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Fuzzy subgroups and similarities

F. Formato, G. Gerla, L. Scarpati

Abstract Given a set S, we show that there is a strict relation between the notion of similarity on S and the one of fuzzy subgroup of transformations in S. Such a relation enables us to extablish a connection between fuzzy subgroups and distances.

Key words Similarity, Fuzzy subgroups, distances.

1 Introduction

Just as the notion of fuzzy subset generalizes that of the classical subset, the concept of similarity can be considered as a many-valued generalization of the classical notion of equivalence. The definition of fuzzy relation from X to Y as a fuzzy subset of $X \times Y$ was first proposed by Zadeh (see for example [14]). Subsequently, many authors, such as Chakraborty and Das [2] Valverde, Trillas and Jacas [5, 12, 13], Ovchinikov [8, 9] have widely studied the similarities in various contexts. Another basic notion is the one of fuzzy subgroups proposed by Rosenfeld in [10].

In this paper we study the link existing between the notion of fuzzy subgroup and the one of similarity. In fact, given a set S, we will show that any fuzzy subgroup of transformations in S is associated with a similarity on S and, conversely, any similarity on S is associated with a fuzzy subgroup of transformations on S. More precisely, in Sect. 2, we give some preliminary notions. In Sect. 3, we show how it is possible to define an equivalence relation starting from a group of transformations and conversely how to define a group of transformations starting from an equivalence relation. In Sects. 4 and 5, the results of the previous paragraph are extended to the fuzzy case, i.e. to the case in which we have similarities and fuzzy subgroups. In Sect. 6 we recall the well-known relationship between the notion of similarity and that of distance (see for examples Ruspini [11] and Valverde [13]). Finally, in Sect. 7 we use this to state a connection between fuzzy subgroups and distances. For the sake of simplicity, we refer to the lattice [0, 1], but most of our results can be extended to every complete and completely distributive lattice.

F. Formato
DMI University of Salerno, 84084 Baronissi (SA), Italy
e-mail: formato@udsab.dia.unisa.it

G. Gerla
DMI University of Salerno, 84084 Baronissi (SA), Italy
e-mail: gerla@matna2.dma.unina.it

L. Scarpati
DMI University of Salerno, 84081 Baronissi (SA), Italy
e-mail: scarpati@matna3.dma.unina.it

Correspondence to: F. Formato

2 Preliminaries

Let S be a set, then a fuzzy subset of S is any map s from S to [0, 1]. We denote by $\mathscr{F}(S)$ the class of fuzzy subsets of S. The fuzzy subsets of $S \times S$ are also called *fuzzy relations*. Given $x \in S$, the value $s(x) \in [0, 1]$ is understood as the degree or truth value of x being an element of the subset s. The basic notions of set theory are extended to the fuzzy subsets as follows. The *inclusion relation* is defined by setting, for every pair s and s' of fuzzy subsets

$$s \subseteq s' \iff s(x) \leqslant s'(x) \quad \text{for every } x \in S. \tag{1}$$

Denote by $\vee$ and $\wedge$ the maximum and the minimum operator in [0, 1] and, by $\neg$ the map defined by setting $\neg(x) = 1 - x$. Then, the *union* $s \cup s'$ and the *intersection* $s \cap s'$ of s and s' are defined by setting for every $x \in S$

$$(s \cup s')(x) = s(x) \vee s'(x) \quad \text{and} \quad (s \cap s')(x) = s(x) \wedge s'(x). \tag{2}$$

In a similar way, one defines the union and the intersection of a family of fuzzy subsets. Finally, we define the *complement* $\sim s$ of s by setting

$$(\sim s)(x) = \neg s(x). \tag{3}$$

In this way, $F(S)$ becomes an algebraic structure ($\mathscr{F}(S)$, $\cup$, $\cap$, $\sim$) that is a complete lattice with an involution. As a matter of fact, such a structure is the direct power of the structure ([0, 1], $\vee$, $\wedge$, $\neg$) with index set S. We say that a fuzzy subset s is "crisp" provided that $s(x) \in \{0, 1\}$ for every $x \in S$. We can identify the class of subsets of S with the class of crisp subsets of S in an obvious way. Given a fuzzy subset s of S, for every $\lambda \in [0, 1]$ the subsets

$$C(s, \lambda) = \{x \in S \mid s(x) \geqslant \lambda\} \text{ and } O(s, \lambda) = \{x \in S \mid s(x) > \lambda\} \tag{4}$$

are called the *closed λ-cut* and the *open λ-cut* of s, respectively. The main properties of the cuts are given in the following proposition.

