mathscr{C}$, is associated with an order-preserving family $(\mathscr{C}_\lambda)_{\lambda \in U}$ of classical closure systems. Then,*

$$\mathscr{H}(\mathscr{C}, \lambda) = \mathscr{C}_\lambda \tag{8.5}$$

for every $\lambda \in U$.

Proof. By hypothesis $\mathscr{C} = \{s \in \mathscr{F}(S) \mid C(s, \lambda) \in \mathscr{C}_\lambda\}$. It is immediate that $\mathscr{H}(\mathscr{C}, \lambda) \subseteq \mathscr{C}_\lambda$, to prove the converse inclusion, let X be an element of $\mathscr{C}_\lambda$ and consider the fuzzy subsets $s_X = \lambda \vee \chi_X$. We have that

$$C(s_X, \mu) = \begin{cases} S & \text{if } \mu \leq \lambda \\ X & \text{if } \mu > \lambda \end{cases}$$

and, since $(\mathscr{C}_\lambda)_{\lambda \in U}$ is order-preserving, $C(s_X, \mu) \in \mathscr{C}_\mu$ for every $\mu \in U$. So, s_X belongs to $\mathscr{C}$ and this proves (8.5). ∎

Observe that (8.5) does not hold, in general. As a matter of fact such an equality is equivalent to saying that

$$\forall \lambda \in U\ \forall X \in \mathscr{C}_\lambda \text{ a continuous chain } (X_\mu)_{\mu \in U} \text{ exists}$$
$$\text{such that } X_\mu \in \mathscr{C}_\mu,\ X_\lambda = X.$$

EXAMPLES. Let S be an Euclidean space, $\mathscr{S}$ the class of closed subsets, and $\mathscr{R}$ the class of closed convex subsets of S. Since $\mathscr{R} \subseteq \mathscr{S}$, we obtain an order-preserving family $(\mathscr{C}_\lambda)_{\lambda \in U}$ by setting

$$\mathscr{C}_\lambda = \begin{cases} \mathscr{R} & \text{if } \lambda \leq 0.5 \\ \mathscr{S} & \text{otherwise.} \end{cases}$$

If $\mathscr{C}$ is the fuzzy closure system associated with this family then $s \in \mathscr{C}$ if and only if $C(s, \lambda)$ is closed and convex if $\lambda \leq 0.5$ and $C(s, \lambda)$ is closed otherwise. In accordance with Proposition 8.6 and 8.7, $\mathscr{C} = \mathscr{C}^* = \overline{Q(\mathscr{C})}$.

Note that $\mathscr{C}$ extends the class of the closed and convex subsets of S. Now, let us change $\mathscr{R}$ with $\mathscr{S}$ in defining $(\mathscr{C}_\lambda)_{\lambda \in U}$, that is, set

$$\mathscr{C}_\lambda = \begin{cases} \mathscr{S} & \text{if } \lambda \leq 0.5 \\ \mathscr{R} & \text{otherwise.} \end{cases}$$

Such a family is order-reversing and $s \in \mathscr{C}$ if and only if $C(s, \lambda)$ is closed if $\lambda \leq 0.5$ and closed and convex otherwise. Also in this case $\mathscr{C}$ is an extension of the class of the closed and convex subsets of E. Obviously, $\mathscr{C} = \mathscr{C}^*$. We claim that

$$Q(\mathscr{C}) \not\subseteq \mathscr{C}, \qquad \mathscr{H}(\mathscr{C}, \lambda) = \mathscr{C}_\lambda.$$

Indeed, let X and Y be two disjoint closed subsets such that Y is convex and define s by

$$s(x) = \begin{cases} 1 & \text{if } x \in Y \\ 0.5 & \text{if } x \in X \\ 0 & \text{otherwise.} \end{cases}$$

Then it is immediate that $s \in \mathscr{C}$ and that, if $\lambda \leq 0.5$, $\lambda \vee C(s, \lambda) = \lambda \vee (X \cup Y)$. Now, $C(\lambda \vee X \cup Y, \mu) = X \cup Y$ for every $\mu > \lambda$, and therefore $C(\lambda \vee X \cup Y, \mu) \notin \mathscr{C}_\mu$ for every $\mu > 0.5$. Thus, $\lambda \vee C(s, \lambda) \notin \mathscr{C}$ and therefore $Q(\mathscr{C}) \not\subseteq \mathscr{C}$.