Proposition 1 Let s and s' be fuzzy subsets, then for every $\lambda \in [0, 1]$

(a) $C(s, 0) = S$,
(b) $\lambda \leqslant \lambda' \ \Rightarrow\ C(s, \lambda) \supseteq C(s, \lambda')$

(c) $s \subseteq s' \;\Rightarrow\; C(s, \lambda) \subseteq C(s', \lambda)$
(d) $C(s, \lambda) = \cap_{\mu < \lambda} O(s, \mu)$
(e) $C(s \cup s', \lambda) = C(s, \lambda) \cup C(s', \lambda)$,
(f) $C(s \cap s', \lambda) = C(s, \lambda) \cap C(s', \lambda)$.

We can interpret the connective "and" by a suitable binary operation on [0, 1]. Usually, one refers to the class of continuous t-norms defined as follows.

Definition 2 Let $T : [0, 1]^2 \to [0, 1]$ be a binary operation. Then T is called a *triangular norm*, in brief a *t-norm*, if the following properties hold:

(i) T is associative,
(ii) T is commutative,
(iii) T is nondecreasing in both variables,
(iv) $T(x, 1) = x \;\forall x \in [0, 1]$.

A t-norm T is called *continuous* provided that it preserves the least upper bounds. T is called *Archimedean* if $T(x, x) < x$ for any $x \in [0, 1]$.

Given a continuous triangular norm T, we can define the *implication* and the *equivalence*, respectively, as follows:

$$x \to_T y = \max\{z \mid T(x, z) \leqslant y\}, \tag{5}$$

$$x \leftrightarrow_T y = T((x \to_T y), (y \to_T x)). \tag{6}$$

Table 1 provides examples of t-norm with the related implication and equivalence. In particular, in the first column we indicate the Lukasiewicz logical connectives.

In the following, $*$ always denotes a continuous t-norm and we write $x*y$ instead of $*(x, y)$.

The notion of similarity or fuzzy equivalence is on the basis of fuzzy set theory and the whole theory of fuzzy sets can be based on such a notion. In fact, we can define the degree with which an element x belongs to a fuzzy set as the degree of similarity of x to a suitable "prototype" in a given set of prototypes (see for example [13, 12]).

Definition 3 *A $*$-similarity*, in brief *similarity*, on a domain S is a fuzzy relation $\mathscr{R} : S \times S \to [0, 1]$ of S such that the following properties hold:

(i) $\mathscr{R}(x, x) = 1$ for every $x \in S$ (reflexivity)
(ii) $\mathscr{R}(x, y) \geqslant \mathscr{R}(y, x)$ for every $x, y \in S$ (symmetry)
(iii) $\mathscr{R}(x, z) \geqslant \mathscr{R}(x, y) * \mathscr{R}(y, z)$ for every $x, y, z \in S$ ($*$-transitivity).

The set $E(S)$ of similarities is ordered by the inclusion relation. Moreover, we have the following proposition.

Proposition 4 Let $E(S)$ be the set of all similarities on S. Then $(E(S), \subseteq)$ is a complete lattice.

If $*$ corresponds to the minimum we can characterize the just given notion in terms of cuts. Indeed, one proves that only $\mathscr{R}$ is a similarity on a set S if and only if every closed cut $C(\mathscr{R}, \lambda)$ is an equivalence relation on S.

3 A natural correspondence between equivalences and subgroups

Let S be a set and denote by Σ_S the group of transformations of S. We will show that, given a subgroup of Σ_S, we can define an equivalence on S and, conversely, that, given an equivalence on S, we can define a subgroup of Σ_S. Indeed, let $G \subseteq \Sigma_S$. Then we can set:

$$\equiv_G = \{(x, y) \in S \times S \mid \exists g \in G : g(x) = y\}. \tag{7}$$

In other words, x is related with y provided that a transformation in G exists such that y is the image of x through it. The proof of the following proposition is a matter of routine.

Proposition 5 Let $G \subseteq \Sigma_S$ be a subgroup of Σ_S. Then $\equiv_G$ is an equivalence relation.

The complete classes of equivalence module $\equiv_G$ are called "orbits". Then the orbit of x is $[x]_G = \{g(x) | g \in G\}$.

Conversely, given a binary relation $\equiv$ in S, we can define the set of transformations

$$G_{\equiv} = \{f \in \Sigma_S | x \equiv f(x) \quad \text{for any } x \in S\}. \tag{8}$$

Then a transformation f belongs to $G_{\equiv}$ provided that every element x is related with its image $f(x)$. The proof of the following proposition is a matter of routine.

Proposition 6 Let $\equiv$ be an equivalence relation. Then $G_{\equiv}$ is a subgroup of Σ_S.

Given a subgroup G of Σ_S, we can consider the equivalence $\equiv_G$ and successively the subgroup $G_{\equiv_G}$. Also, given an equivalence relation $\equiv$ on S we can define the subgroup $G_{\equiv}$ and therefore the equivalence $\equiv_{G_{\equiv}}$. The following proposition shows the connection between G and $G_{\equiv_G}$ and between $\equiv$ and $\equiv_{G_{\equiv}}$.

Table 1.

$T(\alpha, \beta)$	$\max\{\alpha+\beta-1, 0\}$	$\min\{\alpha, \beta\}$	$\alpha \cdot \beta$
$\alpha \to \beta$	$\min\{1-\alpha+\beta, 1\}$	$\begin{cases} 1 & \text{if } \alpha \leqslant \beta, \\ \beta & \text{otherwise,} \end{cases}$	$\begin{cases} 1 & \text{if } \alpha \leqslant \beta, \\ \frac{\beta}{\alpha} & \text{otherwise,} \end{cases}$
$\alpha \leftrightarrow \beta$	$1-\lvert\alpha-\beta\rvert$	$\begin{cases} 1 & \text{if } \alpha = \beta, \\ \min\{\alpha, \beta\} & \text{otherwise,} \end{cases}$	$\begin{cases} 1 & \text{if } \alpha = \beta, \\ \frac{\min\{\alpha,\beta\}}{\max\{\alpha,\beta\}} & \text{otherwise,} \end{cases}$
$\neg\alpha$	$1-\alpha$	$\begin{cases} 1 & \text{if } \alpha = 0, \\ 0 & \text{otherwise,} \end{cases}$	$\begin{cases} 1 & \text{if } \alpha = 0, \\ 0 & \text{otherwise.} \end{cases}$

Proposition 7 For every subgroup G of Σ_S, we have that $G_{\equiv_G} \supseteq G$. For every equivalence relation $\equiv$ on S, the relation $\equiv_{G_\equiv}$ coincides with $\equiv$.

Proof. We have

$$\equiv_G = \{(x,y) \mid \exists g \in G : g(x) = y\}$$

and therefore

$$G_{\equiv_G} = \{f \in \Sigma_S \mid \forall x\ x \equiv_G f(x)\} = \{f \in \Sigma_S \mid \forall x\ \exists g \in G : g(x) = f(x)\} \supseteq G.$$

Also, given an equivalence relation $\equiv$, we have

$$x \equiv_{G_\equiv} y \Leftrightarrow \exists g \in G_\equiv : g(x) = y \Leftrightarrow \exists g : \forall z\ z \equiv g(z)$$

and $g(x) = y \Rightarrow x \equiv y$.

$\Rightarrow x \equiv y$.

Conversely, let $x \equiv y$ and let's define the following transformation:

$$g(z) = \begin{cases} x & \text{if } z = y, \\ y & \text{if } z = x, \\ z & \text{otherwise.} \end{cases}$$

Then we have that $g \in G_\equiv$ and $g(x) = y$. Thus $x \equiv_{G_\equiv} y$.

Observe that $G_{\equiv_G} \neq G$, in general. For example, if we assume that S is the set of points of the Euclidean plane and G is the group of the translations, then it is immediate that $G_{\equiv_G}$ is the whole group Σ_S.