To prove that $\mathscr{C}_\lambda = \mathscr{H}(\mathscr{C}, \lambda)$, let X be an element of $\mathscr{C}_\lambda$. Now if $\lambda > 0.5$ then X is closed and convex and therefore $X \in \mathscr{C}$ and $X \in \mathscr{H}(\mathscr{C}, \lambda)$. If $\lambda \leq 0.5$, it is immediate that $\lambda \wedge X$ is an element of $\mathscr{C}$ such that $X = C(\lambda \wedge X, \lambda)$ and therefore $X \in \mathscr{H}(\mathscr{C}, \lambda)$.

Set $S = [0, 1]$ and let, for every $\lambda \in U, \mathscr{C}_\lambda$ equal the closure system $\{\emptyset, [\lambda, 1], S\}$ and denote by $\mathscr{C}$ the related fuzzy closure system. The family $(\mathscr{C}_\lambda)_{\lambda \in U}$ is neither order-preserving nor order-reversing and we claim that

$$Q(\mathscr{C}) \not\subseteq \mathscr{C} \qquad \text{and} \qquad \mathscr{H}(\mathscr{C}, \lambda) = \mathscr{C}_\lambda.$$

Indeed, at first observe that the empty set and the identity map id: $U \to U$ are elements of $\mathscr{C}$. Then, the fuzzy subset $s = C(\text{id}, \lambda) \vee \lambda$ belongs to $Q(\mathscr{C})$ but, since $C(s, \mu) = [\lambda, 1] \notin \mathscr{C}_\mu$ for every $\mu > \lambda$, $s \notin \mathscr{C}$. Also, since $\emptyset$ is the λ-cut of the empty set and $[\lambda, 1]$ the λ-cut of the identity map, $\mathscr{C}_\lambda \subseteq \mathscr{H}(\mathscr{C}, \lambda)$ and therefore $\mathscr{H}(\mathscr{C}, \lambda) = \mathscr{C}_\lambda$.

This example shows that in Proposition 8.7 the hypothesis $(\mathscr{H}(\mathscr{C}, \lambda))_{\lambda \in U}$ order-preserving is essential to prove the equality. Also, it proves that in Proposition 8.8 the hypothesis $(\mathscr{H}(\mathscr{C}, \lambda))_{\lambda \in U}$ order-preserving is not necessary. If we modify such an example by setting $\mathscr{C}_{1/2} = \{\emptyset, \{0\}, [\frac{1}{2}, 1], S\}$, then it is immediate that no continuous chain $(X_\beta)_{\beta \in U}$ exists such that $X_{1/2} = \{0\}$ and $X_\beta \in \mathscr{C}_\beta$.

9. FUZZY CLOSURE OPERATOR ASSOCIATED WITH A FAMILY OF CLOSURE OPERATORS

We have early observed that there is a strict connection between the closure systems and the closure operators. So, the question arises of giving definitions and results like the ones in Section 8 but with reference to the closure operators. At first we have to introduce a new concept. We say that a fuzzy operator J is an *almost closure* operator, in brief *a-c-operator*, if it satisfies only the first two conditions for a closure operator, that is

$$\text{(i)}\ \ s_1 \subseteq s_2 \Rightarrow J(s_1) \subseteq J(s_2); \qquad \text{(ii)}\ \ J(s) \supseteq s$$

for s_1, s_2, s in $\mathscr{F}(S)$. Every a-c-operator is associated with a closure operator $\bar{J}$ defined by $\bar{J} = J_{\mathscr{C}_J}$. In other words, for every fuzzy subset s, $\bar{J}(s)$ is the least fixed point of J greater or equal to s. If J is a closure operator then $\bar{J} = J$. It is immediate that $\mathscr{C}_J = \mathscr{C}_{\bar{J}}$, that is, J and $\bar{J}$ have the same fixed points.

PROPOSITION 9.1. *Let $(J_\lambda)_{\lambda \in U}$ be a family of a-c-operators and define the fuzzy operator J by*

$$J(s)(x) = \mathrm{Sup}\{\lambda \in U \mid x \in J_\lambda(C(s, \lambda))\}, \tag{9.1}$$

then J is an a-c-operator.

Proof. It is immediate that J satisfies (i). To prove (ii), observe that, since $C(s, \lambda) \subset J_\lambda(C(s, \lambda))$, we have

$$\begin{aligned} s(x) &= \mathrm{Sup}\{\lambda \in U \mid x \in C(s, \lambda)\} \\ &\le \mathrm{Sup}\{\lambda \in U \mid x \in J_\lambda(C(s, \lambda))\} = J(s)(x). \quad \blacksquare \end{aligned}$$

DEFINITION 9.2. Let $(J_\lambda)_{\lambda \in U}$ be a family of closure operators, then we say that the operator J defined by (9.1) is the *a-c-operator associated with* $(J_\lambda)_{\lambda \in U}$ and that $\bar{J}$ is the *closure operator associated* to $(J_\lambda)_{\lambda \in U}$.

The following proposition shows that if $(J_\lambda)_{\lambda \in U}$ is order-reversing then J is a closure operator and therefore $J = \bar{J}$.

PROPOSITION 9.3. *If $(J_\lambda)_{\lambda \in U}$ is an order-reversing family of closure operators then its associated a-c-operator is a closure operator, that is, $J = \bar{J}$.*

Proof. To prove that $J(J(s)) = J(s)$ it is sufficient to prove that every cut $C(J(s), \lambda)$ is a fixed point for J_λ. Indeed, in this case

$$\begin{aligned} J(J(s))(x) &= \mathrm{Sup}\{\lambda \in U \mid x \in J_\lambda(C(J(s), \lambda))\} \\ &= \mathrm{Sup}\{\lambda \in U \mid x \in C(J(s), \lambda)\} = J(s)(x). \end{aligned}$$

Now, observe that $(J_\lambda(C(s,\lambda)))_{\lambda\in U}$ is an order-reversing family of subsets of S. In fact, if $\lambda \le \lambda'$ then $J_\lambda(C(s,\lambda)) \supseteq J_\lambda(C(s,\lambda')) \supseteq J_{\lambda'}(C(s,\lambda'))$. Also, observe that if $\mu \le \lambda$, then every fixed point for J_μ is a fixed point for J_λ. In particular, $J_\mu(C(s,\mu))$ is a fixed point for J_λ. By recalling that the intersection of a class of fixed points for J_λ is a fixed point for J_λ and that, by Lemma 2.1, $C(J(s),\lambda) = \bigcap_{\mu<\lambda} J_\mu(C(s,\mu))$, we conclude that $C(J(s),\lambda)$ is a fixed point for J_λ. ∎

Obviously, (9.1) generalizes the formula for the canonical extension of a classical closure operator. From Lemma 2.1 we may derive that, if $(J_\lambda)_{\lambda\in U}$ is order-reversing then, for every $\mu \in U$

$$O(J(s),\mu) = \bigcup_{\lambda>\mu} J_\lambda(C(s,\lambda)) \subseteq J_\mu(O(s,\mu)) \subseteq J_\mu(C(s,\mu))$$
$$\subseteq C(J(s,\mu)) = \bigcap_{\lambda<\mu} J_\lambda(C(s,\lambda)).$$