4
Any fuzzy group defines a similarity

In this section and in the next one we extend the just given definitions and results to the fuzzy subgroups and similarities.

The notion of fuzzy subgroups is defined as a many valued extension of the classical notion of subgroups (as an example see [1, 10]).

Definition 8 Let $(G, \cdot, ^{-1}, e)$ be a group. Then a *$\ast$-fuzzy subgroup*, in brief a *fuzzy subgroup*, of G is a fuzzy subset $\tilde{G}$ of G such that the following properties hold:

(i) $\tilde{G}(e) = 1$,
(ii) $\tilde{G}(x^{-1}) \geqslant \tilde{G}(x)$ for every $x \in G$,
(iii) $\tilde{G}(x \cdot y) \geqslant \tilde{G}(x) \ast \tilde{G}(y)$ for every $x, y \in G$.

As in the classical case, the following proposition holds.

Proposition 9 Let $F(G)$ be the class of fuzzy subgroups of G. Then $(F(G), \subseteq)$ is a complete lattice.

Given a fuzzy subset $\tilde{G} : \Sigma_S \to [0, 1]$ of Σ_S, we define a fuzzy relation $\mathscr{R}_{\tilde{G}}$ in the following way:

$$\mathscr{R}_{\tilde{G}}(x, y) = \sup_{g \in \Sigma_S} \{\tilde{G}(g) \mid g(x) = y\}. \tag{9}$$

We can consider $\mathscr{R}_{\tilde{G}}(x, y)$ as a multivalued valuation of the claim that a transformation g in $\tilde{G}$ exists such that $g(x) = y$.

Proposition 10 Let $\tilde{G}$ be a fuzzy subset of Σ_S. If the identity i_s belongs to $\tilde{G}$, i.e., $\tilde{G}(i_s) = 1$, then $\mathscr{R}_{\tilde{G}}$ is reflexive. If $\tilde{G}(g^{-1}) \geqslant \tilde{G}(g)$ for every $g \in \Sigma_S$, then $\mathscr{R}_{\tilde{G}}$ is symmetric. If $\tilde{G}(h \circ g) \geqslant \tilde{G}(g) \ast \tilde{G}(h)$, then $\mathscr{R}_{\tilde{G}}$ is $\ast$-transitive.

Proof. Assume $\tilde{G}(i_s) = 1$. Then

$$\mathscr{R}_{\tilde{G}}(x, x) = \sup_{g \in \Sigma_S} \{\tilde{G}(g) \mid g(x) = x\} = \tilde{G}(i_s) = 1.$$

Assume $\tilde{G}(g^{-1}) \geqslant \tilde{G}(g)$ for every $g \in \Sigma_S$. Then

$$\begin{aligned} \mathscr{R}_{\tilde{G}}(x, y) &= \sup_{g \in \Sigma_S} \{\tilde{G}(g) \mid g(x) = y\} \\ &\leqslant \sup_{g \in \Sigma_S} \{\tilde{G}(g^{-1}) \mid g^{-1}(y) = x\} = \mathscr{R}_{\tilde{G}}(y, x) \end{aligned}$$

for every $x, y \in S$. Finally, assume that $\tilde{G}(h \circ g) \geqslant \tilde{G}(g) \ast \tilde{G}(h)$, then

$$\begin{aligned} \mathscr{R}_{\tilde{G}}(x, y) \ast \mathscr{R}_{\tilde{G}}(y, z) &= \sup_{g \in \Sigma_S} \{\tilde{G}(g) \mid g(x) = y\} \ast \sup_{h \in \Sigma_S} \{\tilde{G}(h) \mid h(y) = z\} \\ &= \sup_{g, h \in \Sigma_S} \{\tilde{G}(g) \ast \tilde{G}(h) \mid g(x) = y \text{ and } h(y) = z\} \\ &\leqslant \sup_{g, h \in \Sigma_S} \{\tilde{G}(h \circ g) \mid h(g(x)) = z\} = \mathscr{R}_{\tilde{G}}(x, z). \end{aligned}$$

From Proposition (10), we have the following theorem.

Theorem 11 Let $\tilde{G}$ be a $\ast$-fuzzy subgroup of Σ_S, then $\mathscr{R}_{\tilde{G}}$ is a $\ast$-similarity in S.

In accordance with the classical case, given $a \in S$ we call *orbits* of a through $\tilde{G}$ the fuzzy set $[a]_{\tilde{G}} : S \to [0, 1]$ defined by setting

$$[a]_{\tilde{G}}(x) = \sup_{g \in \Sigma_S} \{\tilde{G}(g) \mid g(a) = x\}. \tag{10}$$

Following [8], given a similarity $\mathscr{R}$, we define an *$\mathscr{R}$-class* of an element $a \in S$ as the fuzzy subset $[a]_{\mathscr{R}}$ of S whose membership function is

$$[a]_{\mathscr{R}}(x) = \mathscr{R}(a, x). \tag{11}$$

In the case $\mathscr{R}$ is determined by a fuzzy subgroup $\tilde{G}$, an $\mathscr{R}$-class coincides with an orbit.

5
Any similarity defines a fuzzy group

Given a fuzzy relation $\mathscr{R}$, we can define a fuzzy subset $\tilde{G}_{\mathscr{R}}$ of Σ_S by setting, for any $f \in \Sigma_S$

$$\tilde{G}_{\mathscr{R}}(f) = \inf_{x \in S} \mathscr{R}(x, f(x)). \tag{12}$$

We can consider $\tilde{G}_{\mathscr{R}}(f)$ as a multivalued valuation of the claim that every $f(x)$ is related with x. In a geometrical interpretation, if $\mathscr{R}(x, y)$ is a valuation of the claim that x is "near" to y, then $\tilde{G}_{\mathscr{R}}$ is the class of transformations f such that $f(x)$ is "near" to x for any point x.

The following proposition gives the conditions under which $\tilde{G}_{\mathscr{R}}$ can be considered as a fuzzy subgroup.

Proposition 12 Let $\mathscr{R}$ be a fuzzy relation on S and $f, g \in \Sigma_S$. If $\mathscr{R}$ is reflexive, then $\tilde{G}_{\mathscr{R}}(i_s) = 1$. If $\mathscr{R}$ is symmetric, then $\tilde{G}_{\mathscr{R}}(f) \leqslant \tilde{G}_{\mathscr{R}}(f^{-1})$. If $\mathscr{R}$ is $\ast$-transitive, then $\tilde{G}_{\mathscr{R}}(f) \ast \tilde{G}_{\mathscr{R}}(g) \leqslant \tilde{G}_{\mathscr{R}}(g \circ f)$.

Proof. If $\mathcal{R}$ is reflexive, we have $\tilde{G}_{\mathcal{R}}(i_s) = \inf_{x \in S} \mathcal{R}(x, x) = 1$. Assume that $\mathcal{R}$ is symmetric, then

$$\tilde{G}_R(f) = \inf_{x \in S} \mathcal{R}(x, f(x)) = \inf_{y \in S} \mathcal{R}(f^{-1}(y), f(f^{-1}(y)))$$

$$= \inf_{y \in S} \mathcal{R}(f^{-1}(y), y) \leqslant \inf_{y \in S} \mathcal{R}(y, f^{-1}(y)) = \tilde{G}_{\mathcal{R}}(f^{-1}).$$

Finally, assume that $\mathcal{R}$ is transitive, then

$$\tilde{G}_{\mathcal{R}}(f) \star \tilde{G}_{\mathcal{R}}(g) = (\inf_{x \in S} \mathcal{R}(x, f(x))) * (\inf_{y \in S} \mathcal{R}(y, g(y)))$$

$$= \inf_{x \in S} \mathcal{R}(x, f(x)) * \inf_{x' \in S} \mathcal{R}(f(x'), g(f(x')))$$

$$= \inf_{x, x' \in S} (\mathcal{R}(x, f(x)) \star \mathcal{R}(f(x'), g(f(x'))))$$

$$\leqslant \inf_{x \in S} (\mathcal{R}(x, f(x)) \star \mathcal{R}(f(x), g(f(x))))$$

$$\leqslant \inf_{x \in S} \mathcal{R}(x, g(f(x))) = \tilde{G}_{\mathcal{R}}(g \circ f)$$

As an immediate consequence, we have the following theorem.