Remark. The fuzzy operator J defined by (9.1) is not an extension of a classical closure operator, in general. As a matter of fact, if $(J_\lambda)_{\lambda\in U}$ is order-reversing, then

$$J \text{ extension of a classical operator}$$
$$\Leftrightarrow J_\mu = J_\nu \text{ for every } \mu \neq 0,1 \text{ and } \nu \neq 0,1.$$

To prove this, assume that J is an extension of a classical closure operator and let ν and μ be two elements of U different from 1 and assume, for example, that $0 < \nu < \mu$. Then, given any subset X of S, by hypothesis $J_\nu(X) \supseteq J_\mu(X)$. Assume that $x \in J_\nu(X)$, then $J(X)(x) = \operatorname{Sup}\{\lambda \in U \mid x \in J_\lambda(X)\} \neq 0$ and therefore $\operatorname{Sup}\{\lambda \in U \mid x \in J_\lambda(X)\} = 1$. Since $\mu \neq 1$, this entails that $\lambda \in U$ exists such that $x \in J_\lambda(X)$ and $\lambda \ge \mu$. Since $J_\lambda(X) \subseteq J_\mu(X)$, it is also $x \in J_\mu(X)$ and we may conclude that $J_\nu(X) \subseteq J_\mu(X)$ and therefore that $J_\nu(X) = J_\mu(X)$. The converse implication is immediate.

PROPOSITION 9.4. *Let $(J_\lambda)_{\lambda\in U}$ be any family of closure operators and $\bar{J}$ the associated fuzzy closure operator. Besides, let $(\mathscr{C}_{J_\lambda})_{\lambda\in U}$ be the corresponding family of closure systems and $\mathscr{C}$ the associated fuzzy closure system. Then, $\bar{J} = J_{\mathscr{C}}$, that is*

$$\begin{array}{ccc} (J_\lambda)_{\lambda\in U} & \to & (\mathscr{C}_{J_\lambda})_{\lambda\in U} \\ \downarrow & & \downarrow \\ \bar{J} & \leftarrow & \mathscr{C} \end{array}.$$

Proof. To prove that $\bar{J} = J_{\mathscr{C}}$ is equivalent to proving that $\mathscr{C} = \mathscr{C}_{\bar{J}}$ and therefore that $\mathscr{C} = \mathscr{C}_J$. Let s be an element of $\mathscr{C}$, then every cut $C(s, \lambda) \in \mathscr{C}_{J_\lambda}$ and therefore is a fixed point for J_λ. Then it is immediate that $J(s) = s$ and therefore that $s \in \mathscr{C}_J$. Conversely, assume $J(s) = s$, then, for every $x \in S$, $\operatorname{Sup}\{\lambda \in U \mid x \in J_\lambda(C(s, \lambda))\} = s(x)$. In other words, $x \in J_\lambda(C(s, \lambda))$ implies $\lambda \leq s(x)$ and therefore $x \in C(s, \lambda)$. Then $J_\lambda(C(s, \lambda))$ is contained in $C(s, \lambda)$ and therefore $C(s, \lambda)$ is a fixed point for J_λ. Thus $s \in \mathscr{C}$. ∎

PROPOSITION 9.5. *Let $(\mathscr{C}_\lambda)_{\lambda \in U}$ be a family of closure systems and $\mathscr{C}$ the associated fuzzy closure system. Moreover, consider the corresponding family $(J_{\mathscr{C}_\lambda})_{\lambda \in U}$ of closure operators and denote by $\bar{J}$ the associated fuzzy closure operator. then, $\mathscr{C} = \mathscr{C}_{\bar{J}}$, that is,*

$$\begin{array}{ccc} (\mathscr{C}_\lambda)_{\lambda \in U} & \rightarrow & (J_{\mathscr{C}_\lambda})_{\lambda \in U} \\ \downarrow & & \downarrow \\ \mathscr{C} & \leftarrow & \bar{J}. \end{array}$$