Theorem 13 Let $\mathcal{R}$ be a $\star$-similarity in S. Then $\tilde{G}_{\mathcal{R}}$ is a $\star$-fuzzy subgroup of Σ_S.

Theorems 11 and 13 show that there is a strict connection between the fuzzy subgroups of Σ_S and the similarities on S. Notice that it is not necessary to refer to Σ_S and that also a fuzzy subgroup of any (abstract) group defines a similarity. This is because any group G is isomorphic to a suitable group of transformations in G. Indeed, denote by $\langle a \rangle$ the element of Σ_G defined by setting $\langle a \rangle(x) = ax$ for every x in G. Then, it immediately follows that the map $h: G \to \Sigma_G$ defined by setting $h(a) = \langle a \rangle$ is an embedding of G in Σ_G. Let $\tilde{G}: G \to [0, 1]$ be any fuzzy subgroup of G. Then $\tilde{G}$ defines also a fuzzy subgroup $\tilde{G}': \Sigma_G \to [0, 1]$ of Σ_G by setting $\tilde{G}'(f) = \tilde{G}(a)$ if f coincides with $\langle a \rangle$ and $\tilde{G}'(f) = 0$ otherwise. It immediately proves that the similarity $\mathcal{R}$ in G associated with $\tilde{G}'$ in accordance with Theorem 11 satisfies the simple equality $\mathcal{R}(x, y) = \tilde{G}(xy^{-1})$. Then, if $\tilde{G}$ is crisp $\mathcal{R}$ is the well-known equivalence associated with a subgroup.

6 Distances and similarities

In this section we recall some well-known results establishing a close relation between the notion of similarity and that of distance (see for examples [13, 12]). More specifically, we call *extended real-valued pseudometric* or simply *extended pseudometric*, on a set S, a map $d: S \times S \to [0, +\infty]$ such that

(i) $d(x, x) = 0$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leqslant d(x, z) + d(z, y)$ for any $x, y, z \in S$,

where we assume that $x + (+\infty) = (+\infty) + x = +\infty$ for any $x \in [0, +\infty]$. In order to explicit such a relation, at first we recall the basic representation theorem for continuous Archimedean t-norms (see for example [6]). In the following, given an injective map $f: [0, 1] \to [0, +\infty]$ we denote by $f^{[-1]}$ the *pseudoinverse* of f, i.e.

$$f^{[-1]}(x) = \begin{cases} f^{-1}(x) & \text{if } x \in f([0, 1]), \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Theorem 14 Let $\star$ be a function from $[0, 1]^2$ into $[0, 1]$. Then $\star$ is a continuous Archimedean t-norm if and only if there exists a continuous strictly decreasing function $f: [0, 1] \to [0, +\infty]$ with $f(1) = 0$ such that for all x, y in $[0, 1]$

$$x \star y = f^{[-1]}(f(x) + f(y)). \tag{14}$$

If the conditions of Theorem 14 hold then we say that f is an *additive generator* of $\star$.

For example, the functions $f(x) = -\ln x$ and $f(x) = 1 - x$ are additive generators for the product and Lukasiewitz t-norms, respectively.

The duality between pseudometrics and similarities, in the case of continuous Archimedean t-norms, has been established by Valverde in [13].

Theorem 15 Let d be an extended pseudometric on S and f a continuous and strictly decreasing function $f: [0, 1] \to [0, +\infty]$ with $f(1) = 0$. Then the fuzzy relation $\mathcal{R}_{\mathrm{d}}$ defined by setting

$$\mathcal{R}_{\mathrm{d}}(x, y) = f^{[-1]}(d(x, y)) \tag{15}$$

is a $\star$-similarity on S, where $\star$ is the continuous, Archimedean t-norm generated by f.

The converse also holds.

Theorem 16 Let $\star$ be a continuous Archimedean t-norm, f an additive generator of $\star$ and $\mathcal{R}$ a $\star$-similarity on S. Then the function $d_{\mathcal{R}}$ defined by setting

$$d_{\mathcal{R}}(x, y) = f(\mathcal{R}(x, y)) \tag{16}$$

is an extended pseudometric on S.

For example, if d is an extended pseudometric on S and $f(x) = -\ln(x)$, then the t-norm $\star$ generated by f is the usual product and a similarity R_{d} is obtained by setting

$$\mathcal{R}_{\mathrm{d}}(x, y) = \mathrm{e}^{-d(x,y)}, \quad \forall x, y \in S. \tag{17}$$

Conversely, if $\mathcal{R}$ is a similarity with respect to the product and we set

$$d_{\mathcal{R}}(x, y) = -\ln(\mathcal{R}(x, y)), \tag{18}$$

then $d_{\mathcal{R}}$ is an extended pseudometric.

Furthermore, if $d: S \times S \to [0, 1]$ is a pseudometric and $f(x) = 1 - x$, then the t-norm $\star$ generated by f is the t-norm of Lukasiewicz and a similarity $\mathcal{R}_{\mathrm{d}}$ is obtained by setting

$$\mathcal{R}_{\mathrm{d}}(x, y) = 1 - d(x, y), \quad \forall x, y \in S. \tag{19}$$

Conversely, if $\mathcal{R}$ is a similarity with respect to the t-norm of Lukasiewicz and we set $d_{\mathcal{R}}(x, y) = 1 - \mathcal{R}(x, y)$, then $d_{\mathcal{R}}$ is a pseudometric.

The minimum is not Archimedean, obviously, so we cannot apply the just proven theorems. Nevertheless, the following proposition holds where we call *ultrapseudometric* a pseudometric such that, for any $x, y, z \in S$, $d(x, y) \leqslant d(x, z) \vee d(z, y)$.

Theorem 17 Let $\star$ be the t-norm of the minimum, $d: S \times S \to [0, 1]$ a map and set

$$\mathcal{R}(x, y) = 1 - d(x, y), \quad \forall x, y \in S. \tag{20}$$

Then $\mathscr{R}$ is a $\ast$-similarity if and only if d is an ultrapseudometric.

Proof. We confirm ourselves to observe that, if $\mathscr{R}$ is $\ast$ -transitive, then

$$1-d(x,z)\geqslant(1-d(x,y))\wedge(1-d(y,z))$$
$$=1-(d(x,y)\vee d(y,z))$$

and therefore

$$d(x,z)\leqslant d(x,y)\vee d(y,z)\quad\text{for all } x,y,z \text{ in } S.$$

7 Distances and fuzzy subgroups

In account of the results exposed in the previous sections, we can establish a relation between the notions of fuzzy subgroup of Σ_S and distance in S.

Consider at first the case of continuous Archimedean t-norms.

Theorem 18 Let d be an extended pseudometric on $S, f:[0,1]\to[0,+\infty]$ a continuous and strictly decreasing function and $\ast$ the t-norm generated by f. Then the fuzzy subset $\tilde{G}_d$ of Σ_s defined by setting, for any $g\in\Sigma_S$

$$\tilde{G}_d(g)=f^{[-1]}\left(\sup_{x\in S} d(x,g(x))\right)\tag{21}$$

is a $\ast$-fuzzy subgroup.

Proof. Let $\mathscr{R}_{\mathrm{d}}(x,y)=f^{[-1]}(d(x,y))$. By Theorem 15, $\mathscr{R}_{\mathrm{d}}$ is a $\ast$-similarity.
By Proposition 12, we have that the fuzzy subset $\tilde{G}_{\mathrm{d}}(g)$

$$\tilde{G}_{\mathrm{d}}(g)=\inf_{x\in S}\mathscr{R}_{\mathrm{d}}(x,g(x))=\inf_{x\in S}f^{[-1]}(d(x,g(x)))$$
$$=f^{[-1]}\left(\sup_{x\in S} d(x,g(x))\right)$$

is a $\ast$-fuzzy subgroup.

As an example, if d is the usual euclidean distance and $f(x)=-\ln x$, then the equality

$$\tilde{G}_{\mathrm{d}}(g)=\mathrm{e}^{-\sup_{x\in S} d(x,g(x))}=\inf_{x\in S} e^{-d(x,g(x))}\tag{22}$$

defines a fuzzy subgroup with respect to the t-norm of the product. For such a fuzzy subgroup, if g is a translation of length h, then $\tilde{G}_{\mathrm{d}}(g)=e^{-h}$, if g is a rotation $\tilde{G}_{\mathrm{d}}(g)=e^{-\infty}=0$. In the same manner, if d is a bounded pseudometric and $f(x)=1-x$, then the equality

$$\tilde{G}_{\mathrm{d}}(g)=1-\sup_{x\in S}(x,g(x))\tag{23}$$

is a fuzzy subgroup with respect to the t-norm of Lukasiewitz.