Proof. Recall that by (3.3) of Proposition 3.1 the closure system corresponding to closure operator $J_{\mathscr{C}_\lambda}$ is $\mathscr{C}_\lambda$. As a consequence, by applying Proposition 9.4 to the family $(J_{\mathscr{C}_\lambda})_{\lambda \in U}$, we obtain that $\bar{J} = J_{\mathscr{C}}$ and therefore that $\mathscr{C} = \mathscr{C}_{\bar{J}}$. ∎

Given a fuzzy closure operator J, we may define a family $(K(J, \lambda))_{\lambda \in U}$ of operators by setting

$$K(J, \lambda)(X) = C(J(\lambda \wedge X), \lambda) \tag{9.2}$$

for every $\lambda \in U$. If we interpret J as a deduction operator, then $K(J, \lambda)(X)$ is the set of formulas that are consequences at degree λ of the formulas in X assumed at degree λ. The following proposition shows that K and $\mathscr{H}$ are related in accordance with the following diagrams.

$$\begin{array}{ccc} J & \rightarrow & \mathscr{C}_J \\ \downarrow & & \downarrow \\ K(J, \lambda) & \leftarrow & \mathscr{H}(\mathscr{C}_J, \lambda) \end{array} \qquad \begin{array}{ccc} \mathscr{C} & \rightarrow & J_{\mathscr{C}} \\ \downarrow & & \downarrow \\ \mathscr{H}(\mathscr{C}, \lambda) & \leftarrow & K(J_{\mathscr{C}}, \lambda) \end{array}.$$

PROPOSITION 9.6. (Castro [5]). *Given a fuzzy closure operator J, $(K(J, \lambda))_{\lambda \in U}$ is a family of closure operators. Namely, we have*

$$K(J, \lambda) = J_{\mathscr{H}(\mathscr{C}_J, \lambda)}. \tag{9.3}$$

Given a fuzzy closure system $\mathscr{C}$, *we have*

$$\mathscr{H}(\mathscr{C}, \lambda) = \mathscr{C}_{K(J_{\mathscr{C}}, \lambda)}. \tag{9.4}$$

Proof. Let X be a subset of S and assume that $x \in J_{\mathscr{H}(\mathscr{C}_J, \lambda)}(X)$. Then $x \in C(s, \lambda)$ for every $s \in \mathscr{C}_J$ such that $X \subseteq C(s, \lambda)$. Taking $s = J(\lambda \wedge X)$, since $s \in \mathscr{C}_J$ and

$$X \subseteq C(\lambda \wedge X, \lambda) \subseteq C(J(\lambda \wedge X), \lambda) = C(s, \lambda),$$

we have $x \in C(J(\lambda \wedge X), \lambda) = K(J, \lambda)(X)$. Conversely, assume that $x \in K(J, \lambda)(X)$, then $J(\lambda \wedge X)(x) \geq \lambda$ and therefore, for any $s \in \mathscr{C}_J$ such that $s \supseteq \lambda \wedge X$, we have $x \in C(s, \lambda)$. Thus, since $s \supseteq \lambda \wedge X$ if and only if $C(s, \lambda) \supseteq X$, for every $s \in \mathscr{C}_J$ such that $C(s, \lambda) \supseteq X$, we have $x \in C(s, \lambda)$. This proves that $x \in J_{\mathscr{H}(\mathscr{C}_J, \lambda)}(X)$.

To prove (9.4), we apply (9.3) to the fuzzy closure operator $J_{\mathscr{C}}$ by obtaining

$$K(J_{\mathscr{C}}, \lambda) = J_{\mathscr{H}(\mathscr{C}, \lambda)}$$

that is equivalent to (9.4). ∎

We conclude this section by noting that, like the fuzzy closure systems, given a fuzzy closure operator J it is possible to build up a new fuzzy closure operator J^* by applying Definition 9.2 to the family $(K(J, \lambda))_{\lambda \in U}$.