Notice that the cuts of the so-defined fuzzy subgroups are related with the notion of ε-translation introduced in [4]. Indeed, given an extended pseudometric d and an additive generator f of a t-norm, a transformation $g\in\Sigma_S$ is an ε-translation if and only if $\tilde{G}_{\mathrm{d}}(g)\geqslant\delta$, with $f(\delta)=\varepsilon$ and $\delta\in[0,1]$.

Conversely, any fuzzy subgroup defines an extended pseudometric.

Theorem 19 Let $\tilde{G}$ be a $\ast$-fuzzy subgroup of Σ_S where $\ast$ is a strict continuous and Archimedean t-norm. Then, given an additive generator f for $\ast$,

$$d(x,y)=f\left(\sup_{g\in\Sigma_S}\{\tilde{G}(g)\mid g(x)=y\}\right)\tag{24}$$

is an extended pseudometric on S.

Proof. By Theorem 11,

$$\mathscr{R}_{\tilde{G}}(x,y)=\sup_{g\in\Sigma_S}\{\tilde{G}(g)\mid g(x)=y\}$$

is a $\ast$-similarity. Since $\ast$ is a continuous and Archimedean t-norm, by Valverde's theorem, $f(\mathscr{R}_{\tilde{G}}(x,y))$ is a pseudometric on S, where f is an additive generator of $\ast$.

As an example, if $\ast$ is the t-norm of the product and $\tilde{G}$ is a $\ast$-fuzzy subgroup, then the function d defined by setting for any $x,y\in S$

$$d(x,y)=-\ln\left(\sup_{g\in\Sigma_S}\{\tilde{G}(g)\mid g(x)=y\}\right)\tag{25}$$

is an extended pseudometric. If $\ast$ is the t-norm of Lukasiewitz and $\tilde{G}$ is a $\ast$-fuzzy subgroup, then by setting

$$d(x,y)=1-\left(\sup_{g\in\Sigma_S}\{\tilde{G}(g)\mid g(x)=y\}\right),\tag{26}$$

we obtain a pseudometric.

In the case that $\ast$ is the t-norm of the minimum, $\ast$-fuzzy subgroups give rise to ultra-pseudometrics, and conversely, as stated in the following propositions.

Proposition 20 Let $\ast$ be the t-norm of the minimum and let $\tilde{G}$ be a fuzzy $\ast$-subgroup. Then the following function $d:S\times S\to[0,1]$

$$d(x,y)=1-\sup_{g\in\Sigma_S}\{\tilde{G}(g)\mid g(x)=y\}\tag{27}$$

is an ultra-pseudometric.

Proof. By Theorem 11

$$\mathscr{R}_{\tilde{G}}(x,y)=\sup_{g\in\Sigma_S}\{\tilde{G}(g)\mid g(x)=y\}$$

is a $\ast$-similarity. Since $\ast$ is the t-norm of the minimum, the thesis follows by Theorem 17.

Proposition 21 Let $d:S\times S\to[0,1]$ be an ultra-pseudometric and let $\ast$ be the t-norm of the minimum. Then the fuzzy subset $\tilde{G}$ defined by setting

$$\tilde{G}(f)=1-\sup_{x\in S} d(x,f(x))\tag{28}$$

for any $f\in\Sigma_S$, is a fuzzy $\ast$-subgroup of Σ_S.

Proof. By Theorem 17 the relation $\mathscr{R}(x,y)=1-d(x,y)$ is a $\ast$-similarity and therefore, by Theorem 13 $\tilde{G}_{\mathscr{R}}$ is a fuzzy $\ast$-subgroup. Moreover,

$$\tilde{G}_{\mathscr{R}}(f)=\inf_{x\in S}\mathscr{R}(x,f(x))=\inf_{x\in S}(1-d(x,f(x)))$$
$$=1-\sup_{x\in S} d(x,f(x)).$$

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