10. TWO EXAMPLES: FUZZY PREORDERS AND FUZZY HERBRAND MODELS

Fuzzy Preorder. As observed by Trillas and Alsina, if $(S, \leq)$ is any preorder then the equality

$$J^{\leq}(X) = \{z \in S \mid \exists x \in X, x \leq z\} \tag{10.1}$$

defines a closure operator $J^{\leq}: \mathscr{P}(S) \to \mathscr{P}(S)$. The idea is that $x \leq z$ means that x entails z. Now, we can extend the notion of preorder as follows. A *fuzzy preorder* (relative to $\wedge$) is a fuzzy relation $I: S \times S \to [0, 1]$ in S such that

(a) $I(x, x) = 1$ (reflexivity)

(b) $I(x, y) \wedge I(y, z) \leq I(x, z)$ (transitivity).

If $I(x, y) = \lambda$ we say that x *implies* y *at degree* λ. This definition immediately suggests the question of defining the fuzzy closure operator associated with a fuzzy preorder as in (10.1).

PROPOSITION 10.1. *Given a fuzzy preorder I, every cut $R_\lambda = \{(x, y) \mid I(x, y) \geq \lambda\}$ is a preorder and therefore defines a closure operator J_λ. The fuzzy closure operator associated with the family $(J_\lambda)_{\lambda \in [0,1]}$ can be defined by setting*

$$J(s)(z) = \mathrm{Sup}\{s(x) \wedge I(x, z) \mid x \in S\} \tag{10.2}$$

for every $s \in \mathscr{F}(S)$ and $x \in S$.

Herbrand Models. Recall that if $\mathscr{L}$ is a first order language with some constants, then a *ground term* of $\mathscr{L}$ is a term not containing variables and the *Herbrand universe* $U_{\mathscr{L}}$ for $\mathscr{L}$ is the set of ground terms of $\mathscr{L}$. Similarly, a *ground atom* is an atomic formula not containing variables and the set $B_{\mathscr{L}}$ of ground atoms is called the *Herbrand base* for $\mathscr{L}$. An *Herbrand interpretation* for $\mathscr{L}$ is any subset M of $B_{\mathscr{L}}$ and the name is justified by the fact that M defines an interpretation of $\mathscr{L}$ in which:

—the domain is the Herbrand universe

—constants in $\mathscr{L}$ are assigned themselves

—any $n$ary function symbol f in $\mathscr{L}$ is interpreted as the map from $(U_{\mathscr{L}})^n$ into $U_{\mathscr{L}}$ defined by associating to terms $t_1, \ldots, t_n$ the term $f(t_1, \ldots, t_n)$

—any $n$ary predicate symbol p is interpreted by $\{(t_1, \ldots, t_n) \mid p(t_1, \ldots, t_n) \in M\}$.

A *definite program clause* is either an atom or a formula of the form $\beta_1 \wedge \cdots \wedge \beta_n \to \beta$ where β is an atom and each β_i is an atom or a negation of an atom. A *ground instance* of a program clause is a closed formula obtained from this clause by suitable substitutions of the free variables by closed terms. A *definite program* is a finite set P of definite program clauses. It is well known that the class $\mathscr{C}_P$ of Herbrand models for P is a closure system in $B_{\mathscr{L}}$ and that, if J_P is the associated closure operator, then for every $X \in \mathscr{P}(B_{\mathscr{L}})$

$$J_P(X) = \{\alpha \in B_{\mathscr{L}} \mid P \cup X \vdash \alpha\}.$$

$J_P(X)$ is named the *least Herbrand model* for P containing X and $J_P(0)$ the *least Herbrand model* for P; we denote it by M_P. The Herbrand models of P coincide with the fixed points of J_P, obviously.

Passing to the fuzzy framework, we call *fuzzy program* any fuzzy subset p of definite program clauses. Now, for every $\lambda \in U$ the cut $C(p, \lambda)$ is a classical program and we may consider the closure operator $J_\lambda = J_{C(p,\lambda)}$ defined by this program and the related class $\mathscr{C}_\lambda$ of Herbrand models. Then, it is natural to consider the fuzzy closure operator $J_P : \mathscr{F}(B_l) \to \mathscr{F}(B_l)$

defined by the order-reversing family $(J_\lambda)_{\lambda \in U}$ and, in correspondence, the closure system $\mathscr{C}_p$ associated with the order-preserving family $(\mathscr{C}_\lambda)_{\lambda \in U}$. We call *fuzzy Herbrand model* for p every fixed point of J_p or, equivalently, every element of $\mathscr{C}_p$. In other words, a fuzzy subset s is a fuzzy Herbrand model for p if and only if every cut $C(s, \lambda)$ is an Herbrand model for $C(p, \lambda)$. Also we say that $J_p(s)$ is the *minimal fuzzy Herbrand model containing* s and, in particular, we call the *least fuzzy Herbrand model* M_p of p the fuzzy subset $J_p(0)$. It is immediate that

$$J^*(s)(\alpha) = \mathrm{Sup}\{\lambda \in U \mid C(s \cup p, \lambda) \vdash \alpha\}. \tag{10.3}$$

From the point of view of expert systems theory, we may interpret the number $J^*(s)(\alpha)$ as a valuation of the truth degree of α, given the "general theory" p and the available fuzzy information s. In this case it is very natural to assume that both s and p are finite, and therefore if $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ are the elements of the codomain of $s \cup p$ different from zero, in (10.3) we have to consider only the programs $C(s \cup p, \lambda_1), \dots, C(s \cup p, \lambda_n)$.

REFERENCES

1. L. Biacino, Generated envelopes, *J. Math. Anal. Appl.* **172** (1993), 179–190.
2. L. Biacino and G. Gerla, Closure systems and L-subalgebras, *Inform. Sci.* **32** (1984), 181–195.
3. L. Biacino and G. Gerla, Generated necessities and possibilities, *Internat. J. Intelligent Systems* **7** (1992), 445–454.
4. L. Biacino and G. Gerla, Closure operators for fuzzy subsets, *in* "EUFIT '93, First European Congress on Fuzzy and Intelligent Technologies, Aachen, 1993," pp. 1441–1447.
5. J. L. Castro, Fuzzy logics as families of bivalued logics, unpublished paper.
6. F. Conrad, Fuzzy topological concepts, *J. Math. Anal. Appl.* **74** (1980), 432–440.
7. A. Di Nola and G. Gerla, Lattice valued algebras, *Stochastica* **11** (1987) 137–150.
8. D. Dubois and H. Prade, "Possibility Theory," Plenum, New York/London, 1988.
9. G. Gerla, Code theory and fuzzy subsemigroups, *J. Math. Anal. Appl.* **128** (1987), 362–369.
10. G. Gerla, An extension principle for fuzzy logic, *Math. Logic Quart.*, to appear.
11. V. Murali, Lattice of fuzzy subalgebras and closure systems in I^X, *Fuzzy Sets and Systems* **41** (1991), 101–111.
12. C. V. Negoita and D. A. Ralescu, Representation theorems for fuzzy concepts, *Kybernetes* **4** (1975), 169–174.
13. Z. Pawlak, Rough sets, *Internat. Inform. and Comput. Sci.* **11** (1982), 341–356.
14. A. Rosenfeld, Fuzzy groups, *J. Math. Anal. Appl.* **35** (1971), 512–517.
15. E. Trillas and C. Alsina, Logic: Going farther from Tarski?, *Fuzzy Sets and Systems* **53** (1993), 1–13.
16. L. A. Zadeh, Fuzzy Sets, *Inform. and Control* **12** (1965), 338–353.